Записки научных
семинаров ПОМИ
Том 300, 2003 г.

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RESONANT PHENOMENA IN SLOWLY IRREGULAR RECTANGULAR WAVEGUIDES

ABSTRACT. The waveguide with a rectangular cross section of size and orientation slowly changing along the waveguide's length is considered. Methods of the canonical perturbation theory to describe the ray dynamics in the waveguide are used. As the size and orientation of cross section slowly changes along the ray trajectory, certain resonance conditions can be satisfied. The phenomena of scattering on a resonance and capture into a resonance is studied. These phenomena lead to destruction of adiabatic invariance in the system.

In view of optical-mechanical analogy [1], propagation of rays in inhomogeneous medium can be described by a Hamiltonian system. This makes geometrical optics of waveguides a natural object of nonlinear dynamics (see, e.g., [2]). In a slowly irregular waveguide, the Hamiltonian function slowly depends on the coordinate along the waveguide. Such a system can be studied using adiabatic invariants and methods of the canonical perturbation theory ([3–7]). In [5], the canonical perturbation theory was used in the case of a planar slowly irregular waveguide. This is a 2 d.o.f. slow-fast Hamiltonian system, where resonances have exponentially small effect. In the present paper we consider effects of capture into resonance and scattering on resonances in ray dynamics in a waveguide with a rectangular cross section whose size and orientation slowly changes along the waveguide's length. We use the methods of analysis of these resonant phenomena developed for Hamiltonian systems with slow and fast variables [8]. These methods were designed and the corresponding theorems were proved for smooth systems. However, in [9, 10] it was demonstrated that they can be also applied adequately for billiard systems, which possess discontinuities. Ray dynamics in a waveguide is another example of a system with discontinuities where one can try to apply these methods. In many aspects the system considered in the present paper is similar to rectangular billiard with slowly moving borders, considered in [9]. The main difference is that here we deal with a 3 d.o.f.

The work was supported in part by grants RFBR 03-01-00158 and INTAS 00-221.

system, while in [9] we considered a $2\frac{1}{2}$ d.o.f. system (i.e., a system with explicit time dependence).

Consider rays propagating in a waveguide of rectangular orthogonal cross section. Let x, y be coordinates in the plane orthogonal to the axis of the waveguide, and z be coordinate along this axis. Let the size and orientation of the rectangle in the cross section of the waveguide slowly depend on z. The propagation of rays in the waveguide is described with the following Hamiltonian function:

$$H = p_x^2 + p_y^2 + p_z^2 - n^2(x, y, \varepsilon z), \tag{1}$$

where p_x, p_y, p_z are momenta canonically conjugated to coordinates x, y, z, respectively, and $n(x, y, \varepsilon z)$ is the refraction index, $\varepsilon \ll 1$ is a positive small parameter. The system should be considered on the Hamiltonian level $H = 0$. As z-coordinate changes, the rectangular cross section slowly turns around its center by the angle $\alpha = \alpha(\varepsilon z)$. At the same time, the waveguide's axis can slowly bend. Therefore, the refraction index can be expressed as $n = n(x \cos\alpha(\varepsilon z) + y \sin\alpha(\varepsilon z) + u(\varepsilon z), -x \sin\alpha(\varepsilon z) + y \cos\alpha(\varepsilon z) + v(\varepsilon z), \varepsilon z)$. Functions u and v describe bending of the axis. We make a canonical transformation of variables $(p_x, p_y, p_z, x, y, z) \mapsto (p_1, p_2, p_3, q_1, q_2, \hat{q_3})$ defined by the generating function

$$(x \cos\alpha(\varepsilon z) + y \sin\alpha(\varepsilon z) + u(\varepsilon z))p_1 +$$
$$+(-x \sin\alpha(\varepsilon z) + y \cos\alpha(\varepsilon z) + v(\varepsilon z))p_2 + p_3 z.$$

Introduce also $q_3 = \varepsilon \hat{q_3}$. In the new coordinates, the Hamiltonian function takes the form

$$\begin{aligned} H = p_1^2 + p_2^2 + p_3^2 + 2\varepsilon p_3 \beta(p_1 q_2 - p_2 q_1) - \\ - n^2(q_1, q_2, q_3) + \varepsilon p_3(u' + v') + O(\varepsilon^2), \end{aligned} \tag{2}$$

where canonically conjugated are p_1, p_2, p_3 and $q_1, q_2, \varepsilon^{-1} q_3$; β, u', v' denote derivatives of α, u, v w.r.t. their argument correspondingly. Variables (p_1, q_1) and (p_2, q_2) are fast; they change at a rate of order 1. Variables (p_3, q_3) are slow; they change at a rate of order ε. Assume the following simple coordinate dependence for the normalized refraction index:

$$n(q_1, q_2, q_3) = \begin{cases} n_0 & \text{if } |q_1| < d_1(q_3), \quad \text{and} \quad |q_2| < d_2(q_3), \\ n_1 & \text{if } |q_1| > d_1(q_3) \quad \text{or} \quad |q_2| > d_2(q_3), \end{cases} \tag{3}$$

where $2d_i$ are the lengths of the sides of the rectangular cross section, which depend on q_3; n_0, $n_1 = \text{const}$, $0 < n_1 < n_0$ for a waveguide using total inner reflection and one can formally put $n_1 = 0$ for a waveguide with perfectly reflecting walls. We use a canonical transformation to introduce new variables (I_i, ϕ_i), $i = 1, 2$ such that

$$q_i = \begin{cases} -d_i + 2d_i\phi_i/\pi & \text{if } 0 < \phi_i \leqslant \pi, \\ 3d_i - 2d_i\phi_i/\pi & \text{if } \pi < \phi_i \leqslant 2\pi, \end{cases} \tag{4}$$
$$p_i = \frac{\pi I_i}{2d_i} \,\mathrm{sign}\,(\sin\phi_i).$$

If d_i are constant, and $\beta = 0$, the new variables (I_i, ϕ_i) are just the action-angle variables of the system.

Variables p_i and q_i $(i = 1, 2)$ can be expressed in terms of (I_i, ϕ_i) as the Fourier series:

$$p_i = {\sum_{k=1}^{\infty}}' \frac{2I_i}{d_i k} \sin k\phi_i, \tag{5}$$
$$q_i = -{\sum_{k=1}^{\infty}}' \frac{8d_i}{\pi^2 k^2} \cos k\phi_i,$$

where primes denote summation over odd k. In the new variables the Hamiltonian of the system is:

$$\begin{aligned} \mathcal{H} = \frac{\pi^2}{4}\left(\frac{I_1^2}{d_1^2} + \frac{I_2^2}{d_2^2}\right) + p_3^2 - n_0^2 - \\ -\frac{16\varepsilon p_3\beta}{\pi^2} {\sum_{k_1=1}^{\infty}}' {\sum_{k_2=1}^{\infty}}' \left[\frac{(I_1 d_2^2 k_1 + I_2 d_1^2 k_2)}{d_1 d_2 k_1^2 k_2^2} \sin(k_1\phi_1 - k_2\phi_2) + \right. \\ \left. + \frac{(I_1 d_2^2 k_1 - I_2 d_1^2 k_2)}{d_1 d_2 k_1^2 k_2^2} \sin(k_1\phi_1 + k_2\phi_2)\right] + \\ + \varepsilon\left[E_1(\phi_1, p_3, q_3) + E_2(\phi_2, p_3, q_3)\right] + O(\varepsilon^2) = H_0 + \varepsilon H_1. \end{aligned} \tag{6}$$

Variables p_3, q_3 in (6) are $O(\varepsilon)$-close to p_3, q_3 in (2); for brevity, we use for them the same notation. Terms E_i appear because generating function of transformation (4) depends on slow variables p_3, q_3; the explicit form of these terms is not important.

In the adiabatic approximation, one averages (6) over fast angle variables ϕ_1, ϕ_2 to find that I_1, I_2 are adiabatic invariants of the motion, and the dynamics is approximately described by the Hamiltonian H_0 considered on the energy level $H_0 = 0$ with constant values of I_1, I_2. It is known that I_i equals $d_i \sin\psi_i$ multiplied by a constant factor, where ψ_i is the angle of reflection by the corresponding wall. This implies, in particular, that an initially parallel beam of rays stays parallel in the course of propagation along the waveguide.

This approach, however, is not applicable near resonances, where some terms of the sum in (6) change slowly. Let m, j be relatively prime integers. We call the terms in (6) with $k_1 = rm$, $k_2 = rj$ and integer r (m, j)-resonant, if the following resonance condition is met:

$$\frac{mI_1}{d_1^2} - \frac{jI_2}{d_2^2} = 0. \tag{7}$$

As the time grows, the values of d_i change and the system may approach a resonance. Consider the ray dynamics near an (m, j)-resonance. We make a canonical transformation $(I_1, I_2, \phi_1, \phi_2) \mapsto (R, J, \phi, \psi)$ with the generating function $W = (m\phi_1 - j\phi_2)R - (l_1\phi_1 - l_2\phi_2)J$, where l_1, l_2 are integers, $ml_2 - jl_1 = 1$. Variables p_3, q_3 remain unchanged under this transformation.

The resonance condition takes the form

$$R = R_{\rm res}(J, p_3, q_3) = \frac{jl_2 d_1^2 + ml_1 d_2^2}{m^2 d_2^2 + j^2 d_1^2} J. \tag{8}$$

In the new variables Hamiltonian (6) is $\mathcal{H} = \mathcal{H}_0(R, J, p_3, q_3) + \varepsilon\mathcal{H}_1(R, J, \phi, \psi, p_3, q_3)$. The canonically conjugated pairs of variables are (R, ϕ), (J, ψ) and $(p_3, \varepsilon^{-1}q_3)$. It follows from the form of the Hamiltonian that ψ is a fast variable, and one can average the equations of motion over it. In the averaged system, $J = -jI_1 + mI_2$ is the integral of motion and below we consider J as a parameter. We use another canonical transformation $(R, \phi) \mapsto (P, \phi)$ with generating function $W_1 = (P + R_{\rm res}(J, \bar{p}_3, q_3))\phi + \varepsilon^{-1}\bar{p}_3 q_3$ to introduce new "action" variable $P = R - R_{\rm res}$; the conjugated "angle" variable is ϕ. In a small neighborhood of the resonance where P is of order $\sqrt{\varepsilon}$, the Hamiltonian takes the following form (bars omitted):

$$\mathcal{H} = \Lambda(J, p_3, q_3) + F_0 + O(\varepsilon^2), \quad F_0 = \frac{1}{2} g P^2 + b\chi(\phi) + a\phi, \tag{9}$$

where Λ is $\mathcal{H}_0$ on the resonance, and

$$\begin{aligned} g &= \left(\frac{\partial^2 \mathcal{H}_0}{\partial R^2}\right)_{R = R_{\rm res}} = \frac{\pi^2}{2}\left(\frac{m^2}{d_1^2} + \frac{j^2}{d_2^2}\right), \\ b &= -4\frac{\varepsilon p_3 \beta d_1 d_2 J}{mj(m^2 d_2^2 + j^2 d_1^2)}, \end{aligned} \tag{10}$$

$$\chi(\phi)=\frac{8}{\pi^2}{\sum_{r=1}^{\infty}}'\frac{1}{r^3}\sin r\phi=\begin{cases}\phi-\phi^2/\pi & \text{if}\quad 0<\phi<\pi,\\ \phi+\phi^2/\pi & \text{if}\quad -\pi<\phi\leqslant 0.\end{cases}$$

For the coefficient a we have the following expression (see [8]):

$$a=(\omega_2/m)(L/\det A),\quad L=\omega_2\{\omega_1,H_0\}-\omega_1\{\omega_2,H_0\},$$
$$A=\begin{pmatrix}\partial^2H_0/\partial I^2 & \omega^T\\ \omega & 0\end{pmatrix}, \tag{11}$$

where $\omega=(\omega_1,\omega_2)$, $\omega_1=(\pi^2/2)(I_1/d_1^2)$, $\omega_2=(\pi^2/2)(I_2/d_2^2)$, and $\{f,g\}$ denotes the Poisson bracket of f, g w.r.t. variables p_3, q_3. Values of d_i, ω_i, L, $\det A$ in expressions (10) and (11) are calculated on the resonant surface.

The dynamics near the resonance depends on the properties of the resonance Hamiltonian F_0. Consider this Hamiltonian at frozen values of J, p_3, q_3. Corresponding phase portraits of the resonance Hamiltonian on the (ϕ,P)-plane are similar to those of a pendulum under the action of a constant torque. If $|a|<|b|$, the phase portrait has a separatrix and an oscillatory domain. If $|a|>|b|$, there is no such a domain on the phase portrait. To be definite, assume $a<0$, $b>0$. The area S of the oscillatory domain (in the case $|a|<|b|$) is a function of J, p_3, q_3, and can be expressed as follows:

$$S=2\sqrt{\frac{2}{g}}\int_{\phi_l}^{\phi_r}\sqrt{F_0^s-b\chi(\phi)-a\phi}\,\mathrm{d}\,\phi, \tag{12}$$

where F_0^s is the value of F_0 on the separatrix

$$F_0^s=\frac{\pi}{4}\frac{(a+b)^2}{b}, \tag{13}$$

and ϕ_l, ϕ_r are the roots of the integrand (in particular, ϕ_r is the coordinate of the saddle point). The explicit expression for ϕ_l depends on the relation between a and b. Consider the case $a<(2\sqrt{2}-3)b$ (the expression for the opposite sign in this inequality is cumbersome, and we do not present it here). We have

$$\phi_l=-\frac{\pi}{2}\frac{a+b}{b}(1+\sqrt{2}),\quad \phi_r=\frac{\pi}{2}\frac{a+b}{b}. \tag{14}$$

For the area S one finds:

$$S=\sqrt{\frac{2}{g}}\frac{(a+b)^2}{2}\left(1+\frac{3\pi}{4}\right)\left(\frac{\pi}{b}\right)^{3/2}. \tag{15}$$

Now we take into account a slow evolution of variables p_3, q_3 in the system. Suppose that due to this evolution the area of the oscillation region S grows with time. In this case, additional space appears inside the oscillatory domain. Hence, phase points can cross the separatrix and change their mode of motion from rotational to oscillatory. This phenomenon is called *capture into a resonance*. A captured phase point leaves a vicinity of the straight line $R = \text{const}$, $J = \text{const}$ in the space (I_1, I_2, p_3, q_3) and continues its motion following approximately the resonant curve $R = R_{\text{res}}(J, p_3, q_3)$, $J = \text{const}$. In this case, direction of ray propagation is not anymore determined by equations $I_{1,2} = \text{const}$, and can be found from equations $R = R_{\text{res}}$, $J = \text{const}$. If later the value of S starts decreasing with time, phase points previously captured can cross the separatrix again and leave the oscillatory domain. This is escape from the resonance.

Phase points that cross the resonance without being captured undergo a jump of adiabatic invariant R of order $\sqrt{\varepsilon}$ (and, therefore, jumps of $I_{1,2}$, since $I_1 = mR - l_1 J$, $I_2 = -jR + l_2 J$). This phenomenon is called *scattering on a resonance*. Both capture into a resonance and scattering on a resonance are probabilistic phenomena, and they are usual in systems with resonance crossings (see [8, 11, 12]). If a phase point M_0 is an initial point of a phase trajectory crossing the resonance surface, the probability to be captured at the resonance crossing is given by the following formula:

$$\Pr(M_0) = \begin{cases} \sqrt{\varepsilon}\,\{S, \Lambda\}/(2\pi|a|) & \text{if} \quad \{S, \Lambda\} > 0, \\ 0 & \text{if} \quad \{S, \Lambda\} < 0. \end{cases} \tag{16}$$

Expression in the r.h.s. of (16) is calculated at the point where the adiabatic trajectory (i.e., one with $I_{1,2} = \text{const}$) starting at M_0 crosses the resonant surface. Therefore, if $d_i = d_i(q_3)$ are periodic functions with period of order 1, then along the distance $\Delta q_3 \sim \varepsilon^{-1/2}$ (corresponding to $\Delta z \sim \varepsilon^{-3/2}$) phase trajectories with initial conditions of total measure ~ 1 undergo captures. Along the corresponding phase trajectories, adiabatic invariants I_i change by values of order 1. As a result, initially parallel beam of rays in a waveguide with reflective walls can be completely destroyed. In a waveguide using total inner reflection, this can produce large losses due to penetration of rays through the walls.

A jump in R can be considered as random with a mean value of order $\sqrt{\varepsilon}$ and standard deviation also of order $\sqrt{\varepsilon}$. This jump produces a jump of order $\sqrt{\varepsilon}$ in $I_{1,2}$. These jumps due to multiple scattering (without captures) on resonances result in drift and diffusion of $I_{1,2}$. The diffusion produces slow divergence of an initially parallel beam of rays. Along a distance $\Delta q_3 \sim \varepsilon^{-1}$ (corresponding to $\Delta z \sim \varepsilon^{-2}$) the diffusion of adiabatic invariants $I_{1,2}$ leads to complete destruction of the beam.

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Поступило 30 ноября 2002 г.