Записки научных
семинаров ПОМИ
Том 300, 2003 г.

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INVERSE CASCADE SOLUTIONS OF THE EULER EQUATIONS

Abstract. The motion of an ideal incompressible fluid in a 2-dimensional domain M is considered. The initial velocity field is supposed to be small-scaled, i.e. its Fourier transform is concentrated at high frequences. The extreme case of flows corresponding to solutions of the Euler equations starting from the ZERO scale is studied. The main result of this work is that such solution exists. Its construction uses variational principle, generalized flows and continual braids.

1. Introduction: what is inverse cascade. Consider the motion of an ideal incompressible fluid in a 2d domain M (in what follows M is a 2d torus $\mathbb{T}^2$). It is described by the Euler equations

$$\frac{\partial u}{\partial t} + (u, \nabla)u + \nabla p = 0, \tag{1}$$

$$\nabla \cdot u = 0, \tag{2}$$

where $u(x,t)$ is velocity, $p(x,t)$ is pressure, $x \in M$, $0 \leqslant t \leqslant T$. Initial conditions:

$$u(x,0) = u_0(x). \tag{3}$$

It is known from numerous observations, experiments and computer simulations that if $u_0(x)$ is a small-scale, homogeneous and isotropic velocity field (i.e. its Fourier transform is concentrated at high frequencies), then the scale of the velocity field $u(x,t)$ becomes larger, as t grows, and the flow ends up as a global vortex (or a few of such vortices), having the scale of the whole domain M, and never returns to small scales.

This paradoxical behavior is a manifestation of *inverse (energy) cascade*, i.e. an irreversible energy flow from small-scale to large-scale motions. Inverse cascade is believed to play a principal role in the dynamics of atmosphere and ocean (see, for example, [3]).

The trouble is that nobody has proved the existence of solutions of the Euler (as well as Navier-Stokes) equations where we can see the inverse cascade. All the concept of inverse cascade has been a pure speculation. The goal of this work is to construct a sort of solutions having inverse-cascade behavior.

2. Weak solutions. I am going to push the situation to its extreme and construct a solution $u(x,t)$ starting at $t=0$ from a *zero* scale. This means that $u(x,t)$ becomes more and more oscillating, and thus tends weakly to zero, as $t \to 0$. Such solution cannot be smooth; it is necessarily a weak solution of the Euler equations.

To define a weak solution, let us choose a test-field $v(x,t) \in C_0^\infty$, such that $\nabla \cdot v = 0$, and a scalar test-function $\varphi(x,t) \in C_0^\infty$. Suppose $u(x,t)$ is a smooth solution of (1), (2). Let us multiply both parts of (1) and (2) resp. by $v(x,t)$ and $\varphi(x,t)$ and integrate by parts. We obtain identities

$$\int\int \Big[(u, \frac{\partial v}{\partial t}) + (u, \nabla v \cdot u)\Big] dx dt = 0; \tag{4}$$

$$\int\int (u, \nabla \varphi) dx dt = 0. \tag{5}$$

Definition 1. *Vector field $u(x,t)$, satisfying (4), (5) for any test-functions $v(x,t), \varphi(x,t)$ is called a weak solution of the Euler equations.*

The main result is

Theorem 1. *There exists a weak solution $u(x,t)$ such that*
(i) *$u(x,t) \to 0$ weakly in L^2, as $t \to 0$;*
(ii) *The kinetic energy*

$$E(t) = \int\limits_M \frac{1}{2}|u(x,t)|^2 dx = \text{const}. \tag{6}$$

3. The Least Action Principle. We are going to use the Least Action Principle for construction of the flow. In the classical hydrodynamics it is formulated as follows [1]. Consider the group $\mathcal{D}(M) = \mathcal{D}$ of volume-preserving diffeomorphisms of the flow domain M. It is a configuration space for the fluid, i.e. the space of its Lagrangian positions. Any element $g \in \mathcal{D}$ is a mapping of M into itself; if $x \in M$, then $g(x)$ is a position of a fluid particle having the label $x \in M$. So, there is a 1-1 correspondence between fluid particles and points of the flow domain M.

For every fluid trajectory $g_t \in \mathcal{D}, 0 \leqslant t \leqslant T$, we define its *action*

$$J\{g_t\} = \int_0^T \int_M \frac{1}{2} |\dot{g}_t(x,t)|^2 dx dt. \tag{7}$$

If g_t is the actual flow without external forces, then it is a geodesic on $\mathcal{D}$, i.e.

$$\delta J\{g_t\} = 0, \tag{8}$$

provided $\delta g_0 = 0$, $\delta g_T = 0$.

It is tempting to construct global solutions of the Euler equations by fixing g_0 and $g_T \in \mathcal{D}$, and looking for a flow g_t in $\mathcal{D}$ which *minimizes* the action $J\{g_t\}$. We can always take $g_0 = Id$, the identity map, and $g_T = g \in \mathcal{D}$.

Unfortunately, this method fails: for most $g \in \mathcal{D}$ there is no smooth trajectory g_t connecting Id and g and minimizing the action [4].

4. Generalized flows. Y. Brenier [2] has found a generalized solution of the above variational problem. He has defined a new object called *generalized flow*. Let us fix $g \in \mathcal{D}$. Let Ω be a standard probability space with σ-algebra $\mathcal{F}$ and probability measure μ, so that $\mu(\Omega) = 1$.

Definition 2. *Generalized flow* $\mathbf{G}$, *connecting* Id *and* $g \in \mathcal{D}$, *is a mapping from* $\Omega \times [0,T]$ *into* M, $(\omega, t) \mapsto x(\omega, t)$, *satisfying the following conditions.*

(i) *For any measurable* $A \subset M$ *and any* t,

$$\mu\{\omega \mid x(\omega,t) \in A\} = \operatorname{mes} A.$$

(This property may be called incompressibility).

(ii) *Let us define the action of* $\mathbf{G}$,

$$J\{\mathbf{G}\} = \int_\Omega \mu(d\omega) \int_0^T \frac{1}{2} |\dot{x}(\omega,t)|^2 dt. \tag{10}$$

Then $J\{\mathbf{G}\} < \infty$.

(iii) *For almost all* (ω, x), $x(\omega,T) = g(x(\omega,0))$ *(the endpoint condition).*

Relaxed variational problem (Brenier [2]): given $g \in \mathcal{D}$, find a generalized flow $\mathbf{G}$, satisfying (i)-(iii) and minimizing the action $J\{\mathbf{G}\}$.

Theorem 2. *For any bounded domain M and for any $g \in \mathcal{D}(M)$, there exists a generalized flow* $\mathbf{G}$*, minimizing the action* $J\{\mathbf{G}\}$.

(Proof is based on considerations of weak compactness of the space of measures and upper semicontinuity of the action functional.)

5. Examples of minimizing generalized flows. These examples were found by Y. Brenier [2]. Suppose $M \subset \mathbf{R}^2$ is a unit disk $|x| < 1$, and g is an antipodal map, i.e. $g : x \mapsto -x$. Then there exist 3 independent minimizing generalized flows, connecting Id and g:

1. Uniform clockwise rotation;
2. Uniform counterclockwise rotation;
3. The third solution found by Brenier: at $t = 0$ all fluid particles explode and give rise to a continuum of particles moving in all directions. Each particle moves along an arc of an ellipse, as if in the field of quadratic potential, and all particles reach the opposite point at the same moment $t = T$.

Every convex combination of these basic generalized flows is also a solution.

Unfortunately, these generalized flows do not look as real fluid flows, and they hardly describe any real turbulence, as it was hoped in the beginning.

6. Braids [5]**.** In 2-d case there is an additional structure. Consider first a smooth flow g_t connecting Id and $g \in \mathcal{D}(M)$. In the cylinder $Q = M \times [0, T]$ consider trajectories of fluid particles $\gamma_x : t \mapsto g_t(x)$. They do not meet one another for different $x \in M$, and thus form a smooth *braid* $\mathbf{B}$ comprising continuum of threads, filling Q with constant density.

Let us fix $g \in \mathcal{D}(M)$ and a smooth braid $\mathbf{B}_0$ connecting Id and g. We may look for a braid $\mathbf{B}$, connecting Id and g and having minimal action *among the braids isotopic to* $\mathbf{B}_0$. Solution of this problem, if exists, gives a smooth solution of the Euler equations. But this problem can have no smooth solution, and we have to introduce generalized solutions.

7. Generalized braids. Let us fix $g \in \mathcal{D}(M)$ and a smooth braid $\mathbf{B}_0$ connecting Id and g (the braid $\mathbf{B}_0$ consists of trajectories of a smooth flow g_t).

Definition 3. *Generalized braid* $\mathbf{B}$*, weakly isotopic to* $\mathbf{B}_0$*, is a generalized flow* $x(\omega, t)$*,* $\omega \in \Omega$*, connecting* Id *and* g *and such that for any* N*,* *for almost all* $(\omega_1, \dots, \omega_N) \in \Omega^N$, *the* N*-braid* $\Big(x(\omega_1, t), \dots, x(\omega_N, t)\Big)$ *is isotopic to the braid* $\Big(g_t(x_1), \dots, g_t(x_N)\Big)$, *where* $x_i = x(\omega_i, 0)$.

(Note that these N-braids have the same sets of endpoints.)

8. Minimal generalized braids. Let $g \in \mathcal{D}(M)$ and a smooth braid $\mathbf{B}_0$, connecting Id and g, be given. We are looking for a braid $\mathbf{B}$, connecting Id and g, weakly isotopic to $\mathbf{B}_0$ and having minimal action $J\{\mathbf{B}\}$.

Theorem 3. (i) *There exists a braid* $\tilde{\mathbf{B}}$ *with minimal action among all braids* $\mathbf{B}$ *weakly isotopic to* $\mathbf{B}_0$.

(ii) *There exists a vector field* $u(x,t) \in L^2$, *such that for almost all* (ω, t),

$$\dot{x}(\omega, t) = u(x(\omega, t), t), \tag{11}$$

where $x(\omega, t)$ *is a mapping corresponding to the braid* $\tilde{\mathbf{B}}$, *regarded as a generalized flow.*

(iii) *Vector field* $u(x,t)$ *is a weak solution of the Euler equations, such that the kinetic energy*

$$E(t) = \int\limits_M \frac{1}{2}|u(x,t)|^2 = \text{const}. \tag{12}$$

9. Semiflows. Semiflow is a family f_s^t, $T \geqslant s \geqslant t \geqslant 0$ of continuous maps of M into itself, such that

(i) f_s^t depends continuously on s and t, $s \geqslant t$;

(ii) $f_s^s = Id$;

(iii) for any $T \geqslant r \geqslant s \geqslant t \geqslant 0$, $\quad f_s^t \circ f_r^s = f_r^t$;

(iv) f_s^t is a continuous measure preserving endomorphism of M, $T \geqslant s \geqslant t \geqslant 0$.

(Note that in this definition time is going back.)

10. Example of a semiflow. Assume that $M = \mathbb{T}^2$. Let $U(x,t)$, $\frac{1}{2} < t < 1$, be a smooth incompressible field on M. Define a new field $V(x,t)$ by

$$V(x,t) = U(2^i x, 2^i t), \quad \text{if} \;\; 2^{-i-1} < t < 2^{-i}. \tag{13}$$

Let $f_s^t(x)$ be a flow in M generated by the field $V(x,t)$. It has the following self-similarity:

$$f_{2^{-i}}^{2^{-i-1}}(x) = 2^{-i} f_1^{1/2}(2^i x). \tag{14}$$

Let us define $V(x,t)$ as

$$V(x,t) = \begin{cases} (a \sin x_2, 0), & \text{if } 1/2 < t < 3/4, \\ (0, a \sin x_1), & \text{if } 3/4 < t < 1, \end{cases} \tag{15}$$

and extend to all t by (13).

Lemma 1. *There exists $a_0 > 0$, such that for $0 \leqslant a \leqslant a_0$ mapping f_1^0 is invertible, and $(f_1^0)^{-1}$ is Hölder continuous; for $a > a_0$, for a.a. $x \in M$, the set $Z_x = (f_1^0)^{-1}(x)$ has a positive Hausdorff dimension.*

11. Semibraids. Semiflow f_s^t defines a *semibraid* $\mathbf{S}B_0$; it is an incompressible braid, but some threads join together at $t = 0$.

The flow domain M is partitioned into sets $Z_x = (f_1^0)^{-1}(x)$, $x \in M$. If $x(\omega, t)$, $\omega \in \Omega$ is any incompressible generalized flow in M, we can define a partition of the particle space Ω into sets $\Xi_x: \ \omega \in \Xi_x$, if $x(\omega, 1) \in Z_x$. Let $\chi_x(d\omega)$ be the conditional measure on Ξ_x, defined for a.a. $x \in M$.

Definition 4. *Generalized flow $x(\omega, t)$, $\omega \in \Omega$, is called a semibraid weakly isotopic to $\mathbf{S}B_0$, if*

(i) *for any k, for a.a. $x \in M$, for a.a. $(\omega_1, \dots, \omega_k) \in \Xi_x^k$, $x(\omega_1, 0) = \dots = x(\omega_k, 0)$;*

(ii) *for any N, for any $k_1, \dots, k_N$, for a.a. $(x_1, \dots, x_N) \in M^N$, for a.a. $\omega_{n,k} \in \Xi_{x_n}$, $1 \leqslant n \leqslant N$, $1 \leqslant k \leqslant k_n$, finite braid comprising the lines $\delta_{n,k} : t \mapsto \Big(x(\omega_{1,1}, t), \dots, x(\omega_{N,k_N}, t)\Big)$ is isotopic to the braid $(\gamma_{n,k})$, where $\gamma_{n,k} : t \mapsto f_1^t(x_{n,k})$, $x_{n,k} = x(\omega_{n,k}, 1)$. (During the isotopy there should be $\delta_{n,k_1}(0) = \dots = \delta_{n,k_n}(0)$ for all n.)*

12. Minimal semibraids. Consider the following problem: among all semibraids $\mathbf{S}B$, weakly isotopic to $\mathbf{S}B_0$, find a semibraid $\tilde{\mathbf{S}}B$ with minimal action $J\{\tilde{\mathbf{S}}B\}$.

Theorem 4. (i) *There exists a semibraid $\tilde{\mathbf{S}}B$ which is weakly isotopic to $\mathbf{S}B_0$ and has minimal action among all such semibraids.*

(ii) *For this semibraid there exists a vector field $u(x,t) \in L^2$, such that for a.a. (ω, t), $\dot{x}(\omega, t) = u(x(\omega, t), t)$.*

(iii) *The field $u(x,t)$ is a weak solution of the Euler equations, and $u(x,t) \to 0$ weakly in L^2 as $t \to 0$.*

(iv) *The kinetic energy*

$$E(t) = \int_M \frac{1}{2}|u(x,t)|^2 = \mathrm{const}. \tag{16}$$

Thus Theorem 1 is proved.

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Поступило 30 ноября 2002 г.