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TWO-DIMENSIONAL CHAOS: THE BAKER MAP UNDER CONTROL

Abstract. Some results on the stochastic control of two-dimensional chaotic map, i.e., baker map are presented. The approach is based on probabilistic coupling of the controlled dynamics with a controlling system and subsequent lift of the coupled dynamics in a suitable functional space. The lifted dynamics is described in terms of probability densities and is governed by linear Perron–Frobenius and Koopman operators. Sufficient condition of controllability and estimation for time to achieve control for a given accuracy in terms of spectral decomposition of Perron-Frobenius operator are obtained.

1. Introduction.

It is well-known that chaotic dynamical systems pose a great challenge for the problem of controlling their periodic orbits. It is due to the fact that such systems possess positive Lyapunov exponents and therefore any trajectory of chaotic system is intrinsically unstable. Any controller eager to stabilize an orbit using conventional methods of feedback control has to exert perturbations to the system continuously monitoring its state and adjusting control. Hence conventional control methods such as OGY method [2] are inherently complex from the computational point of view and this drawback can not be overcome without introducing new ideas.

That is why a novel approach to controlling chaotic dynamical systems was proposed recently in [6]. The method is called probabilistic control of chaos.

The main idea of the method [6] is to couple probabilistically a given dynamical system S with another (controlling) system C which represents a contraction on a subset of a phase space. The resulting dynamics is then lifted to a suitable rigged Hilbert space and studied using Frobenius–Perron and Koopman operators.

In [6] the approach is illustrated on one-dimensional system, namely Renyi map which is considered to be a prototype of one-dimensional chaotic systems. It is also announced that it can be applied to a wider class of systems, namely two-dimensional ones. However the spectral details are not discussed. In the present paper we apply the method of probabilistic control to the baker map which is a model two-dimensional chaotic system. We find out what are the subtleties of a two-dimensional case compared with one-dimensional one and what modifications in the scheme of probabilistic coupling are needed in order to achieve the desired result of control.

2. The Baker map.

Two-dimensional baker map is defined on the square $X = [0,1] \times [0,1]$ by relations

$$F : (x,y) \mapsto F(x,y) = \begin{cases} \left(2x, \dfrac{y}{2}\right), & 0 \leqslant x \leqslant \dfrac{1}{2} \\ \left(2x-1, \dfrac{y+1}{2}\right), & \dfrac{1}{2} < x \leqslant 1. \end{cases}$$

It is known [1] that this is a one-to-one map of X on itself which defines mixing measure-preserving dynamical system and even a K-automorphism. It is easy to see that the projection of the F map onto x axis is a Bernoulli shift:

$$x \mapsto f(x) = \begin{cases} 2x, & 0 \leqslant x \leqslant \dfrac{1}{2} \\ 2x-1, & \dfrac{1}{2} < x \leqslant 1. \end{cases}$$

Because the baker map is measure-preserving the Koopman (K_b) and Perron–Frobenius (U_b) operators for the baker map are given by equations

$$(K_b B)(x,y) = B(F(x,y)), \quad (U_b A)(x,y) = A(F^{-1}(x,y))$$

correspondingly, where $A(x,y)$ is a distribution density on the square $X = [0,1] \times [0,1]$ and $B(x,y)$ is an observable. Initially the operators U_b and K_b are defined in Hilbert space $H = L^2(X)$, and they can be lifted into rigged Hilbert space $\Phi \subset H \subset \Phi^X$ such that $A(x,y) \in \Phi$ and $B(x,y) \in \Phi^X$. Here the space Φ^X is a dual space of Φ, and Φ must satisfy the following conditions:

1) Φ is a locally convex topological vector space with a topology stronger than that of H; 2) $[\Phi]_H = H$; 3) $K_b \Phi \subset \Phi$.

The choice of space Φ is determined by physical considerations and can be made in several ways. In the next section we choose the actual spaces Φ and Φ^X in the stochastic control problem. After fixing the spaces Φ and Φ^X we modify the dynamics of the initial system, construct the

Perron–Frobenius operator U_p for the modified system and analyze it in the rigged Hilbert Space $\Phi \subset H \subset \Phi^X$.

The operator U_b can be decomposed as $U_b = \overline{U}_x R \overline{U}_y^+$, where $\overline{U}_x$ is Perron–Frobenius operator for the one-dimensional Bernoulli shift which acts in x variable; R is an operator of multiplication by function $1 + r(x)\, r(y)$, where $r(x) \equiv 1$ for $0 \leqslant x < 1/2$ and $r(x) \equiv -1$ for $1/2 \leqslant x \leqslant 1$; $\overline{U}_y^+$ is Koopman operator for the one-dimensional Bernoulli shift acting in y variable. This relation gives rise to the decomposition of U_b:

$$U_b = \overline{U}_x\,(1 + r(x)\,r(y))\,\overline{U}_y^+ = \overline{U}_x\,\overline{U}_y^+ + \overline{U}_x\,r(x)\,r(y)\,\overline{U}_y^+ = U_{b0} + \delta U_b \quad (1)$$

into diagonal U_{b0} and off-diagonal δU_b parts.

3. Controlled Baker map.

In this section we construct the so-called controlled baker map. We follow the ideas proposed in [6].

We select some periodic orbit of period N, i.e., the sequence of points $\{(x_s, y_s)\}_{s=1}^N$ such that $(x_{s+1}, y_{s+1}) = F(x_s, y_s)$ and $F(x_N, y_N) = (x_1, y_1)$.

We specify also a set $\{\Delta_s \subset X\}_{s=1}^N$ of rectangular neighborhoods (control windows) of points (x_s, y_s) such that: 1) $(x_s, y_s) \in \Delta_s$; 2) $\Delta_i \cap \Delta_k = \varnothing$, $i \neq k$; 3) $\Delta \equiv \bigcup\limits_{s=1}^{N} \Delta_s \subset X$.

Now we construct the controlled map F_p. If $(x, y) \in \Delta_s$ $(s = 1, 2, \ldots, N)$ then F_p acts by definition as $F(x, y)$ with probability p and as $C(x, y)$ with probability $1 - p$ where the function $C(x, y)$ is given by:

$$C(x, y) = (a_s x + (1 - a_s) x_s, b_s y + (1 - b_s) y_s), \quad (x, y) \in \Delta_s.$$

Here the real numbers a_s, b_s satisfy $|a_s| \leqslant 1$, $|b_s| \leqslant 1$ so that C is a contraction with stationary points (x_s, y_s) $(s = 1, 2, \ldots, N)$. If $(x, y) \in X \setminus \Delta$ then $F_p(x, y) \equiv F(x, y)$.

In order to simplify calculations above assume that $\Delta = X$. Then the Perron–Frobenius operator U_p for the coupled system F_p will act on a density ρ as

$$\begin{aligned} U_p\,\rho &= p \cdot U_b \rho + \\ &+ (1 - p) \cdot \left[\sum_{s=1}^{N} \rho \left(\frac{x - (1 - a_s) x_s}{a_s}, \frac{y - (1 - b_s) y_s}{b_s} \right) \chi_{C(\Delta_s)} \right] = \\ &= p \cdot U_b \rho + (1 - p) \cdot \tilde{U} \rho. \end{aligned} \quad (2)$$

Now the Perron–Frobenius operator U_p specifies dynamics in the space of densities for the coupled system F_p. The operator U_p itself as well as its spectral characteristics depend on parameters p, $a_1, \dots, a_N$, $b_1, \dots, b_N$. The task of control is to choose their values in such a way that any initial density ρ_0 under the dynamics U_p^t would converge to some equilibrium density $\rho_{\rm inv}$, i.e., the density $\rho_{\rm inv}$ would represent a global attractor in the space of densities. If this turns out to be impossible but the iterations $U_p^t \rho_0$ converge to the $\rho_{\rm inv}$ for any ρ_0 belonging to some subspace of the densities space then we speak of *partial* control.

4. Analysis of the coupled system.

As we pointed out in Sec. 3 projections of the chosen control windows Δ_s generate partition of the unit interval $[0,1]$ of the abscissa axis. We will denote elements of this latter partition also as Δ_s in cases when it does not lead to misunderstanding.

We introduce the following system of functions:

$$\big|\beta_n(x)\big\rangle = \sum_{s=1}^{N} (x - x_s)^n \cdot \chi_{\Delta_s}(x), \quad n = 0, 1, \dots . \tag{3}$$

In the space of piecewise-analytic functions on interval $[0,1]$ with respect to the partition Δ_s and in a dual space we define the system of functionals

$$\big\langle \tilde\beta_m(x)\big| = \frac{1}{N} \sum_{s=1}^{N} \frac{(-1)^m}{m!}\, \delta^{(m)}(x - x_s), \quad m = 0, 1, \dots . \tag{4}$$

These systems are mutually dual (or biorthogonal), i.e., $\big\langle \tilde\beta_m(x)\big|\beta_n(x)\big\rangle = \delta_{mn}$. In this pair of bases the matrix elements of the operator U for Bernoulli shift have the form

$$U_{mn} = \big\langle \tilde\beta_m \big| U \big| \beta_n \big\rangle = \\ = \begin{cases} \dfrac{1}{2^n}, & n = m, \\ \dfrac{1}{N\, m!\, 2^{m+1}} \displaystyle\sum_{i,j=1}^{N} \frac{d^m}{dx^m}(x - x_{j+1})^n \Big|_{x_i \in S(\Delta_j)}, & n > m, \\ 0, & n < m. \end{cases} \tag{5}$$

Below we present the pair of bases which will serve for the analysis of the Perron–Frobenius operator U_p for the controlled baker map. We take the system of tensor products $\{\big|\beta_i(x), \tilde\beta_j(y)\big\rangle\}_{i,j=0}^{\infty}$ to be the basis in the space Φ, while for the dual basis in Φ^X we take the system of products $\{\langle\tilde\beta_i(x),\beta_j(y)|\}_{i,j=0}^{\infty}$. Indeed,

$$\begin{aligned}\langle\tilde\beta_i,\,\beta_j\,|\beta_{i'},\,\tilde\beta_{j'}\rangle &= \langle\tilde\beta_i\,|\,\beta_{i'}\rangle\langle\beta_j\,|\,\tilde\beta_{j'}\rangle = \\ &= \langle\tilde\beta_i\,|\,\beta_{i'}\rangle\langle\tilde\beta_{j'}\,|\,\beta_j\rangle^* = \delta_{i,i'}\,\delta_{j,j'}.\end{aligned} \tag{6}$$

We define the spaces Φ and Φ^X as closures of the corresponding linear sets in weak convergence topology.

Using (2) we see that matrix elements of the operator U_p in this pair of bases are

$$\begin{aligned}\langle\tilde\beta_i,\,\beta_j\,|U_p|\beta_{i'},\,\tilde\beta_{j'}\rangle &= p\,\langle\tilde\beta_i,\,\beta_j\,|U_b|\beta_{i'},\,\tilde\beta_{j'}\rangle + \\ &+ (1-p)\langle\tilde\beta_i,\,\beta_j\,|\tilde U|\beta_{i'},\,\tilde\beta_{j'}\rangle.\end{aligned} \tag{7}$$

Using (1) we calculate matrix elements separately for diagonal and off-diagonal parts of operator U_b. For the diagonal part U_{b0} we obtain:

$$\begin{aligned}&\langle\tilde\beta_i,\,\beta_j\,|U_{b0}|\beta_{i'},\,\tilde\beta_{j'}\rangle = \langle\tilde\beta_i,\,\beta_j\,|\overline{U}_x\,\overline{U}_y^{+}|\beta_{i'},\,\tilde\beta_{j'}\rangle = \\ &= \langle\tilde\beta_i|\overline{U}_x|\beta_{i'}\rangle\,\langle\beta_j|\overline{U}_y^{+}|\tilde\beta_{j'}\rangle = \langle\tilde\beta_i|\overline{U}_x|\beta_{i'}\rangle\,\langle\tilde\beta_{j'}|\overline{U}_y|\beta_j\rangle^*.\end{aligned} \tag{8}$$

Using (5) one has

$$\langle\tilde\beta_i,\,\beta_j\,|U_{b0}|\beta_{i'},\,\tilde\beta_{j'}\rangle = \begin{cases}\dfrac{1}{2^{i+j'}}, & i=i', \quad j=j', \\ U_{i\,i'}\,U_{j'\,j}, & i'>i, \quad j>j', \\ 0, & \text{otherwise.}\end{cases} \tag{9}$$

For the off-diagonal part δU_b we obtain

$$\begin{aligned}&\langle\tilde\beta_i,\,\beta_j\,|\delta U_b|\beta_{i'},\,\tilde\beta_{j'}\rangle = \langle\tilde\beta_i,\,\beta_j\,|\overline{U}_x\,r(x)\,r(y)\,\overline{U}_y^{+}|\beta_{i'},\,\tilde\beta_{j'}\rangle = \\ &= \langle\tilde\beta_i|\overline{U}_x\,r(x)|\beta_{i'}\rangle\langle\beta_j|r(y)\,\overline{U}_y^{+}|\tilde\beta_{j'}\rangle = \langle\tilde\beta_i|\overline{U}_x\,r(x)|\beta_{i'}\rangle\langle\tilde\beta_{j'}|\overline{U}_y\,r(y)|\beta_j\rangle^*.\end{aligned} \tag{10}$$

In order to calculate each of the similar factors in the r.h.s. of (10) we make use of intertwining relation for the operator $\overline{U}$:

$$\frac{d^m}{dx^m}\,\overline{U}\,f = \frac{1}{2^m}\,\overline{U}\,\frac{d^m}{dx^m}f. \tag{11}$$

Therefore

$$\langle\tilde\beta_i|\overline{U}_x\,r(x)|\beta_{i'}\rangle = \frac{1}{N\,i!}\sum_{s=1}^{N}\frac{d^i}{dx^i}|\overline{U}_x\,r(x)|\beta_{i'}\rangle\Big|_{x_s} =$$

$$= \frac{1}{2\,N\,i!} \sum_{s,q=1}^{N} \frac{d^i}{dx^i}[(x/2 - x_q)^{i'} - (x/2 + 1/2 - x_q)^{i'}]\Big|_{x_s \in S(\Delta_q)}. \tag{12}$$

If we denote by $t_{i\,i'}$ the values of $\langle\tilde\beta_i|\overline{U}_x\, r(x)|\beta_{i'}\rangle$ obtained in (12) we see immediately that $t_{i\,i'} = 0$ when $i \geqslant i'$.

Summing up the obtained relations we arrive to the following: the matrix elements of the operator U_b in our pair of bases have the form:

$$\langle\tilde\beta_i,\, \beta_j|U_b|\beta_{i'},\, \tilde\beta_{j'}\rangle = \begin{cases} \dfrac{1}{2^{i+j'}}, & i = i', \quad j = j', \\ U_{i\,i'}\,U_{j'\,j} + t_{i\,i'}\,t_{j'\,j}, & i' > i, \quad j > j', \\ 0, & \text{otherwise.} \end{cases} \tag{13}$$

Matrix elements of U_b possess the so-called nonreccurrence property [4], meaning that for nonnegative δi and δj generally $\langle\tilde\beta_{i-\delta i},\, \beta_{j+\delta j}|U_b|\beta_i,\, \tilde\beta_j\rangle \neq 0$, but at the same time necessarily $\langle\tilde\beta_i,\, \beta_j|U_b|\beta_{i-\delta i},\, \tilde\beta_{j+\delta j}\rangle = 0$. In other words, the matrix of the operator U_b is in some sense upper-triangular; which means precisely that upper-left submatrix of U_b of any finite order can be made strictly upper-triangular by renumerating the bases elements. The nonreccurrence property is the key point for the spectral analysis of the operator U_b.

Because of the validity of nonreccurrence property for the operator U_b we can apply the general diagonalization scheme based on the so-called Subdynamics [4, 5].

Namely, we introduce the projectors

$$P_n = \sum_{i+j=n} |\beta_i,\, \tilde\beta_j\rangle\langle\tilde\beta_i,\, \beta_j|, \qquad Q_n = I - P_n. \tag{14}$$

Each projector P_n selects $(n+1)$-dimensional subspace which is eigenspace of the U_{b0} operator with eigenvalue $\frac{1}{2^n}$.

The matrix U_b has numbers $\lambda_m = \frac{1}{2^m}$ as its diagonal elements. These numbers represent spectrum of operator U_b in the following sense: in a specially chosen pair of bases $|\beta'_{i'},\, \tilde\beta'{}_{j'}\rangle$ and $\langle\tilde\beta'{}_i,\, \beta'_j|$ the matrix of the operator U_b splits into direct sum of Jordan blocks. Each block is generated by a single projector P_m and every projector P_m generates one block. The total correlation function can be decomposed into the sum of independently evolving modes $\langle\tilde\beta'{}_i,\, \beta'_j|U_p^t|\beta'_{i'},\, \tilde\beta'{}_{j'}\rangle$ with exponentially decaying asymptotics $\lambda_{i,j}^t$ when $t \to \infty$, where $\lambda_{i,j}$ are given by (17). Off-diagonal elements of the Jordan block in a subspace generated by

P_m produce polynomial corrections to the exponential decay. The pair of bases which provide such a decomposition is given by the Creation $\mathcal{C}_m(z)$ and Destruction $\mathcal{D}_m(z)$ operators [4, 5]:

$$\left|\beta'_{i'},\, \tilde{\beta}'_{j'}\right\rangle = [P_m + \mathcal{D}_m(z)]\left|\beta_{i'},\, \tilde{\beta}_{j'}\right\rangle \tag{15}$$

$$\left\langle\tilde{\beta}'_{i},\, \beta'_{j}\right| = [P_m + \mathcal{C}_m(z)]\left\langle\tilde{\beta}_{i},\, \beta_{j}\right|. \tag{16}$$

Now we calculate the last term in (7). Using explicit expression for action of $\tilde{U}$ one can obtain:

$$\left\langle\tilde{\beta}_i,\, \beta_j\left|\tilde{U}\right|\beta_{i'},\, \tilde{\beta}_{j'}\right\rangle = \frac{1}{N^2}\sum_{s,q=1}^{N}\frac{b_q^{j+1}}{a_s^i}\,\delta_{i'\,i}\,\delta_{j'\,j}\,.$$

Thus we see that matrix of the addend $\tilde{U}$ is strictly diagonal. Therefore the matrix of the coupled operator U_p can be deduced from the matrix of $p\,U_b$ just by modifying diagonal terms. So the scheme of diagonalization of the U_b operator can be carried out for the U_p substituting $[z-\lambda_{i,j}]$ for the $[z-2^{m-k+l}]$ in the recurrent equations for related matrix elements of U_b, $\mathcal{C}_m(z)$ and $\mathcal{D}_m(z)$, where $\lambda_{i,j}$ are given by

$$\lambda_{i,j} = p\cdot\frac{1}{2^{(i+j)}} + (1-p)\cdot\frac{1}{N^2}\sum_{s,q=1}^{N}\frac{b_s^{j+1}}{a_q^i}, \tag{17}$$

Now from (17) we are able to deduce the control condition. According to [6] the sufficient condition for the system to become controllable is that all eigenvalues of the Perron–Frobenius operator (except $\lambda_{0,0}=1$) lie outside the unit disc, i.e., $|\lambda_{i,j}|>1, \quad i,j\geqslant 1$. It is easy to see that in general we are unable to achieve this condition because for any fixed value of i it is possible to make $|\lambda_{i,j}|$ arbitrary small choosing appropriate j if $|b_s|<1$. That is why we are forced to put all b_s to be equal to 1. This means that our controlling map C does not move points in y coordinate leaving them all stationary. In this case our eigenvalues modify to

$$\lambda_{i,j} = p\cdot\frac{1}{2^{(i+j)}} + (1-p)\cdot\frac{1}{N}\sum_{q=1}^{N}\frac{1}{a_q^i}, \tag{18}$$

and the system will be controllable if $|\lambda_{i,j}|>1$. Restricting our considerations with the most transparent case when for all q contracting factors $a_q = a$ we obtain the simplified spectrum in the form $\lambda_{i,j} = p\cdot\frac{1}{2^{(i+j)}} + (1-p)\cdot\frac{1}{a^i}$. The controllability condition now reads:

$|\lambda_{i,j}| = \left|p \cdot \frac{1}{2} + (1-p)\,\frac{1}{a}\right| > 1$. This condition corresponds to the region I in plane of parameters a and p depicted in Fig. 1. If parameters are chosen in this region then the controllability condition holds, all eigenvalues of the operator U_p lie outside the unit disc and this means that any initial density under the dynamics U_p^t converges to the singular measure which represents equilibrium state of the system. This situation is interpreted as global control of the system since the mentioned measure serves as a global attractor in space of densities.

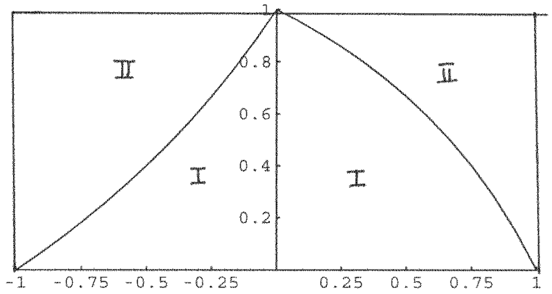

Fig. 1. Regions of control. Dependence of control on parameters p and a. The vertical axis is the value of parameter p. The horizontal axis is the value of parameter a. The region I corresponds to the global control and is located inside the curvilinear triangle. The region II corresponds to partial control and is located outside the curvilinear triangle.

If we select the point (a, p) in region II then some finite number of the eigenvalues $\lambda_{i,j}$ lie inside the unit disc and in order for the trajectory to be stabilized by the stochastic control the initial density ρ_0 must lie in some subspace of the space of densities. This subspace is distinguished by orthogonality condition $\left\langle \tilde{\beta'}_i,\, \beta'_j \middle| \rho_0 \right\rangle = 0$ for those functionals $\left\langle \tilde{\beta'}_i,\, \beta'_j \right|$ which correspond to the eigenvalues $|\lambda_{i,j}| < 1$. This situation can be interpreted as partial control.

In case of global control the operator U_p can be diagonalized in a specially chosen pair of bases (16)–(15). This means that its generalized spectral decomposition can be written in the form

$$U_p\, \rho = \sum_{i,j} \lambda_{i,j} \left\langle \tilde{\beta'}_i,\, \beta'_j \middle| \rho \right\rangle \left| \beta'_i,\, \tilde{\beta'}_j \right\rangle, \tag{19}$$

and the dynamics in t is given by

$$U_p^t\, \rho = \sum_{i,j} \lambda_{i,j}^t \left\langle \tilde{\beta'}_i,\, \beta'_j \middle| \rho \right\rangle \left| \beta'_i,\, \tilde{\beta'}_j \right\rangle. \tag{20}$$

According to the general ideas presented in [8] the characteristic time of the system is given by $T \cong \frac{1}{K}\ln\left(\frac{1}{\varepsilon}\right)$, where ε is a given accuracy with which the trajectory is observed and K is the Kolmogorov entropy. If we denote by z_1 the least eigenvalue $\lambda_{i,j}$ which is greater than 1 then the analog of K will be $\ln z_1$ and the formula for the characteristic time will be $T \cong \frac{1}{\ln z_1}\ln\left(\frac{1}{\varepsilon}\right)$, which agrees well with the expression written in [6]. This value has meaning of the transient time to achieve control, i.e., the time which is needed for a system to settle on a given trajectory subject to stabilization.

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