Записки научных
семинаров ПОМИ
Том 300, 2003 г.

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EMBEDDED SPECTRUM ON A METRIC GRAPH (AN OBSERVATION)

ABSTRACT. An elementary example of a simplest star-like graph which shows that the analytical disjunction may occur not on only at isolated points of the spectral parameter, but also on whole intervals, are suggested.

1. INTRODUCTION

It is well know, see for instance [1], that the standard description of solutions of homogeneous Schrödinger equation

$$-u'' + q(x)u = \lambda u$$

on graphs based on Cauchy problem at given point fails even in case of simplest starlike graphs, since existence of solution is guaranteed only locally, on the arc considered, but cannot be continued through the vertex. Another effect connected with vertex is existence of embedded eigenvalues on some graphs containing cycles, see therein. This effect should be taken into account in scattering problems or in control problems. This effect for Kirchhoff boundary conditions at the vertex 0 with n adjacent arcs Γ_l, $l = 1, 2, \ldots, n$,

$$u_1(0) = u_2(0) = \cdots = u_n(0),\ u_1'(0) + u_2'(0) + \cdots + u_n'(0) = 0,$$

appears as existence of a non-zero solution of the above equation with some zero components of Cauchy data

$$u_m(0) = u_m'(0) = 0,\ \ m = m_1,\, m_2,\, \ldots, m_k,\ k < n,$$

for certain value λ_0 of the spectral parameter. Then, due to uniqueness theorem, the solution u is equal to zero on the corresponding edges Γ_{m_1}, $\Gamma_{m_2}, \ldots \Gamma_{m_k}$. Then we say, that the arcs Γ_l, $l = 1, 2, \ldots, n$, are *analytically disjoint* at the vertex a for the spectral parameter λ_0. Probably most important question concerning the analytical disjunction is description of all boundary conditions at the vertex for which the analytical disjunction is impossible in certain interval of the spectral parameter. Answer to this question would imply study of behavior of the corresponding DN-map and will be discussed elsewhere.

In actual note we suggest an elementary example of a simplest star-like graph which shows that the analytical disjunction may occur not only at isolated points of the spectral parameter, but also on whole intervals.

2. String with point masses

We begin with auxiliary problem of distorted reflection on the semi-infinite string fastened at the origin, and bearing a point-mass M at the point a. The wave-process in this system is described by the wave equation, as before, if $x \neq 0$, the same initial and boundary conditions, and some new boundary conditions at the point a where the point mass is attached; the solution should be continuous at this point and the jump of the x-derivative of the solution u at this point

$$\left[\frac{\partial u}{\partial x}\right]_{x=a} := \frac{\partial u}{\partial x}(a+0,t) - \frac{\partial u}{\partial x}(a-0,t)$$

should be proportional with coefficient M to the displacement $u(a,t)$ of the string at the point a:

$$\left[\frac{\partial u}{\partial x}\right]_{x=a} + Mu(a,t) = 0.$$

The stationary modes X of the string with a point mass M attached may be obtained as continuous solutions of the homogeneous equation

$$-X'' = p^2 X,\ 0 < x < a \text{ and } a < x < \infty,$$

with the boundary conditions

$$X(0) = 0,\ \left[\frac{\partial X}{\partial x}\right]_a + Mp^2 X(a) = 0.$$

For $p > 0$ the stationary mode X on the interval (a, ∞) is presented as

$$X(x,p) = e^{ipx} + Re^{-ipx},$$

with reflection coefficient $R = R(p)$, and is equal to

$$\alpha \sin px$$

on the interval $(0, a)$. Then the continuity condition at the point a permits to eliminate α, $\alpha = (e^{ipa} + Re^{-ipa})/(\sin pa)$, and the jump condition may be considered as an equation for R:

$$i\left(e^{ipa} - Re^{-ipa}\right) = \frac{e^{ipa} + Re^{-ipa}}{\sin pa}\left[\cos ap - Mp\sin pa\right],$$

which has the solution

$$R(p) = e^{2ipa}\frac{i - [\cot ap - Mp]}{i + [\cot ap - Mp]} = -e^{2ipa}\frac{e^{-ipa} - pM\sin pa}{e^{ipa} - pM\sin pa}.$$

One can easily see that if the mass is absent, $M = 0$, then the reflection coefficient R is equal to -1. Vice versa, if the mass is infinite, then the reflection coefficient is equal to $-e^{2pia}$, which just corresponds to fastening the string at the point a. But more interesting phenomenon is the selective *resonance reflection* which may be observed at the points $p = n\pi/a$, where $\sin pa = 0$: at these points the reflection coefficient is -1, the same as for the string *without mass*. This fact is true for any, even large, mass.

Similar effect of the resonance reflection may be observed for the string with Neumann boundary condition at the origin. The corresponding spectral problem is

$$-Y'' = p^2 Y,\ \ 0 < x < a \text{ and } a < x < \infty,$$

solutions are continuous at $x = a$ and fulfill the boundary conditions

$$Y'(0) = 0,\ \left[\frac{\partial Y}{\partial x}\right]_a + Mp^2 Y(a) = 0.$$

The solution for $x > a$ is equal to

$$Y(x,p) = e^{ipx} + e^{2ipa}\frac{e^{-ipa} - iMp\cos pa}{e^{ipa} + iMp\cos pa}e^{-ipx}.$$

The function $Q(p) = e^{2ipa}(e^{-ipa} - iMp\cos pa)/(e^{ipa} + iMp\cos pa)$ is the corresponding reflection coefficient.

Note that the solution X of the problem with zero boundary condition at the origin may be continued as an odd function $\mathbf{X}$ on the whole string $(-\infty, \infty)$ and fulfills the differential equation $-X'' = p^2 X$ and the boundary conditions at the points $\pm a$:

$$\mathbf{X} = \begin{cases} e^{ikx} + Re^{-ikx} & \text{for } \ x > a, \\ -e^{-ikx} - Re^{ikx} & \text{for } \ x < -a. \end{cases}$$

Similarly, the solution Y of the problem with the boundary condition $Y'(0) = 0$ at the origin may be continued as an even function $\mathbf{Y}$ on the whole string $(-\infty, \infty)$ and fulfills the differential equation $-Y'' = p^2 Y$ and the boundary conditions at the points $\pm a$:

$$\mathbf{Y} = \begin{cases} e^{ikx} + Qe^{-ikx} & \text{for} \quad x > a, \\ e^{-ikx} + Qe^{ikx} & \text{for} \quad x < -a. \end{cases}$$

From now on we assume that both solutions are already continued on the whole axis as odd and even functions.

Consider the bilaterally infinite string $-\infty < x < \infty$ with two equal masses M attached at the points $\pm a$. The *right scattered waves* are the stationary modes $\overleftarrow{\Psi}$ which fulfill the equation

$$-\overleftarrow{\Psi}'' = p^2 \overleftarrow{\Psi} \text{ with } x \neq \pm a,$$

are continuous and fulfill the boundary conditions at the points $\pm a$

$$[\overleftarrow{\Psi'}]_{\pm a} + Mp^2 \overleftarrow{\Psi} = 0,$$

and have the form of the combination of incoming from the right e^{ikx} and reflected $\overrightarrow{R} e^{-ikx}$ wave on (a, ∞):

$$\overleftarrow{\Psi}\big|_{(a,\infty)} = e^{ikx} + \overrightarrow{R} e^{-ikx}$$

and transmitted wave on the interval $(-\infty, -a)$:

$$\overleftarrow{\Psi}\big|_{(-\infty,-a)} = \overleftarrow{T} e^{ikx}.$$

We may find the transmission and reflection coefficients $\overleftarrow{T}$, $\overrightarrow{R}$ and interpret the behavior of them in dependence of M and p similarly to our study of the resonance reflection in the previous problem. Now the solution $\overleftarrow{\Psi}$ of the formulated problem may be obtained as

$$\overleftarrow{\Psi} = \frac{\mathbf{X} + \mathbf{Y}}{2}.$$

The transmission and reflection coefficients are equal to

$$\overleftarrow{T} = \frac{Q - R}{2}, \quad \overrightarrow{R} = \frac{Q + R}{2}.$$

Behavior of the above transmission and refection coefficients in dependence of M and p reveals the phenomenon of the resonance transmission.

Similarly the left scattered waves may be constructed. They are defined by the behavior on the intervals $(-\infty, -a)$

$$\overrightarrow{\Psi}\Big|_{(-\infty,a)} = e^{-ikx} + \overleftarrow{R}\, e^{ikx}$$

and (a, ∞)

$$\overrightarrow{\Psi}\Big|_{(a,\infty)} = \overrightarrow{T}\, e^{-ikx}.$$

One can construct also the scattering matrix $\mathbf{S}$ from the obtained transmission and reflection coefficients as

$$\mathbf{S}(p) = \begin{pmatrix} \overleftarrow{T}(p) & \overleftarrow{R}(p) \\ \overrightarrow{R}(p) & \overrightarrow{T}(p) \end{pmatrix}.$$

This matrix transfers the pair of incoming waves

$$\begin{pmatrix} \overleftarrow{a} \\ \overrightarrow{b} \end{pmatrix} \sim \begin{pmatrix} \overleftarrow{a}\, e^{ipx} & \text{on} & (a, \infty) \\ \overrightarrow{b}\, e^{-ipx} & \text{on} & (-\infty, -a) \end{pmatrix}$$

into the pair of outgoing waves

$$\mathbf{S}(p) \begin{pmatrix} \overleftarrow{a} \\ \overrightarrow{b} \end{pmatrix} \sim \begin{pmatrix} [\overleftarrow{T}\,\overleftarrow{a} + \overleftarrow{R}\,\overrightarrow{b}]e^{ipx} & \text{on} & (-\infty, -a) \\ [\overrightarrow{R}\,\overleftarrow{a} + \overrightarrow{T}\,\overrightarrow{b}]e^{-ipx} & \text{on} & (a, \infty) \end{pmatrix}.$$

Now we are able to construct an example of a network of three strings joined at the origin, which exhibits the analytical disjunction effect on the interval $0 < \lambda < \infty$. Consider the stationary string equation on the network $\mathbf{R} \cup \mathbf{R}_+$

$$-y''(x) = p^2 y(x),\ \ 0 < |x| < a \text{ and } a < |x| < \infty,$$

$$-z''(\xi) = p^2 \xi,\ \ 0 < \xi < \infty,$$

with the boundary conditions (point masses) at $\pm a$ on $\mathbf{R}$

$$[y']_{\pm a} + Mp^2 y(\pm a) = 0,$$

no boundary conditions on $\mathbf{R}_+$, and Kirchhoff boundary condition at the junction:

$$y(0) = z(0), \quad [y']_{x=0} + z'(0) = 0.$$

It is clear that on the whole interval $0 < \lambda < \infty$ the semi-infinite string is analytically disjoint from the network, since there are eigenfunctions of the operator supported by R only and equal to zero on the adjacent semi-axis. These are the eigenfunctions of the even series obtained by odd continuation of the scattered waves of the above problem with the point mass on the half-string.

Based on the above explicit formulae for the scattered waves on $\mathbf{R}$ one may easily construct the scattered waves on the whole network and explore the resonance scattering in case when the semi-axis $\mathbf{R}_+$ is analytically disjoint. It will be done elsewhere.

References

1. V. Bogevolnov, A. Mikhailova, B. Pavlov, A. Yafyasov, *About scattering on the ring.* in: Operator Theory: Advances and Application, **124**, Birkhauser Verlag, Basel (2001), pp. 155–187.

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Поступило 30 ноября 2002 г.