Записки научных
семинаров ПОМИ
Том 300, 2003 г.

O. Zubelevich

ON EXPONENTIALLY SMALL EFFECTS IN DYNAMICAL SYSTEMS WITH A SMALL PARAMETER

Abstract. In the present paper we obtain a theorem which enables us to treat different exponentially small effects of dynamics from a unified point of view. As an example, we discuss the problem of fast phase averaging in non-autonomous Hamiltonian system with 3/2 degrees of freedom.

1. Introduction

Exponentially small effects appear in different kinds of dynamical systems with a small parameter (denoted below by $\varepsilon > 0$). Here we mean that some important dynamical quantities can be of order

$$O(e^{-c/\varepsilon^\beta}), \tag{1.1}$$

where c and β are positive constants.

For instance, exponentially small effects take place in the problem of fast phase average [2, 3, 4], and in the problem of the change of an adiabatic invariant [1, 5]. Though these problems lead to similar estimations, they are different by their dynamical substance and considered by different methods.

In the present paper we obtain a theorem which enables to treat given instances by the single viewpoint. However the precision of results obtained by means of this theorem slightly concedes for methods of articles [1, 3, 5]. These methods are more particular and therefore more precise then the present theorem.

These articles contain non-improvable estimates of constant c at (1.1). Such estimates are essentially beyond the bounds of the technique built below.

This work was partially supported by the grants INTAS 00-221 and Russian Foundation of Basic Research 00-15-99269.

2. Main result

Let $\{(E_s, \|\cdot\|_s)\}_{0<s\leqslant 1}$ be a scale of normed spaces over the field $\mathbf{R}$ or $\mathbf{C}$:

$$E_{s+\delta} \subseteq E_s, \quad \|\cdot\|_s \leqslant \|\cdot\|_{s+\delta}, \quad 0<\delta\leqslant 1-s.$$

We denote by

$$B_s(R) = \{x \in E_s \mid \|x\|_s \leqslant R\}$$

the ball in the space E_s with centre at zero. The following inclusion is obvious: $B_{s+\delta}(R) \subseteq B_s(R)$.

The parameter ε belongs to the interval $\mathcal{E}_\rho = \{\varepsilon \in \mathbf{R} \mid 0 < \varepsilon < \rho\}$.

Consider a map

$$P(\cdot,\varepsilon) : E_{s+\delta} \to E_s, \quad \varepsilon \in \mathcal{E}_\rho, \tag{2.1}$$

such that for all $\varepsilon \in \mathcal{E}_\rho$ and $x', x'' \in B_{s+\delta}(R)$ the following inequalities hold:

$$\|P(0,\varepsilon)\|_s \leqslant L, \quad s<1, \tag{2.2}$$

$$\|P(x',\varepsilon) - P(x'',\varepsilon)\|_s \leqslant \frac{M}{\delta^p}\|x'-x''\|_{s+\delta}, \quad p>0. \tag{2.3}$$

Constants M, L and p do not depend on s and ε.

Generically the equation

$$x = \varepsilon P(x,\varepsilon). \tag{2.4}$$

is not solvable with respect to x when $\varepsilon > 0$, but the following theorem holds.

Theorem 1. *For some positive constants $C_1, C_2, C_3, \varepsilon_0$ (depending only on L, M, p, ρ) and arbitrary $\varepsilon \in \mathcal{E}_{\varepsilon_0}$ there is a point $\hat{x} = \hat{x}(\varepsilon) \in B_{1/2}(R)$, such that*

$$\|\hat{x} - \varepsilon P(\hat{x},\varepsilon)\|_{1/2 - C_1\varepsilon^{1/p}} \leqslant C_2 \exp\left(-\frac{C_3}{\varepsilon^{1/p}}\right). \tag{2.5}$$

This theorem is pure algebraic in its nature. Proof of this theorem does not use any topological terms. In particular the assumption of completeness of spaces E_s is not necessary.

Remark 2.1. Theorem 1 remains valid if we consider $\delta \in [(2M\varepsilon)^{\frac{1}{p}}, 2(2M\varepsilon)^{\frac{1}{p}}]$ in (2.1, (2.3).

2.1. Proof. Substitute $x' = x, \quad x'' = 0$ in (2.3). Using estimate (2.2) and properties of norm we get:

$$\|P(x,\varepsilon)\|_s \leqslant \frac{M}{\delta^p}\|x\|_{s+\delta} + L \tag{2.6}$$

for all $x \in B_{s+\delta}(R)$ and $\varepsilon \in \mathcal{E}_\rho$.

Denote $\gamma = 2^p \cdot M$.

Lemma 2.1. *For sufficiently small positive ε there exists α_ε,*

$$\frac{1}{2(2\gamma)^{1/p}} \leqslant \alpha_\varepsilon \leqslant \frac{1}{(2\gamma)^{1/p}}, \tag{2.7}$$

such that $n = \alpha_\varepsilon/\varepsilon^{1/p} \in \mathbf{N}$.

Proof of this lemma becomes evident after the following trivial remark: for all sufficiently small ε the length of the interval

$$\left[\frac{1}{2(2\gamma\varepsilon)^{1/p}}, \frac{1}{(2\gamma\varepsilon)^{1/p}}\right]$$

is greater than 1.

Let us consider the sequence $\{x_k\}$:

$$x_0 = 0, \quad x_{k+1} = \varepsilon P(x_k, \varepsilon), \quad k = 0, \dots, n-1. \tag{2.8}$$

Below in this section the number n is defined by 2.1. Formulas (2.3) and (2.6) are used with $\delta = 1/(2n)$. Thus using estimate (2.7) we get remark 2.1.

Lemma 2.2. *For sufficiently small positive ε and $k = 1, \dots, n$ the following inclusion holds:*

$$x_k \in B_{1-\frac{k}{2n}}(R).$$

Proof. Let us consider $\varepsilon \leqslant R/(2L)$. It is obvious that $x_1 \in B_{1-\frac{1}{2n}}(R)$. Indeed according to formula (2.2) we get:

$$\|x_1\|_{1-\frac{1}{2n}} = \varepsilon\|P(0,\varepsilon)\|_{1-\frac{1}{2n}} \leqslant \varepsilon L \leqslant R.$$

To finish the proof we check that if $x_j \in B_{1-\frac{j}{2n}}(R)$ then

$$x_{j+1} \in B_{1-\frac{j+1}{2n}}(R), \quad \text{where} \quad j = 1, \dots, n-1.$$

Using formula (2.6) and Lemma 2.1 we obtain the estimates:

$$\|x_{j+1}\|_{1-\frac{j+1}{2n}} = \varepsilon\|P(x_j,\varepsilon)\|_{1-\frac{j+1}{2n}} \leqslant \varepsilon(\gamma n^p\|x_j\|_{1-\frac{j}{2n}} + L) \leqslant \frac{R}{2} + \varepsilon L \leqslant R.$$

Lemma is proved.

Lemma 2.3. *For sufficiently small positive* ε

$$\|x_k - \varepsilon P(x_k,\varepsilon)\|_{1-\frac{k+1}{2n}} \leqslant L\varepsilon(\varepsilon\gamma n^p)^k, \quad k = 1,\dots,n. \tag{2.9}$$

Proof. According to (2.3) we get:

$$\|x_1 - \varepsilon P(x_1,\varepsilon)\|_{1-\frac{1}{n}} = \|\varepsilon P(0,\varepsilon) - \varepsilon P(\varepsilon P(0,\varepsilon),\varepsilon)\|_{1-\frac{1}{n}} \leqslant \varepsilon^2\gamma n^p L.$$

Suppose that inequality (2.9) holds for $k = j, \quad j = 1,\dots,n-1$. Using lemma 2.2 we verify this inequality for $k = j+1$:

$$\begin{aligned}\|x_{j+1} - \varepsilon P(x_{j+1},\varepsilon)\|_{1-\frac{j+2}{2n}} &= \|\varepsilon P(x_j,\varepsilon) - \varepsilon P(\varepsilon P(x_j,\varepsilon),\varepsilon)\|_{1-\frac{j+2}{2n}} \\ &\leqslant \varepsilon\gamma n^p \|x_j - \varepsilon P(x_j,\varepsilon)\|_{1-\frac{j+1}{2n}} \leqslant L\varepsilon(\varepsilon\gamma n^p)^{j+1}.\end{aligned}$$

Lemma is proved.

We substitute $k = n$ in (2.9). According to Lemma 2.1 we estimate the right-hand side of this formula in the following way:

$$L\varepsilon(\varepsilon\gamma n^p)^n \leqslant L\varepsilon\left(\frac{1}{2}\right)^{\alpha_\varepsilon/\varepsilon^{1/p}} \leqslant L\varepsilon\exp\left(-\frac{\log 2}{2(2\gamma)^{1/p}\varepsilon^{1/p}}\right).$$

To conclude the proof we put $\hat{x} = x_n$.

Theorem 1 is proved.

3. Example

As a simplest application of this technique consider the following theorem obtained in [2].

Theorem 2. *Let*

$$\dot{x} = v(x,t,\varepsilon) \tag{3.1}$$

be a dynamical system C^1-smooth and 2π periodic in t and real-analytic in x on a compact real-analytic manifold M.

There exists a C^1-smooth and 2π periodic in t and real-analytic in x change of variables

$$x = y + \varepsilon h(y,t,\varepsilon),$$

such that system (3.1) takes the form

$$\dot{y} = \varepsilon(< v(y,t,0) > + \varepsilon v_1(y,\varepsilon) + \mu(y,t,\varepsilon)), \tag{3.2}$$

where $< f(t) >= \frac{1}{2\pi}\int\limits_0^{2\pi} f(t)dt$, *and*

$$|h(y,t,\varepsilon)| + |v_1(y,\varepsilon)| \leqslant C, \quad |\mu(y,t,\varepsilon)| \leqslant c_1 e^{-c_2/\varepsilon}.$$

Positive constants C, c_1, c_2 *do not depend on* ε.

Regarding heuristically let us try to find change of variables which leads our system to the form:

$$\dot{y} = \varepsilon u(y,\varepsilon).$$

Such a change does not exist generically, but we get an equation:

$$h_t = v(y+\varepsilon h, t, \varepsilon) - \varepsilon h_y u - u.$$

Left-hand side of this equation is of zero mean, hence right hand side has to be of zero mean too. Thus we must put

$$u =< v(y+\varepsilon h, t, \varepsilon) - \varepsilon h_y u > .$$

Both of these equations can be presented as a system:

$$h = \partial_t{}^{-1}[v(y+\varepsilon h, t, \varepsilon) - \varepsilon h_y u], \quad u =< v(y+\varepsilon h, t, \varepsilon) - \varepsilon h_y u >, \quad (3.3)$$

where $[f] = f - < f >$ and by $\partial_t{}^{-1}$ we denoted inversion of the operator ∂_t:

$$\partial_t{}^{-1} \sum_{k\neq 0} f_k e^{ikt} = \sum_{k\neq 0} \frac{f_k}{ik} e^{ikt}.$$

System (3.3) can be reduced to the form (2.4) by some change of the functions h, u.

All equations are considered on the following scale of Banach spaces. Let $M_s = \{x \mid \Re x \in M, \quad |\Im x| < s\}$ be a complex neighborhood of M and $\overline{M_s}$ is its closure. Let E_s be a space of functions $f(x,t)$ that are analytic in M_s and C^1-smooth and 2π-periodic in t. The norm in E_s is

$$\|f\|_s = \max_{0\leqslant t\leqslant 2\pi} \max_{x\in\overline{M_s}} (|f(x,t)| + |f_t(x,t)|).$$

So by theorem 1 theorem 2 is proved.

The author thanks D. Treschev for useful discussions.

References

1. C. Bonnet, D. Sauzin, T. Seara, M. Valencia, *Adiabatic invariant of the harmonic oscillator, complex matching resurgence.* — SIAM J. Math. Anal., **29**, No 6, (1998), 1335–1360.
2. A. I. Neishtadt, *The separation of motions in systems with rapidly rotating phase.* — Prikl. Mat. Mekh. 1984, **48**, No. 2, 197–204; English transl. in J. Appl. Math. Mech. **48**, No. 2 (1984), 133–139.
3. A. Pronin, D. Treschev, *Continuous averaging in multi-frequency slow-fast systems.* — Regular and Chaotic Dynamics, **5**, No. 2 (2000), 157–170.
4. C. Simó, *Averaging under fast quasi-periodic forcing in Hamiltonian mechanics: Integrability and chaotic behavior.* — J. Seimenis ed., NATO Adv. Sci. Inst. Ser. B Phys. 331, Plenum Press, New York (1994), 13–34.
5. *Slitskin A. A. On the motion of one-dimensional oscillator under adiabatic conditions.* — Zhurn. Exper. Teor. Fiz. **45**, No 4 (1963), 978–988.

Moscow State University
E-mail: ozubel@yandex.ru

Поступило 30 ноября 2002 г.