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ON HAMILTONIAN SYSTEMS WITH HOMOCLINIC ORBIT TO A SADDLE-CENTER

Abstract. We consider a real analytic two degrees of freedom Hamiltonian system possessing a homoclinic orbit to a saddle-center equilibrium (two nonzero real and two nonzero imaginary eigenvalues). We take a two-parameter unfolding for such the system and show that in nonresonance case there are countable sets of multi-round homoclinic orbits to a saddle-center. We also find families of periodic orbits, accumulating at homoclinic orbits.

1. Introduction and set-up

Let a real analytic Hamiltonian vector field X_{H_0} with a Hamiltonian H_0 have a saddle-center equilibrium p, $H_0(p) = 0$, i.e., eigenvalues of the corresponding linearization operator are $\pm i\omega$, $\pm\lambda$, $\omega\lambda \neq 0$, ω, $\lambda \in R$. Such equilibrium has two analytic one-dimensional submanifolds, stable, W^s, and unstable, W^u, passing through p. Suppose one component of the set $W^s \backslash \{p\}$ coincides with one component of the set $W^u \backslash \{p\}$ forming a homoclinic loop Γ lying within a three-dimensional level $V_0 = \{H_0 = 0\}$. Since the existence of the loop is a codimension two phenomenon then bifurcations for Hamiltonian systems close to the system with the loop should be studied in a two-parameter unfolding H_μ, $\mu \in R^2$. Let H_μ, $\mu \in R^2$, C^r-smoothly depends on μ, $r \geqslant 3$, and suppose the Hamiltonian H_0 has a saddle-center singular point p with a loop Γ. All our considerations will be done in some small neighbourhood U of Γ.

Without loss of generality we may consider all Hamiltonians H_μ to have the same singular point p (i.e., p does not depend on μ for $\| \mu \|$ small enough) and $H_\mu(p) \equiv 0$. Due to the Moser theorem [6] there is a symplectic chart with coordinates (x_1, x_2, y_1, y_2) such that all H_μ can be written in the normal form

$$H_\mu(x_1, x_2, y_1, y_2) = h(\xi, \eta) = \lambda\eta + \omega\eta + \cdots,$$

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where $\xi = x_1y_1$, $\eta = (x_2^2 + y_2^2)/2$, with λ, ω being functions of μ, $\omega(0)\lambda(0) \neq 0$. After a linear scaling of H_μ (or time), depending on μ, we can assume $\lambda(\mu) \equiv 1$. For definiteness, we suppose henceforth $\gamma(\mu) = \omega(\mu)/\lambda(\mu)$ to be positive. In these coordinates differential equations take the form

$$\dot{x}_1 = -x_1\frac{\partial h}{\partial \xi}, \quad \dot{x}_2 = -y_2\frac{\partial h}{\partial \eta}, \quad \dot{y}_1 = y_1\frac{\partial h}{\partial \xi}, \quad \dot{y}_2 = x_2\frac{\partial h}{\partial \eta}.$$

Thus the local stable submanifold W^s is the axis x_1 and the axis y_1 is the local unstable submanifold W^u. For definiteness we suppose the outgoing part of Γ to coincide locally with the half-axis $y_1 > 0$. Depending on whether Γ returns to p being locally the half-axis $x_1 < 0$ or $x_1 > 0$ we distinguish two cases, I and II. These cases have different topology of levels $V_h = \{H_\mu = h\}$. In the case I all orbits starting near the homoclinic one stay close to it for one round along homoclinic orbit.

In order to study the orbit behaviour near the loop Γ, we use the Poincare map f_μ at some section N. Map f_μ is constructed as a composition of two maps, one local near the singular point, T_μ, and another one global, S_μ, near a global piece of Γ. As local three-dimensional cross-sections we take the disks $N^s = \{x_1 = \pm d_1, \eta \leqslant \eta_0\}$ and $N^u = \{y_1 = d_2, \eta \leqslant \eta_0\}$ for some small positive η_0. These sections are foliated by levels V_h into smooth two-dimensional disks which we denote N_h^s, N_h^u, respectively. Since the sets V_h are invariant under the flow, it is convenient to restrict the maps $T_\mu : N^s \to N^u$ and $S_\mu : N^u \to N^s$ to disks N_h^s, N_h^u.

The local map $T_\mu(h) : N_h^s \to N_h^u$ in the coordinates (x_2, y_2) can be written in the following form [4]:

$$x_2^1 = x_2^0 \cos\Delta_h(\eta_0) - y_2^0 \sin\Delta_h(\eta_0), \quad y_2^1 = x_2^0 \sin\Delta_h(\eta_0) + y_2^0 \cos\Delta_h(\eta_0),$$

where (x_2^1, y_2^1) are coordinates on N_h^u and (x_2^0, y_2^0) are coordinates on N_h^s, $\Delta_h(\eta) = -a_h'(\eta)\ln(\pm d_1 d_2/a_h(\eta))$, $a_h(\eta)$ is the solution of the equation $h(\xi,\eta) = h$ with respect to ξ in some neighbourhood of the point $(0, 0)$, $a_h(\eta) = h - \gamma\eta + \cdots$, $a_h'(\eta) = da_h(\eta)/d\eta$. The global map $S_\mu(h) : N_h^u \to N_h^s$ is some analytic symplectic map smoothly depending on μ:

$$x_2^0 = F_\mu(x_2^1, y_2^1, h), \quad y_2^0 = G_\mu(x_2^1, y_2^1, h),$$

where $\det(D(F_\mu, G_\mu)/D(x_2^1, y_2^1)) \equiv 1, F_0(0, 0, 0) = G_0(0, 0, 0) = 0$ (the latter follows from the loop existence). Henceforth we shall consider such two-parameter unfolding of the system with a loop for which the following genericity condition is satisfied: $\det(D(F_\mu, G_\mu)/D(\mu_1, \mu_2)) \neq 0$ at the point $x_2^1 = y_2^1 = h = \mu_1 = \mu_2 = 0$. Let us denote the Jacobi matrix $D(F_\mu, G_\mu)/D(x_2^1, y_2^1)$ calculated at $x_2^1 = y_2^1 = h = \mu_1 = \mu_2 = 0$ by A.

2. Multi-round homoclinic orbits

Now consider the level V_0 where the equilibrium point is located. Later on we will write f_μ, S_μ, T_μ instead of $f_\mu(0)$, $S_\mu(0)$, $T_\mu(0)$.

Global map, and, therefore A depend on a choice of the cross-sections, determined by constants d_1, d_2. We can choose some positive constants $\tilde{d}_1$ and $\tilde{d}_2$ such that A has the diagonal form $\mathrm{diag}(b, b^{-1})$ [3]. So the global map takes the form:

$$x_2^0 = u_1 + bx_2^1 + O(\| u \|^2 + \| X^1 \|^2),$$

$$y_2^0 = u_2 + b^{-1}y_2^1 + O(\| u \|^2 + \| X^1 \|^2),$$

where $X^1 = (x_2^1, y_2^1)$, $u = R\mu$, R is Jacobi matrix $D(F_\mu, G_\mu)/D(\mu_1, \mu_2)$ calculated at $x_2^1 = y_2^1 = h = \mu_1 = \mu_2 = 0$. Later on we denote new parameters as u, the coordinates (x_2^1, y_2^1) in N_0^u - as (x, y), and the coordinates (x_2^0, y_2^0) in N_0^s - as $(\bar{x}, \bar{y})$.

Evidently, for $u \neq 0$ the loop is already destroyed, but some multi-round loops can exist. The most complete investigation of n-round homoclinic orbits to a saddle-center in a one-parameter unfolding of reversible two-degree-of-freedom Hamiltonian systems was accomplished in [1, 5].

The existence of homoclinic doubling and tripling was proved before [2, 3]:

Theorem 1. *Assume $b \neq 1$. Then there is a positive ε small enough and tubular neighbourhood U of Γ in the level V_0 such that in a neighbourhood $E_\varepsilon = \{u : \| u \| < \varepsilon\}$ there exist four countable families of points*

$$\Omega_2 = \{u_j^i : i = \overline{1,4}, \| u_j^i \| \to 0 \ as \ j \to \infty\},$$

each its point corresponds to a Hamiltonian system with two-round homoclinic orbit to p lying in U, and there exist eight countable families

$$\Omega_3 = \{u_j^i : i = \overline{1,8}, \|u_j^i\| \to 0 \ as \ j \to \infty\},$$

each its point corresponds to a Hamiltonian systems with three-round homoclinic orbit to p lying in U.

Moreover, when $b > (1+\sqrt{5})/2$ there are other families of u-values which correspond to three-round orbits to p.

The results of this paper are concerned with the existence of n-round orbits, $n = 2^k \cdot 3^m$, where $k, m \in Z^+$, $k + m \geqslant 1$. At first we prove the existence of $2 \cdot 2^k$ and $3 \cdot 2^k$-round homoclinic loops.

Theorem 2. *Assume*

(1) $b \neq 1$;

(2) *parameters b and γ do not satisfy the resonance relations of the kth order:*

$$F_j(a,\gamma) = 0, \quad j = \overline{1,k}, \quad a = b - b^{-1}, \tag{1}$$

where

$$F_1(a,\gamma) = a - (-1)^i \gamma(a^2+1), i = 1, 2,$$
$$F_2(a,\gamma) = F_1(2F_1(a,\gamma),\gamma),$$
$$\ldots.$$
$$F_k(a,\gamma) = F_1(2F_{k-1}(a,\gamma),\gamma).$$

Then for any $k \in Z^+$ and $u^* \in \Omega_{2^k}$ there is a tubular neighborhood U of Γ, $U \subset V_0$, and a positive $\varepsilon(k)$ small enough, such that on parameter plane (u_1, u_2) in a neighborhood $E_{\varepsilon(k)} = \{u : \|u - u^*\| < \varepsilon(k)\}$ there exist sets

$$\Omega_n = \left\{u_j^i : i = \overline{1,4}, \|u_j^i - u^*\| \to 0 \text{ as } j \to \infty\right\} \text{ for } n = 2 \cdot 2^k$$

and

$$\Omega_n = \left\{u_j^i : i = \overline{1,8}, \|u_j^i - u^*\| \to 0 \text{ as } j \to \infty\right\} \text{ for } n = 3 \cdot 2^k,$$

corresponding to systems with Hamiltonians H_u, $u \in \Omega_n$, having n-round homoclinic orbit to p.

Proof. This proof has a recurrent character. Consider first $k = 1$. For a given two-round homoclinic orbit corresponding to some u^* from Ω_2 we construct a new global map S_1 along its global piece. Jacobi matrix $DS_1/D(x,y)$ for $x = y = 0, u = u^*$ can become a rotation matrix under some relations between b and γ (the resonance relation). In the nonresonance case, from Theorem 1, we get sets of 4-round and 6-round homoclinic orbits near Γ.

Then choosing Γ_{u^*} as Γ and repeating this proof for $k = 2$ we get existence of 4-round and 6-round homoclinic orbits relatively Γ_{u^*} (or 8-round and 12-round homoclinic orbits relatively Γ).

Repeating this proof for $k = 3, \ldots,$ we get conditions of the theorem about existence of n-round homoclinic orbits for $n = m \cdot 2^k$, $m = 2, 3$ and the resonance relations (1). $\square$

The resonance relations can be used to answer the question about the existence of multi-round homoclinic orbits for a Hamiltonian system with a homoclinic loop with fixed values of parameters b and γ.

Let b and γ be connected by the resonance relation of the order k with some index i (for example $i = 1$). We consider four families of parameter values from Ω_2, $i = \overline{1,4}$, corresponding to those Hamiltonian systems in the unfolding that have two-round homoclinic orbits (they have been obtained in Theorem 1). If we take some $u^* = u^1_{j*}$ (or $u^* = u^3_{j*}$) from Ω_2, then there exists a neighbourhood in the parameter plane $|u - u^*| < \varepsilon$ which consists of countable sets of parameter values, corresponding to Hamiltonian systems with n-round homoclinic orbits of finite roundness $2 \cdot 2^m$, $3 \cdot 2^m$ for $m = \overline{1, k-1}$ only. Choosing a parameter from the remaining family $u^* = u^2_{j*}$ (or $u^* = u^4_{j*}$) we obtain the countable sets of parameter values in some neighbourhood $|u - u^*| < \varepsilon$. The elements of these sets correspond to Hamiltonian systems with multi-round homoclinic orbits of any roundness $2 \cdot 2^m, 3 \cdot 2^m$, $m = 1, 2, \ldots$.

Now we prove the existence of countable sets of parameter values corresponding to systems with 6-round and 9-round homoclinic orbits.

Theorem 3. *Assume*

(1) $b \neq 1$;

(2) *parameters b and γ do not satisfy the resonance relations:*

$$\gamma = (-1)^j \frac{(-1)^{i+1}(b^3 - b^{-3})\sqrt{b^2 + b^{-2} + 1} + 2}{(b^2 + b^{-2} - 1)(b^3 - b^{-3} + 2(-1)^i \sqrt{b^2 + b^{-2} + 1})},$$

$$i, j = 0, 1. \tag{2}$$

Then for any $u^* \in \Omega_3$ there is a tubular neighborhood U of Γ, $U \subset V_0$, and a positive ε small enough, such that on parameter plane (u_1, u_2) in a neighborhood $E_\varepsilon = \{u : ||u - u^*|| < \varepsilon\}$ there exist sets

$$\Omega_6 = \{u^i_j : i = \overline{1,4}, ||u^i_j - u^*|| \to 0 \text{ as } j \to \infty\}$$

and

$$\Omega_9 = \{u_j^i : i = \overline{1,8}, ||u_j^i - u^*|| \to 0 \ as \ j \to \infty\},$$

corresponding to systems having 6-round and 9-round homoclinic orbits to p.

Proof. We take an $u^* \in \Omega_3$, corresponding to the system with three-round homoclinic orbit Γ_{u^*} to p. Let us determine the Poincare map as a composition of two maps, one local near the singular point and another one global near a global piece of Γ_{u^*}.

If Jacobi matrix $DS_1/D(x,y)$ for $x = y = 0, u = u^*$ is a rotation matrix, then, depending on which of eight families u^* belongs to, we will get one from the resonance relations (2). In the nonresonance case, by Theorem 1, in some neighborhood E_ε in parameter plane there exist sets

$$\Omega_6 = \{u_j^i : i = \overline{1,4}, ||u_j^i - u^*|| \to 0 \ as \ j \to \infty\}$$

and

$$\Omega_9 = \{u_j^i : i = \overline{1,8}, ||u_j^i - u^*|| \to 0 \ as \ j \to \infty\},$$

corresponding to the systems with 6-round and 9-round homoclinic orbits. □

On the next stage we take three-round homoclinic orbit Γ_{u^*} as Γ and applying Theorem 3 to this case we get sets Ω_{18} and Ω_{27}. Repeating this procedure we get the existence of n-round homoclinic orbits for $n = m \cdot 3^k$, $m = 2, 3$, and the resonance relations (2), where instead of parameter b we substitute the parameter of new global map b_k.

As consequence of Theorems 2, 3 we obtain the existence of $2^k \cdot 3^m$-round homoclinic orbits. Indeed, choosing a multi-round homoclinic loop with roundnees $3 \cdot 2^k$ (or $2 \cdot 3^k$) and applying Theorem 3 (or Theorem 2) we get $2^k \cdot 3^m$-round (or $3^k \cdot 2^m$-round) homoclinic loops.

3. Periodic orbits

Now we briefly discuss the existence of families of periodic orbits in the level V_0. As was proved earlyer (see [3]), if $b \neq 1$ then there exist four countable families of one-round periodic orbits in some neighbourhood of Γ which accumulate at Γ within the level V_0. The type of these orbits depends on some quantity $s = \gamma(b - b^{-1})$, namely, if $|s| < 2$, then two from these families consist of elliptic periodic orbits and two remaining families consist of hyperbolic ones. If $|s| > 2$ then all the periodic orbits of these families are hyperbolic.

We apply this result to neighbourhoods of n-round homoclinic orbits Γ_{u^*}, $u^* \in \Omega_n$, where $n = 2^k \cdot 3^m$ for some k and m (for definitness, we suppose $k \geqslant 1$, i.e., $n = 2n_0$). As was proved above such homoclinic orbit exists if b and γ are not connected by the resonance relation. This relation appears from the condition that Jacobi matrix $D\tilde{S}/DX$ calculated at $x = y = 0$, $u = u^*$ is not a rotation matrix, where $\tilde{S}$ is the global map near a global piece of n_0-round homoclinic orbit. Therefore $\tilde{b} \neq 1$ and the result [3] can be applied to this case. So the existence of $2n_0$-round homoclinic orbis involves the existence of four countable families of one-round periodic orbits relatively Γ_{n_0}-round homoclinic loop. Theirs types depend on $\gamma(\tilde{b} - \tilde{b}^{-1})$.

Thus in the nonresonance case in some neighbourhood of every multi-round homoclinic orbit, obtained above, we get four countable families of one-round periodic orbits.

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