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CARTESIAN APPROACH FOR NONHOLONOMIC SYSTEMS

Abstract. The aim of this communication is to develop a Cartesian approach to describe nonholonomic systems with constraints which are linear with respect to velocity. The obtained results are applied to study the classical nonholonomic system: the constrained particle in $\mathbb{R}^3$, the Chapliguin-Carathéodory sleigh and the symmetrical top in the Suslov problem.

1. Introduction

In the history of mechanics, there have been two point of views for studying mechanical systems: the Newtonian and the Cartesian. In "Philosophiae Naturalis Principia Mathematica" (1687), Newton considers that movements of celestial bodies can be described by differential equations of the second order. To determine their trajectory, it is necessary to give the initial position and velocity. Descartes in "Principia Philosophiae" (1644) proposed that the behaviour of the celestial bodies be studied from another point of view. He considered that bodies are moved by ether. As a consequence, the equation of motion must be of the first order

$$\dot{\mathbf{x}} = \mathbf{v}(x). \tag{1.1}$$

Hence, to determine the trajectory from Descartes's point of view it is necessary to give only the initial position, but he did not propose a method for determining the vector field $\mathbf{v}$. In [1] the author affirms that it is possible to solve the problem of dynamics in the configuration space. In this monograph Kozlov proved the following assertions:

Theorem [1]. *The manifold $\Sigma_t = \{y - u(x,t) = 0, x \in X\}$ is invariant for the Hamiltonian equations with Hamiltonian $H(x,y,t)$ iff the vector field u satisfies the Lamba equations*

$$\partial_t u(x,t) + \mathrm{rot} u(x,t)\mathbf{v}(x,t) = -\partial_x h(x,t) \tag{1.2}$$

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where $\operatorname{rot} u(x,t) = \partial_x u - (\partial_x u)^T$ *is the antisymmetrical matrix of* $\dim X \times \dim X$,

$$\mathbf{v}(x,t) = \partial_y H|_{y=u(x,t)} \tag{1.3}$$

is a vector field on X, $h(x,t) = H(x,y,t)|_{y=u(x,t)}$.

Theorem [1]. *For small* t, *there is a smooth vector field* u *for* (1.2), *such that* $u(x,0) = u_0(x), \forall x \in X$, *where* X *is a compact manifold without boundary,* u_0 *is a smooth vector field on* X.

Kozlov affirms that in order to solve the problem of dynamics from the Descartes point of view it is first of all necessary to determine the solution of (1.2) and after that to solve the equation (1.1), where $\mathbf{v}$ is the vector field obtained from (1.3). It is clear that if H is a natural Hamiltonian, i.e.,

$$H(x,y,t) = \frac{1}{2}\sum_{k,j} G^{kj}(x,t)p_k p_j + U(x,t),$$

where $G = (G_{kj})$ is a Riemann metric, $G^{-1} = (G^{kj})$, then

$$\mathbf{v}(x,t) = G^{-1}u(x,t). \tag{1.4}$$

The phase trajectories which lie on the submanifold Σ_t can be obtained as a solution of the differential equations (1.1), (1.4).

2. The Lagrangian and Cartesian approach for nonholonomic systems

The aim of the present communication is to developed the Descartes point of view for mechanic systems with constraints which are linear with respect to velocity. At the same time we shall develop the Lagrangian (Newtonian) approach for such systems.

Firstly, we shall introduce the following notation and concept.

Let X be a smooth manifold of the dimension N with local coordinates $x = (x^1, \dots, x^N)$ and equipped by the Riemann metric $G = (G_{kj})$. By $\xi(X)$, $\Lambda(X)$, ∇ we denote respectively the Lie algebra of vector fields on X and the algebra of the 1-form on X, and the connection:

$$\nabla : \xi(X) \times \xi(X) \longmapsto \xi(X),$$
$$(u,v) \mapsto \nabla_u v$$

which is $\mathbb{R}$ lineal with respect to v and C^∞ lineal with respect to v and is compatible with metric G, i.e., $\nabla_u G(v,w) = 0, \quad \forall u,v,w \in \xi(X)$.

The vector field $v \in \xi(X)$ is called the integral element of $\Omega \in \Lambda(X)$ if $\Omega(v) = 0$. We shall denote by $K(\Omega_1, \Omega_2, \dots, \Omega_M)$, $M \le N$ the set of the integral elements of the independent 1-forms $\Omega_1, \Omega_2, \dots, \Omega_M$.

Proposition 1.1. *The most general element of* $K(\Omega_1, \Omega_2, \dots, \Omega_M)$ *admits the following representation*

$$\mathbf{v} = \det \begin{pmatrix} \Omega_1(\partial_1) & \Omega_1(\partial_2) & \dots & \Omega_1(\partial_N) & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \Omega_M(\partial_1) & \Omega_M(\partial_2) & \dots & \Omega_M(\partial_N) & 0 \\ \Omega_{M+1}(\partial_1) & \Omega_{M+1}(\partial_2) & \dots & \Omega_{M+1}(\partial_N) & \lambda_{M+1} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ \Omega_N(\partial_1) & \Omega_N(\partial_2) & \dots & \Omega_N(\partial_N) & \lambda_N \\ \partial_1 & \partial_2 & \dots & \partial_N & 0 \end{pmatrix}, \tag{2.1}$$

where $\partial_k = \frac{\partial}{\partial x^k}$.

We shall consider that $\Omega_1, \Omega_2, \dots, \Omega_M$, $M \le N-1$ are given 1-forms, and $\Omega_{M+1}, \Omega_{M+2}, \dots, \Omega_N$, are arbitrary 1-forms on X. Furthermore, we assume that they are pointwise independent, i.e.,

$$\Upsilon \equiv \Omega_1 \wedge \Omega_2 \dots \wedge \Omega_N(\partial_1, \partial_2, \dots, \partial_N) \ne 0. \tag{2.2}$$

It is important to observe that the arbitrary 1-forms can be determined only from the conditions (1.4). The functions λ_j, $j = M+1, \dots, N$ are arbitrary functions on X.

The vector field $\mathbf{v}$ has the following properties

1. $\Omega_j(\mathbf{v}) = 0$, $j = 1, 2, \dots, M$ $\Omega_j(\mathbf{v}) = -\Upsilon \lambda_j$, $j = M+1, \dots, N$.

2. We deduce that the vector $\mathbf{v}(x) = (v^1(x), \dots, v^N(x))^T$ can be represented as follows $\mathbf{v}(x) = \mathcal{M}^{-1}\lambda$, where $\mathcal{M} = \Big(\Omega_j(\partial_k)_{j,k=1,\dots,N}\Big)$, $\lambda = -\Upsilon\,(0, \dots, 0, \lambda_{M+1}, \dots, \lambda_N)$.

3. Let σ be the 1-form associated with the vector field $\mathbf{v}$, i.e.,

$$\sigma = (\mathbf{v}(x), dx) \equiv \sum_{j,k=1}^{N} G_{jk}(x) v^j(x) dx^k \equiv \sum_{k=1}^{N} v_k dx^k$$

then the 2-form $d\sigma$: $d\sigma = \frac{1}{2}\sum_{j,k=1}^{N} a_{jk}(x)\Omega_j \wedge \Omega_k$, where $A = (a_{jk})$ is a matrix such that

$$a_{jk} = (-1)^{j+k-1}\frac{1}{\Upsilon} d\sigma \wedge \Omega_1 \wedge \dots \wedge \widehat{\Omega}_k \dots \wedge \widehat{\Omega}_j \dots \wedge \Omega_N(\partial_1, \partial_2, \dots, \partial_N), \tag{2.2}$$

$\widehat{\Omega}_j$, $\widehat{\Omega}_k$ means that these elements are omitted.

It is clear that the contraction of $d\sigma$ along $\mathbf{v}$ is

$$\iota_{\mathbf{v}} d\sigma = \sum_{j=1}^{N} \Lambda_j \Omega_j, \quad \text{where} \quad \Lambda \equiv \mathrm{col}(\Lambda_1, \Lambda_2, \ldots, \Lambda_N) = A^T \lambda. \tag{2.3}$$

We shall analyze the differential equations

$$\dot{\mathbf{x}} = \mathcal{M}^{-1} \lambda \tag{2.4}$$

under the conditions

$$\begin{cases} \Lambda_j = 0, \quad j = M+1, \ldots, N \\ \Upsilon = \Omega_1 \wedge \Omega_2 \ldots \wedge \Omega_N (\partial_1, \partial_2, \ldots, \partial_N) \neq 0. \end{cases} \tag{2.5}$$

The following result allowed us to establish the relations between the Cartesian and Newtonian approaches.

Proposition 2.1. *The differential equations* $\dot{\mathbf{x}} = \mathbf{v}(x), \quad x \in X$ *are invariant relationship of the Lagrangian equations with Lagrangian function* L :

$$L = \frac{1}{2} ||\dot{\mathbf{x}} - \mathbf{v}||^2 = \frac{1}{2} \sum_{j,k=1}^{N} G_{jk}(x)(\dot{x}^j - v^j)(\dot{x}^k - v^k).$$

In fact, by derivation we deduce that $\nabla_{\dot{\mathbf{x}}}(\dot{\mathbf{x}} - \mathbf{v}(x)) = 0$, *or,*

$$\nabla_{\dot{\mathbf{x}}}(\partial_{\dot{\mathbf{x}}} L) = 0, \tag{2.6}$$

which are equivalent to Lagrangian equations with the Lagrangian function L *given above.*

It is easy to show that (2.6) admits the representation

$$\nabla_{\dot{\mathbf{x}}}(\partial_{\dot{\mathbf{x}}^j} T) = \omega(\partial_j) + \nabla_{\dot{\mathbf{x}} - \mathbf{v}(\mathbf{x})} v_j \tag{2.7}$$

where $\omega = d\frac{||\mathbf{v}||^2}{2} + \iota_{\mathbf{v}} d\sigma$, σ is the 1-form associated with the vector field $\mathbf{v}$.

Kozlov in [1] studied the case when $\mathbf{v} : \iota_{\mathbf{v}} d\sigma = -dh(x)$. We shall study the case when (2.4) and (2.5) hold. The differential equations which describe the behaviour of such mechanical systems under these restrictions can be represented as follows

$$\nabla_{\dot{\mathbf{x}}}(\partial_{\dot{\mathbf{x}}^j} T) = \frac{\partial \frac{1}{2} ||\mathbf{v}||^2}{\partial x^k} + \sum_{j=1}^{M} \Lambda_j \Omega_j(\partial_k). \tag{2.8}$$

These differential equations can be interpreted as the equations of motion of nonholonomic mechanical systems with an active potential field of force with potential U :

$$U = \frac{1}{2}||\mathbf{v}(x)||^2 + U_0, \quad U_0 = \mathrm{const},$$

and with the reactive forces which have the components,

$$\Big(\sum_{j=1}^{M} \Lambda_j \Omega_j(\partial_1), \sum_{j=1}^{M} \Lambda_j \Omega_j(\partial_2), \dots, \sum_{j=1}^{M} \Lambda_j \Omega_j(\partial_N)\Big),$$

generated by the constraints $\sum_{k=1}^{N} \Omega_j(\partial_k)\dot{x}^k = 0$.

The studying of the behaviour of the nonholonomic systems by using the equations (2.4)–(2.5) and (2.8) we shall call Cartesian and Lagrangian approach respectively.

3. The Cartesian approach for nonholonomic systems with 3 degrees of freedom and one constraint

First of all, for $N = 3$ and $M = 1$, by considering the case

$$\iota_{\mathbf{v}} d\sigma = \Lambda_1\Omega_1 + \Lambda_2\Omega_2 + \Lambda_3\Omega_3$$

where

$$\Lambda_1 = \lambda_2\Omega_3(\mathrm{rot}\mathbf{v}) - \lambda_3\Omega_2(\mathrm{rot}\mathbf{v}), \quad \Lambda_2 = \lambda_3\Omega_1 rot\mathbf{v}),$$
$$\Lambda_3 = -\lambda_2\Omega_1(\mathrm{rot}\mathbf{v}),$$

we obtain that the system (2.4)–(2.5) take the form respectively

$$\dot{\mathbf{x}} = [\Omega_1, \lambda_2\Omega_3 - \lambda_3\Omega_2] \tag{3.1}$$

under the conditions

$$\begin{cases} \Upsilon \neq 0, \\ \Omega_1(\mathrm{rot}\mathbf{v}) = 0, \end{cases} \tag{3.2}$$

where $\Omega_j(x) = (\Omega_j(\partial_x), \Omega_j(\partial_y), \Omega_j(\partial_z))$, $j = 1, 2, 3$; $[,]$ is the vector product on $\mathbb{R}^3$.

In the assertions below, we shall illustrate the above results on concrete examples.

The nonholonomic constrained particle in $\mathbb{R}^3$

Let us suppose that $X = \mathbb{R}^3$, $G = \mathrm{diag}(1,1,1)$ $\Omega_1 = dx + a(x,y,z)dy$. By choosing $\Omega_2 = dy$, $\Omega_3 = dz$ we deduce that $\Upsilon = 1$ and

$$\begin{cases} \mathbf{v} = a\lambda_2\partial_x - \lambda_2\partial_y - \lambda_3\partial_z, \\ \lambda_2\Omega_1(\mathrm{rot}\mathbf{v}) = \partial_z((1+a^2)\lambda_2^2) + (a\partial_x\lambda_3 - \partial_y\lambda_3)\lambda_2 = 0, \\ \Lambda_1 = -\lambda_2(\partial_y(a\lambda_2) + \partial_x\lambda_2) - \lambda_3(\partial_x\lambda_3 + \partial_z(a\lambda_2)). \end{cases}$$

We shall study the case when $a = a(z)$, $\lambda_2 = \frac{A}{\sqrt{a^2+1}}$, $\lambda_3 = b_2(z)$, where A is an arbitrary constant, b_2 is an arbitrary function.

The equations generated by the vector field $\mathbf{v}$ in this case are

$$\begin{cases} \dot{x} = \dfrac{a(z)A}{\sqrt{1+a^2(z)}}, \\ \dot{y} = -\dfrac{A}{\sqrt{1+a^2(z)}}, \\ \dot{z} = -b_2(z). \end{cases} \tag{3.3}$$

Consequence 3.1. *All trajectories of the equation of motion of the constrained Lagrangian system*

$$< \mathbb{R}^3,\, L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(z),\, \{\dot{x} + a(z)\dot{y} = 0\} >$$

coincide with the solutions of (3.3) [5].

The Chapliguin-Carathéodory sleigh

We shall now analyse one of the most classical nonholonomic systems: Chapliguin–Carathéodory's sleigh [3, 4]. The idealized sleigh is a body that has three points of contact with the plane. Two of them slide freely but the third, A, behaves like a knife edge subjected to a constraining force $\mathbf{R}$ which does not allow transversal velocity. More precisely, let yoz be an inertial frame and $\xi\, A\eta$ be a frame moving with the sleigh. Take as generalized coordinates the Cartesian coordinates of the center of mass C of the sleigh and the angle x between the y and the ξ axis. The reaction force $\mathbf{R}$ against the runners is exerted laterally at the point of application A in such a way that the η component of the velocity is zero. Hence, one has the constrained system $\mathcal{M}$ with the configuration space $X = S^1 \times \mathbb{R}^2$, with the kinetic energy $T = \frac{m}{2}(\dot{y}^2 + \dot{z}^2) + \frac{I_c}{2}\dot{x}^2$, and with the constraint

$a\dot{x} + \sin x\dot{y} - \cos x\dot{z} = 0$, where m is the mass of the system and I_c is the moment of inertia about a vertical axis through C and $a = |AC|$. Observe that the "javelin" (or arrow or Chapliguin's skate) is a particular case of this mechanical system and can be obtained when $a = 0$.

To apply the Cartesian approach for this system, first we introduce the 1-form Ω_1, Ω_2, Ω_3 in such a way that the determinant $\Upsilon \neq 0$. In this subcase, we achieve this condition if

$$\Omega_1 = adx + \sin xdy - \cos xdz, \quad \Omega_2 = \sin xdz + \cos xdy, \quad \Omega_3 = dx.$$

Under these restrictions we obtain that $\Upsilon = 1$ and it is easy to show that the condition (2.5) in this case holds if

$$\begin{cases} \partial_y \lambda_3 = \dfrac{am}{J_C + a^2 m} \partial_z \lambda_2, \\ \partial_z \lambda_3 = -\dfrac{am}{J_C + a^2 m} \partial_y \lambda_2, \\ \partial_x \lambda_2 = a\lambda_3. \end{cases} \tag{3.4}$$

After some calculations we can prove that the functions

$$\begin{cases} \lambda_2 = \cos\alpha\, V_1(y,z,a) + \sin\alpha\, V_2(y,z,a) + a \displaystyle\int K(x,a)dx, \\ \lambda_3 = \dfrac{am}{J_C + a^2 m}\Big(\cos\alpha\, V_2(y,z,a) - \sin\alpha\, V_1(y,z,a)\Big) + K(x,a), \\ \alpha = \dfrac{a^2 m x}{J_C + a^2 m} \end{cases} \tag{3.5}$$

are solutions of (3.4), where K is an arbitrary function and V_1, V_2 are functions which satisfy the Cauchy-Riemann conditions:

$$\begin{cases} \partial_y V_1(y,z,a) = \partial_z V_2(y,z,a), \\ \partial_z V_1(y,z,a) = -\partial_y V_2(y,z,a). \end{cases}$$

The aim of the assertions below is to study the differential equations generated by the vector field $\mathbf{v}$:

$$\begin{cases} \dot{x} = \lambda_3(x,y,z,a), \\ \dot{y} = \lambda_2(x,y,z,a)\cos x - a\lambda_3 \sin x, \\ \dot{z} = -\lambda_2(x,y,z,a)\sin x - a\lambda_3 \cos x. \end{cases} \tag{3.6}$$

Consequence 3.2 [5]. *All trajectories of the Chapliguin skate* ($a = 0$) *and the Chapliguin-Carathéodory sleigh by inertia* ($K = 0$, $V_1 = A$, $V_2 = B$, $A, B = const$) *can be obtained from* (3.6), (3.5).

Consequence 3.3 [5]. *Let us introduce the following notation* $\mathbf{v}_\infty = \lim_{a\to\infty} \mathbf{v}$ *and let us suppose that* $K = 0$. *Then the equations generated by the vector field* $\mathbf{v}_\infty$ *are*

$$\begin{cases} \dot{x} = 0, \\ \dot{w} = ie^{ix} F(w) \end{cases}$$

where $w = y + iz$ *and* $F = V_1 + iV_2$ *is a holomorphic function on* $D \subset \mathbb{C}$.

The Lagrangian approach in this case gives the differential equations

$$\begin{cases} J_c \ddot{x} = 0, \\ m\ddot{y} = \partial_y \dfrac{1}{2} \left| \dfrac{dF(w)}{dw} \right|^2, \\ m\ddot{z} = \partial_z \dfrac{1}{2} \left| \dfrac{dF(w)}{dw} \right|^2. \end{cases}$$

The Cartesian approach for the symmetrical body in the Suslov problem

In this section we study one classical problem of nonholonomic dynamics formulated by Suslov. In this problem we consider the rotational motion of a rigid body around a fixed point. The motion of the body is restricted by one nonholonomic constraints $(\mathbf{a}, \omega) = 0$ where ω is the body angular velocity. We assume that the vector constant $\mathbf{a} = (0, 0, 1)$ [2]. The aim of this part is to propose the Cartesian approach for this case. For simplicity we shall suppose that the body is symmetrical.

Let us suppose that $X = SO(3)$, with the Riemann metric

$$G = \begin{pmatrix} I_3 & I_3 \cos z & 0 \\ I_3 \cos z & I_1 \sin^2 z + I_3 \cos^2 z & 0 \\ 0 & 0 & I_1 \end{pmatrix},$$

where $I_1 = I_2$, I_3 are constants (the inertial moment of the body).

The given 1-form is the following $\Omega_1 = dx + \cos z dy$. By choosing properly the 1-form Ω_2, Ω_3 we obtain that $\Upsilon = 1$, equations (3.1) and conditions (3.2) take the form

$$\dot{x} = \frac{\cos z\, v_2}{I_1 \sin^2 z}, \quad \dot{y} = \frac{-v_2}{I_1 \sin^2 z}, \quad \dot{z} = -\frac{v_3}{I_1}, \tag{3.7}$$

$$\partial_y v_3 - \partial_z v_2 - \cos z \partial_x v_3 = 0, \tag{3.8}$$

respectively. We shall study these equations under the restriction $||\mathbf{v}||^2 = 2h + \sin z(b_1 \sin x + b_2 \cos x)$, where h, b_1, b_2 are constants.

Consequence 3.4. *Let us suppose that the solutions of* (3.8) *are such that:*

i)

$$\begin{cases} v_2 = A = const, \\ v_3 = \sqrt{\dfrac{-b_3 u^3 - h u^2 + b_3 u + \frac{h-A}{I_1}}{1-u^2}} = v_3(z), \quad u = \cos z. \end{cases}$$

Then $b_1 = b_2 = 0$ *and the solution of* (3.7) *determine all trajectories of the symmetrycal top in the Suslov-Kozlov case* [2].

ii)

$$\begin{cases} v_2 = \sin z(b_2 \sin x - b_1 \cos x)\sqrt{\dfrac{2h + \sin z(b_1 \sin x + b_2 \cos x)}{b_1^2 + b_2^2}}, \\ v_3 = (b_1 \sin x + b_2 \cos x)\sqrt{\dfrac{2h + \sin z(b_1 \sin x + b_2 \cos x)}{b_1^2 + b_2^2}} \end{cases}$$

then $b_3 = 0$ *and the solution of* (3.7) *determine all trajectories of the symmetrycal top in the Suslov–Kharlamova–Zabelina case* [6].

The trivial case when $v_j = \text{const}$ corresponds to the Suslov case [2].

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