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HYPERBOLIC STRUCTURE FOR THE QUADRATIC MAP

ABSTRACT. The quadratic map of the real plane is considered. A possible construction of a hyperbolic structure on some non compact set is suggested and discussed.

§0. INTRODUCTION

Let M be a Riemannian manifold, $G : M \longrightarrow M$ be a diffeomorphism. Let $M_g \subset M$ be an invariant set with respect to the map G (i.e. $G(M_g) = M_g$). One says, that the set M_g possesses a weak uniform hyperbolic structure with respect to the map G, if for each point $x \in M_g$ the tangent space T_xM can be represented as a direct sum:

$$T_xM = E_s(x) \dot{+} E_u(x)$$

and for each constant $\lambda > 1$ there exists a constant $c > 0$ such that

$$\forall v \in E_s \ \ ||(G^n(x))' v|| \leqslant c\lambda^{-n}||v||, \ n = 1, 2, ...,$$

$$\forall v \in E_u \ \ ||(G^{-n}(x))' v|| \leqslant c\lambda^{-n}||v||, \ n = 1, 2, ...,$$

E_s, E_u are called stable and unstable subspaces (or directions if they are 1-dimensional).

We consider a generic area-preserving quadratic map of the real plane $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined as:

$$x_1 = x^2 - y + C, \tag{1}$$

$$y_1 = x, \tag{2}$$

$$F(x, y) = (x_1, y_1), \tag{3}$$

where C is a real parameter.

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Let M_g be an open subset of $\mathbb{R}^2$ invariant with respect to the map F. The main question we study is how to describe such a subset M_g being as wide as possible, which possesses a weak uniform hyperbolic structure with respect to the map F.

Consider an open set D consisting of points of the plane which tend to infinity under both forward and backward iterates. We prove the following results:

1. Set D is "wide enough".

2. There are stable and unstable directions at every point of the set D.

3. There exists a subset Bad of the set D, which is closed in the topology of D and of zero Lebesgue measure with the following property: the stable and unstable directions at each point of Bad coincide to each other: $E_s = E_u$. The situation in the set Bad is worse for us. But if we consider $E = D \backslash Bad$, we can construct a hyperbolic structure on E.

Here is the list of notations which might produce a confusion.

1. "$\mathbb{K}$" means an element of the set $\{\mathbb{R}, \mathbb{C}\}$,

2. "$BOUND$" means the set of all bounded subsets of the real plane $\mathbb{R}^2$.

§1. Basic facts on the dynamics of the quadratic map

To begin with we introduce some domains of $\mathbb{K}^2$ which possess important properties with respect to the quadratic map. Let A and B be real numbers. Consider a domain in $\mathbb{C}^2$ defined as:

$$\Omega^{\mathbb{C}}_{AB} = \{(x, y) \in \mathbb{C}^2 : |x| > B, |x| > A|y|\}.$$

Consider also a domain in $\mathbb{R}^2$ defined as:

$$\Omega_{AB} = \{(x, y) \in \mathbb{R}^2 : x > B, x > A|y|\}.$$

It is easy to see that under certain conditions on A and B these domains possess some properties.

Theorem 1. *Let real numbers A and B be such that*

$$B > \frac{1}{2}(\frac{1}{A} + A) + \sqrt{\frac{1}{4}(\frac{1}{A} + A)^2 + |C|},\ B > 0,\ A > 1, \tag{4}$$

then for $\Omega \in \{\Omega_{AB}, \Omega^{\mathbb{C}}_{AB}\}$ the following conditions hold:

$$G(\Omega) \subset \Omega, \tag{5}$$

$$\forall z \in \Omega \lim_{n\to+\infty} ||G^n(z)|| = \infty. \tag{6}$$

Moreover for Ω_{AB} one can point out an additional property:

$$\lim_{n\to+\infty} ||G^n(z)|| = \infty \Longrightarrow \exists\; m \in \mathbb{N}:\; G^m(z) \in \Omega_{AB}. \tag{7}$$

Consider the following sets invariant with respect to the quadratic map:

$$\Gamma_t = \{(x,y) \in \mathbb{R}^2 : \lim_{n\to+\infty} ||(x_n, y_n)|| = \infty\},$$

$$\Gamma_b = \{(x,y) \in \mathbb{R}^2 : \lim_{n\to+\infty} ||(x_{-n}, y_{-n})|| = \infty\},$$

$$B_0 = \Gamma_t \cup \Gamma_b,$$

$$A_t = \{(x,y) \in \mathbb{R}^2 : \{(x_n, y_n)\}_{n\in\mathbb{Z}_+} \in BOUND\},$$

$$A_b = \{(x,y) \in \mathbb{R}^2 : \{(x_n, y_n)\}_{n\in\mathbb{Z}_-} \in BOUND\},$$

$$A_0 = A_t \cap A_b, \quad D = \Gamma_t \cap \Gamma_b,$$

$$B_t = \Gamma_t \cap A_b \subset B_0, \quad B_b = \Gamma_b \cap A_t \subset B_0.$$

First of all, it is easy to observe the following topological properties of the defined above sets.

Theorem 2. *The sets B_0, D are open, A_t, A_b are closed. The set A_0 is compact. The sets B_t, B_b are closed in the topology of B_0.*

The next theorem is an important fact about these sets which is needed in the sequel.

Theorem 3. *Let mes be the Lebesgue measure in $\mathbb{R}^2$ then the following equalities are true:*

$$mes B_t = 0, \quad mes B_b = 0. \tag{9}$$

§2. Stable and unstable vector fields on the set D. Weak uniform hyperbolic structure

Let N_g be an open subset of $\mathbb{K}^d$ and let $G : N_g \longrightarrow N_g$ be a diffeomorphism.

Definition 1. *We say that an open set N_g possesses a stable g^s_λ and an unstable g^s_λ vector fields with respect to the map G with a parameter $\lambda > 1$ if the following conditions hold:*

$$g_\lambda^{s(u)} : N_g \longrightarrow \mathbb{K}^d, \tag{10}$$

$$\forall z \in N_g \ ||g_\lambda^{s(u)}|| = 1, \tag{11}$$

$$\forall z \in N_g \ \exists c > 0 : \ \forall n \in \mathbb{N} \ ||(G^n(z))' g^s_\lambda(z)|| \leqslant c\lambda^{-n},$$
$$||(G^{-n}(z))' g^u_\lambda(z)|| \leqslant c\lambda^{-n}. \tag{12}$$

Here is the basic statement of this section.

Theorem 4. *The set D possesses analytic vector fields k^s and k^u which are stable and unstable vector fields with respect to the map F for each parameter $\lambda > 1$.*

Proof. We construct the stable vector field, the unstable one can be constructed analogously. Consider an arbitrary $\lambda > 1$. We need an auxiliary complex-valued function $\Phi_s^{\mathbb{C}}$ defined on the domain $\Omega_{AB}^{\mathbb{C}}$, where $B = \lambda + \frac{3}{4}e_1$ and A is taken such that the condition (4) holds. Here e_1 is an arbitrary number from the interval $(0, 1)$. In order to construct the function $\Phi_s^{\mathbb{C}}$ we take the function Φ defined on $\mathbb{C}^2 \backslash (\{0\} \times \mathbb{C})$ giving us the ratio of the second component to the first one:

$$\Phi(t_1, t_2) = \frac{t_2}{t_1}.$$

Let $(x_0, y_0) \in \Omega_{AB}^{\mathbb{C}}$ be a point and $\{(x_n, y_n)\}_{n\in\mathbb{Z}}$ be its trajectory. Consider a sequence $\{l_n\}_{n\in\mathbb{N}}$ of vectors from $\mathbb{C}^2$ such that $(F^n(x, y))' l_n = (2\lambda + 2e_1, 1)$. It is easy to see that $\Phi(l_n)$ is an n-continuous fraction:

$$\Phi(l_n) = [2x_0; -2x_1, 2x_2, ..., (-1)^{n-1}2x_{n-1}, (-1)^n(2\lambda + 2e_1)].$$

As it follows from the continuous fractions theory, the sequence $\{\Phi(l_n)\}$ converges uniformly relatively $(x_0, y_0) \in \Omega_{AB}^{\mathbb{C}}$ to a limit $\Phi_s^{\mathbb{C}}(x_0, y_0)$, which does not depend on λ and on e_1. Moreover the following formula holds:

$$\Phi_s^{\mathbb{C}}(x_0, y_0) = [2x_0; -2x_1, 2x_2, ..., (-1)^n 2x_n, ...]. \tag{13}$$

So the function $\Phi_s^{\mathbb{C}}$ is constructed. Notice that $\Phi(l_n)$ depends analytically on a point $(x_0, y_0) \in \Omega_{AB}^{\mathbb{C}}$. Since the convergence is uniform, $\Phi_s^{\mathbb{C}}$ depends analytically on $(x_0, y_0) \in \Omega_{AB}^{\mathbb{C}}$. We construct the restriction Φ_s

of the function $\Phi_s^{\mathbb{C}}$ to the real domain Ω_{AB} by setting $\Phi_s = \Phi_s^{\mathbb{C}}|_{\Omega_{AB}}$. Obviously, the function Φ_s is analytic on Ω_{AB}. This implies that the vector k^s defined as $k^s = (\frac{1}{\sqrt{1+\Phi_s^2}}, \frac{\Phi_s}{\sqrt{1+\Phi_s^2}})$ is analytic on Ω_{AB}. Notice that the number Φ_s is the angle coefficient of the vector k^s. Moreover the construction implies that the 1-distribution defined by the field k^s is invariant under the map F, hence the field k^s can be continued analytically on the set D.

We show that the constructed field k^s is a stable vector field with respect to the map F with the parameter λ. It is clear that the conditions (10), (11) are true. We check now the condition (12). Take an arbitrary $z_0 \in \Omega_{AB}$, set $c = 1$. We show that the required inequality holds. Either $\lim_{n\to\infty} \frac{l_n}{||l_n||} = k^s$ or $\lim_{n\to\infty} \frac{l_n}{||l_n||} = -k^s$ holds. We consider the first case only, the second one can be considered in the same way. Take arbitrary $\varepsilon > 0$ and $n \in \mathbb{N}$. There exists a natural number N greater than n such that the following is true: $||(F^n(z_0))^{'}k^s|| = ||(F^n(z_0))'(\frac{l_N}{||l_N||} + (k^s - \frac{l_N}{||l_N||}))|| \leqslant ||(F^n(z_0))^{'}\frac{l_N}{||l_N||}|| + \varepsilon = ||v||/||u|| + \varepsilon$, where $v = [(F^{N-n})'(F^n(z_0))]^{-1}(2\lambda+2e_1, 1)$, $u = [F^n(z_0)]^{-1}v = l_N$. Thus we get:

$$||(F^n(z_0))^{'}k^s|| \leqslant ||v||/||u|| + \varepsilon. \tag{14}$$

By some simple algebraic calculation one sees that $|v_1| > (\lambda + \frac{e_1}{4})|v_2|$ and then $||u||/||v|| > \lambda^n$. Substituting this into the inequality (14) one obtains:

$$||(F^n(z_0))^{'}k^s|| \leqslant \lambda^{-n} + \varepsilon.$$

Since ε is arbitrary, we get:

$$||(F^n(z_0))^{'}k^s|| \leqslant \lambda^{-n}.$$

Thus for $z_0 \in \Omega_{AB}$ the condition (12) holds. Suppose that $z_0 \in D$, consider $k = min\{k_1 \in \mathbb{Z}_+ : \ F^{k_1}(z) \in \Omega_{AB}\}$. Since the 1-distribution defined by the field k^s is invariant under the map F, one gets:

$$\exists \alpha \in \mathbb{R}_+ \backslash \{0\} : \ (F^k)^{'}(z)k^s(z) = \alpha k_s(F^k(z)).$$

Set $c = \alpha\lambda^k$. Then the condition (12) holds, indeed: $||(F^n(z))^{'}k^s(z)|| = ||(F^{n-k})^{'}(F^k(z))(F^k)^{'}(z)k_s(z)|| = |\alpha|||(F^{n-k})^{'}(F^k(z))k^s(F^k(z))|| \leqslant |\alpha|\lambda^{-n}\lambda^k = c\lambda^{-n}$. Q.E.D.

We denote by Φ_s and Φ_u angle coefficients of the vector fields k^s and k^u respectively, where k^s and k^u are as in the last theorem. Notice that the numbers Φ_s and Φ_u can be infinite in some points of the set D.

Consider a real-analytic function S, being the sinus between the stable k^s and the unstable k^u vector fields, defined on the set D as

$$S = k_x^s k_y^u - k_y^s k_x^u. \tag{15}$$

Consider the following set:

$$Bad = \{z \in D : S(z) = 0\}.$$

We formulate an important results about it, which are basic components in the main theorem.

Theorem 5. *The following statements are true:*

$$Cl_D(Bad) = Bad, \tag{16}$$
$$F(Bad) = Bad, \tag{17}$$
$$mes Bad = 0. \tag{18}$$

Proof. The statement (16) follows from the continuity of the map S. The statement (17) follows from the invariance of the 1-distributions defined by the fields k^s, k^u under the map F. Using the formula (13), it is easy to see that, there are regions where $S < 0$ and there are regions where $S > 0$, so the function S analytic on the set D is not identically zero, hence it vanishes on a set of Lebesgue measure zero. This shows the statement (18). Q.E.D.

The following theorem is the main result of this article.

Theorem 6. *There exist an open subset E and a compact subset H of the real plane $\mathbb{R}^2$, invariant with respect to the map F and such that the following statements hold:*

(i) *the set E is contained in $\mathbb{R}^2 \backslash H$ and is of the full Lebesgue measure in $\mathbb{R}2 \backslash H$,*

(ii) *The set E possesses a weak uniform hyperbolic structure with respect to the map F.*

Proof. Set $H = A_0$, $E = D \backslash Bad$. The equalities (16) and (17) and the fact that the set D is open imply that the set E is open and invariant with respect to the map F. The definition of the set E, the theorem 3 and the formula (18) imply the statement (i). The statement (ii) follows from the theorem 4 and the definition of the set Bad. This completes the proof of the theorem. Q.E.D.

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References

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