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THE ANDRONOV—HOPF BIFURCATION WITH 2:1 RESONANCE

ABSTRACT. We consider dissipative dynamical systems in the neighborhood of quasi-periodic n-dimensional invariant tori that are not normally hyperbolic. We assume that the normal spectrum contains precisely two pairs of simple pure imaginary eigenvalues. We investigate the case where the frequencies are in the ratio 2:1. We establish sufficient conditions for the existence of invariant tori of dimension n+p in certain region of the parameter space.

1. INTRODUCTION

One of the central topics in the qualitative theory of differential equations is the study of invariant submanifolds. A number of general theorems establishing the existence and bifurcations and describing the properties of those submanifolds plays a fundamental role in the analysis of nonlinear dynamical systems. A particular example of an invariant submanifold is an invariant torus with parallel dynamics(quasi-periodic torus). The present paper is devoted to the dynamics near quasi-periodic tori that are not normally hyperbolic. In the case when the codimension of the tori in the ambient space exceeds 1, we need the hypothesis that the normal linear dynamics is reducible to Floquet form. Our approach is based on the recent progress in the parametric KAM theory [3, 4] which examines invariant tori in smooth or analytic dynamical systems rather than tori in individual systems. As essential component of the present approach is that the normal forms are developed near non-hyperbolic diohantine tori. Here the dynamics is formally independent of the variables on the tori. In the nonhyperbolic case bifurcations occur, similar to Andronov-Hopf bifurcation as they occur near equilibrium or periodic solution.

Let X be a smooth vector field on a smooth real finite-dimensional manifold M. An invariant n-torus T of vector field X is a submanifold of M invariant under the flow of X and diffeomorphic to the standard n-torus $\mathbf{T}^n = (\mathbf{R}/2\pi\mathbf{Z})^n$. An invariant n-torus T is called quasi-periodic if there exist coordinates $\phi \in T^n$ on $\mathbf{T}$ in which the restriction of $\mathbf{X}$ to $\mathbf{T}$

gets the form $\omega\partial/\partial\varphi$ for some vectors ω if its frequencies $\omega_1, \dots, \omega_n$ are linearly independent over the field $\mathbf{Q}$ of rational numbers. It turns out that all the invariant quasi-periodic tori in the phase space of a generic vector field are usually not only nonresonant but, moreover, their frequencies satisfy certain diophantine conditions.

Before formulating these conditions, we introduce some notations to be used below. By the angle brackets, we will denote the standard inner product of two vectors

$$\langle a, b\rangle = \sum_{j=1}^{N} a_j b_j \quad \text{for} \quad a \in \mathbf{R^N}, b \in \mathbf{R^N}.$$

The symbol $|a|$ will denote the l_1-norm of vector $\mathbf{a}$:

$$|a| = \sum_{j=1}^{N} |a_j| \quad \text{for} \quad a \in \mathbf{C^N}.$$

We will write $\mathbf{O}_n(u)$ instead $\mathbf{O}(|u|^n)$. Finally, $\mathbf{N}$ is the set of positive integers. A quasi-periodic invariant n-torus is said to be diophantine if its frequency vector ω satisfies the infinite system of inequalities

$$|\langle\omega, k\rangle| \geq \gamma |k|^{-\tau} \quad \text{for} \quad \forall k \in \mathbf{Z}\backslash\{0\} \tag{1}$$

for some constants $\tau > 0$ and $\gamma > 0$. Diophantine invariant tori of dynamical systems are the subject of the Kolmogorov–ArnoI'd–Moser (KAM) theory [2, 8, 3, 11].

2. Bifurcation model

The classical Andronov-Hopf bifurcation theorem gives sufficient conditions under which a one-parameter family of periodic solutions bifurcates from an equilibrium point in a one-parameterized family of autonomous ordinary differential equations. One of these conditions is that criticality, the matrix of linearization of the vector field has a simple pair of pure imaginary complex conjugate eigenvalues $\pm i\omega$, $\omega > 0$ and no other eigenvalue of the form $ik\omega$, where k is an integer(non-resonant condition) [1, 10]. Next in the hierarchy is the case where there are two pairs of simple pure imaginary eigenvalues $\pm i\beta_1$ and $\pm i\beta_2$, where β_1, $\beta_2 > 0$. The cases in which β_2/β_1 is rational are known as the resonant cases. Of these, the so-called strong resonances ($\beta_2/\beta_1 = 1$,2 and 3) are the most interesting. The **1:1, 1:2** and **1:3** cases have been studied by many authors (for example [9, 7, 1, 10]. The Andronov–Hopf bifurcation of invariant tori have been considered in [4, 6, 12, 5, 2, 13].

Let us consider a μ-parameterized family of vector fields $\mathbf{X}_\mu(\mathbf{x}, \mathbf{y})$ on the manifold $\mathbf{M} = \mathbf{T^n} \times \mathbf{R^4}$ which for $\mu = \mu_0$ has the torus $\mathbf{T^n} \times \{0\}$ as a nonhyperbolic, diophantine invariant tori, which is in Floquet form. This means that

$$\mathbf{X}_{\mu_0} = \sum_{j=1}^{n} \{\omega_j + O_1(y)\} \frac{\partial}{\partial x_j} + \sum_{l,r=1}^{4} \{A_{lr} y_r + O_2(y)\} \frac{\partial}{\partial y_l}, \qquad (2)$$

where $x = (x_1, \ldots, x_n) (\text{mod } 2\pi) \in \mathbf{T^n}$ and $y = (y_1, \ldots, y_4) \in \mathbf{R^4}$. The parameters μ vary in a open domain $\mathbf{G} \subset \mathbf{R^s}$. Suppose the frequency-vector $\omega = (\omega_1, \ldots, \omega_n) \in \mathbf{R^n}$ satisfies a diophantine conditions (1) for some $\tau > n$ and $\gamma > 0$. We investigate the case of matrix $\mathbf{A}$ when it has pure imaginary eigenvalues $\pm\beta_1 i$; $\pm\beta_2 i$. Here we take $\beta_1, \beta_2 > 0$ and we call $\beta = (\beta_1, \beta_2) \in \mathbf{R^2}$ the vector of normal frequencies. In this paper we study the $2 : \mathbf{1}$ resonance, and assume

$$\beta_2 = 2\beta_1$$

at $\mu = \mu_0$.

3. Normal form and Truncated system

Now consider the normal linear part of the vector field $\mathbf{X}_{\mu_0}$ at the torus $\mathbf{T^n} \times \{0\}$, which is given by

$$\sum_{j=1}^{n} \omega_j \frac{\partial}{\partial x_j} + \sum_{l,r=1}^{4} \{A_{lr} y_r\} \frac{\partial}{\partial y_l}. \qquad (3)$$

This normal linear part generates a $\mathbf{T^{n+2}}$-action on the phase space $\mathbf{M}$. Then our normal form is a formal symmetrization of $\mathbf{X}_\mu(\mathbf{x}, \mathbf{y})$ at the torus $\mathbf{T^n} \times \{0\} \times \{\mu_0\}$ with respect to this action.

Theorem. *Let $\mathbf{X}_\mu(\mathbf{x}, \mathbf{y})$ be a real analytic family of vector fields*

$$\mathbf{X}_\mu = \sum_{j=1}^{n} \{\omega_j + f_j(x, y, \mu)\} \frac{\partial}{\partial x_j} + \sum_{l,r=1}^{4} \{A_{lr} y_r + g_l(x, y, \mu)\} \frac{\partial}{\partial y_l}$$

with $(x, y, \mu) \in \mathbf{T^n} \times R^4 \times R^s$, where $f(x, y, 0) = \mathbf{O}(y)$ and $g(x, y, 0) = \mathbf{O}(y^2)$. Suppose

(i) $\beta_2 = 2\beta_1$,

(ii) $|\langle \omega, k \rangle + p\beta_1| \geq \gamma |k|^{-\tau}$ *for* $\forall k \in \mathbf{Z} \backslash \{0\}, 0 < p < q.$

Then a real analytic transformation map Φ*:*

$$\Phi : (\mathbf{x}, \mathbf{y}, \mu) \to (\mathbf{x} + \phi(\mathbf{x}, \mathbf{y}, \mu), y + \chi(\mathbf{x}, \mathbf{y}, \mu))$$

exists, tangent to the identity map, such that the $\Phi_*(\mathbf{x})$ *has the form:*

$$\begin{cases} \dot{x}_j &= \omega_j + h_j(\sigma_1, \sigma_2, \sigma_3, \mu) + O_q(y) \\ \dot{y}_1 &= (a_1 + b_1 y_3 - b_2 y_4) y_1 - (a_2 - b_1 y_4 - b_2 y_3) y_2 + O_{q+1}(y) \\ \dot{y}_2 &= (a_2 + b_2 y_3 + b_1 y_4) y_1 + (a_1 + b_2 y_4 - b_1 y_3) y_2 + O_{q+1}(y) \\ \dot{y}_3 &= c_1 y_3 - c_2 y_4 + d_1(y_1^2 - y_2^2) - 2 d_2 y_1 y_2 + O_{q+1}(y) \\ \dot{y}_4 &= c_2 y_3 + c_1 y_4 + d_2(y_1^2 - y_2^2) + 2 d_1 y_1 y_2 + O_{q+1}(y), \end{cases} \tag{4}$$

where $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2$ *are real functions of* μ *and*

$$\sigma_1 = y_1^2 + y_2^2 \ , \quad \sigma_2 = y_3^2 + y_4^2 \ , \quad \sigma_3 = y_1^2 y_3 - y_3 y_2 + 2 y_1 y_2 y_4 .$$

Moreover, a_1, a_2, c_1, c_2 *are zero at the origin.*

Proof. To prove the theorem one may follow [2].

We consider the case $q = 3$. To study system (4) we introduce new variables

$$y_1 = \rho_1 \cos\phi_1, \quad y_2 = \rho_1 \sin\phi_1, \quad y_3 = \rho_2 \cos\phi_2, \quad y_4 = \rho_1 \sin\phi_2$$

and obtain

$$\begin{cases} \dot{x}_j = \omega_j + h_j(\rho_1, \rho_2, \phi_1, \phi_2, \mu) + O_3(\rho) \\ \dot{\rho}_1 = \alpha_1(\mu)\rho_1 + q_1 \cos(\phi_2 - 2\phi_1 + \theta_1)\rho_1\rho_2 \\ \qquad + \rho_1(a_{1,1}\rho_1^2 + a_{1,2}\rho_2^2) + O_4(\rho) \\ \dot{\rho}_2 = \alpha_2(\mu)\rho_2 + q_2 \cos(-\phi_2 + 2\phi_1 + \theta_2)\rho_1^2 \\ \qquad + \rho_2(a_{2,1}\rho_1^2 + a_{2,2}\rho_2^2) + O_4(\rho) \\ \dot{\phi}_1 = \beta_1(\mu) + q_1 \sin(\phi_2 - 2\phi_1 + \theta_1)\rho_2 + (b_{2,1}\rho_1^2 + b_{2,2}\rho_2^2) + O_3(\rho) \\ \dot{\phi}_2 = \beta_2(\mu) + q_2 \sin(-\phi_2 + 2\phi_1 + \theta_2)\rho_1^2/\rho_2 + (b_{2,1}\rho_1^2 \\ \qquad + b_{2,2}\rho_2^2) + O_3(\rho), \end{cases} \tag{5}$$

where $q_1 \cos(\theta_1) = b_1(0)$, $q_1 \sin(\theta_1) = b_2(0)$, $q_2 \cos(\theta_2) = d_1(0)$, $q_2 \sin(\theta_2) = d_2(0)$, $\rho = (\rho_1, \rho_2)$, $\phi = (\phi_1, \phi_2)$. We next define $\zeta = \phi_2 - 2\phi_1 - \theta_2$, $\theta_0 = \theta_1 + \theta_2$. To system (5) we associate the truncated system

$$\begin{cases} \dot{x}_j = \omega_j + h_j(\rho, \phi_1, \zeta, \mu) \\ \dot{\phi}_1 = \beta_1(\mu) + q_1 \sin(\zeta + \theta_1)\rho_2 + (b_{2,1}\rho_1^2 + b_{2,2}\rho_2^2) \\ \dot{\rho}_1 = \alpha_1(\mu)\rho_1 + q_1 \cos(\zeta)\rho_1\rho_2 + \rho_1(a_{1,1}\rho_1^2 + a_{1,2}\rho_2^2) \\ \dot{\rho}_2 = \alpha_2(\mu)\rho_2 + q_2 \cos(\zeta + \theta_0)\rho_1^2 + \rho_2(a_{2,1}\rho_1^2 + a_{2,2}\rho_2^2) \\ \dot{\zeta} = \beta_2(\mu) - 2\beta_1(\mu) - q_2 \sin(\zeta + \theta_0)\rho_1^2/\rho_2 - 2q_1 \sin(\zeta)\rho_2 \\ \qquad + (c_{2,1}\rho_1^2 + c_{2,2}\rho_2^2). \end{cases} \tag{6}$$

If the truncated system has invariant quasi-periodic tori, so this implies occurrence of quasi-periodicity in the full system. Similar procedures have been studied by Bibikov, Broer, Takens Chensiner [2, 4, 5].

4. Construction of the Bifurcation Tori

The system (6) is invariant under the symmetry $\rho_1 \to -\rho_1$ which implies that the line $\rho_1 = 0$ is invariant. Because ρ_1, ρ_2 are both polar radii, the solutions of interest are those satisfying $\rho_1, \rho_2 \geq 0$. Equation (6) posseses two types of fixed point, those with $\rho_1 = 0$ corresponding to a classical Andronov–Hopf bifurcation (non-resonant case), and those with $\rho_1 \neq 0$ – non-classical bifurcation. We consider non-classical bifurcation. (The unfolding classical quasi-periodic invariant tori was developed by H. W. Broer, G. B. Huitema and B. L. J. Braaksma [4].) Suppose that $\nu(\mu) = \beta_2(\mu) - 2\beta_1(\mu)$ and rank $\partial(\alpha_1, \alpha_2, \nu)/\partial\mu = 3$. To analyze this bifurcation, we introduce a small parameter δ, and scaling variables ρ_1, ρ_2, ζ and parameters α_1, α_2, ν

$$\rho_1 \to \delta\rho_1, \rho_2 \to \delta\rho_2, \zeta \to \delta\zeta, \alpha_1 \to \delta\alpha_1, \alpha_2 \to \delta\alpha_2, \nu \to \delta\nu.$$

Equations (7) reduce to

$$\begin{cases} \alpha_1(\mu)\rho_1 + q_1 \cos(\zeta)\rho_1\rho_2 = 0 \\ \alpha_2(\mu)\rho_2 + q_2 \cos(\zeta + \theta_0)\rho_1^2 = 0 \\ \nu - q_2 \sin(\zeta + \theta_0)\rho_1^2/\rho_2 - 2q_1 \sin(\zeta)\rho_2 = 0. \end{cases} \tag{7}$$

Lemma. *Assume that* $\tan(\theta_0) \neq 0$. *Then the system of equation* (7) *has* 0, 1 *or* 2 *non-classical small-amplitude solution.*

Proof. We solve the first and the second equations

$$\begin{cases} \alpha_1(\mu)\rho_1 + q_1 \cos(\zeta)\rho_1\rho_2 = 0 \\ \alpha_2(\mu)\rho_2 + q_2 \cos(\zeta + \theta_0)\rho_1^2 = 0 \end{cases}$$

for ρ_2, ρ_1^2/ρ_2 :

$$\rho_2 = -\alpha_1(\mu)/q_1 \cos(\zeta), \quad \rho_1^2/\rho_2 = -\alpha_2(\mu)/q_2 \cos(\zeta + \theta_0).$$

Substitution of ρ_2 and ρ_1^2/ρ_2 into the third equation yields the quadratic equation in $\tan(\zeta)$:

$$\nu(\mu) + \alpha_2(\mu)\tan(\zeta + \theta_0) + 2\alpha_1(\mu)\tan(\zeta) = 0. \tag{8}$$

Generically, a quadratic equation has none, one or two real roots. Therefore, we expect to get open regions in parameter space in which the system (7) has respectively none, one and two real solutions. We can prove quasi-periodic stability of the corresponding tori following Broer, Huitema, Takens, Braaksma [4].

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References

1. V. I. Arnol'd, *Geometrical Methods in the Theory of Ordinary Differential Equations.* — Springer-Verlag (1983).
2. Yu. N. Bibikov, *Multifrequency Nonlinear Oscillations and their Bifurcations.* — Leningrad Univ. Press (1991).
3. H. W. Broer, G. B. Huitema and M. B. Sevryuk, *Quasi-periodic motions in families of dynamical systems. Order amidst chaos.* — Lecture Notes in Mathematics, **1645**, Springer, Berlin (1996).
4. H. W. Broer, G. B. Huitema, F. Takens, B. Braaksma, *Unfoldings and bifurcations of quasi-periodic tori.* — Mem. Amer. Math. Soc., **83**(421) (1990), 1–81.
5. A. Chenciner, G. Looss, *Bifurcations de tores invariants.* — Arch. Rat. Mech. An., **69**(2) (1979), 109–198; **71**(4) (1979), 301–306.
6. D. Flockerzi, *Generalized bifurcation of higher dimensional tori.* — J. Diff. Eq., **55** (1984), 346–367.
7. E. Knobloch, M. Proctor, *The double Hopf bifurcation with 2 : 1 resonance.* — Proc. R. Soc. Lond., **A 415** (1988), 61–90.
8. V. F. Lazutkin, *KAM Theory and Semiclassical Approximations to Eigenfunctions.* — Springer-Verlag (1993).
9. V. G. LeBlanc, W. F. Langford, *Classification and Unfoldings of 1:2 Resonant Hopf Bifurcation.* — Arch. Rational Mech. Anal. **136** (1996), 305–357.
10. J. E. Marsden, M. McCracken, *The Hopf bifurcation and its applications.* — Springer-Verlag (1976).

11. J. Moser, *Stable and Random Motions in Dynamical Systems, with Special Emphasis on Celestial Mechanics.* — Ann. Math. Studies, Vol. 77. Princeton Univ. Press (1973).
12. G. R. Sell, *Bifurcation of higher dimensional tori.* — Arch. Rat. Mech. An., **69**(13) (1979), 199–230.
13. D. Yu. Volkov, *Bifurcation of invariant tori of equilibrium state involving zero characteristic values.* — Vestn. Leningr. Univ., Ser. I 1988, No. 2, 102–103 (1988).

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