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COMPUTATION OF THE YONEDA ALGEBRAS FOR ALGEBRAS OF DIHEDRAL TYPE

1. Introduction

The algebras of dihedral, semidihedral and quaternion type were defined and classified by Karin Erdmann in [1]. They generalize the blocks with dihedral, semidihedral and generalized quaternion defect groups respectively. The classification contains dozens of infinite families of algebras. Each family is defined by a quiver with relations containing some parameters.

In 1987, David Benson and Jon Carlson [2] proposed some diagrammatic methods for modular representations and cohomology, which were further developed by the first author and used in several papers ([3–9]) to compute the Yoneda algebras of some dihedral and semidihedral algebras. This computation contains two steps: to find the projective resolutions of simple modules, and to determine the Yoneda algebra. For the algebras that appear as principal blocks of group algebras, these results allowed to find the cohomology ring of the corresponding groups.

It turns out that for all considered algebras, the minimal projective resolution of a simple module is periodic or can be represented as totalization of an infinite bicomplex. The bicomplex repeats itself in some regular way, but in general is not periodic. To find the structure of the bicomplex, it is often necessary to determine its first 10–20 diagonals. This computation being rather difficult to do by hand, the object of this work is not only to determine the Yoneda algebras for other families of dihedral algebras, but also to use computer-based techniques to find the projective resolutions.

In this paper, we determine the Yoneda algebras for one infinite family of dihedral algebras: the family $D(3\mathcal{L})$ in the notation of [1]. The projective resolutions for this family were computed by an original `C++` program implemented by the second author. The computations made for other dihedral algebras show that this program can be also efficiently used for most of them.

The algorithm of the program finds the algebra defined by the given quiver with relations (with fixed values of parameters) and computes the minimal projective resolutions of the simple modules over this algebra. For every simple module S_i, the program tries to construct a bicomplex lying in the first quadrant of the plane and consisting of projective modules, such that its totalization gives the minimal projective resolution of S_i. After computing a new diagonal of the bicomplex, the program compares the dimensions of the corresponding image and kernel in the totalization to check the exactness.

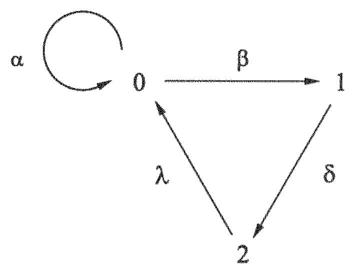
It takes less than one second to compute sufficiently many modules in the bicomplex to see its structure. Running the program for different parameters allows to conjecture the general form of the bicomplex for arbitrary parameters. The conjecture is easy to prove by hand, as the bicomplex contains only finitely many different squares. A more complete presentation of the program will be the object of a separate article.

Let K be a field, Λ be an associative K-algebra with identity, M be a Λ-module (all the considered modules are left modules). The K-module $\mathcal{E}(M) = \bigoplus_{m\geqslant 0} \mathrm{Ext}^m_\Lambda(M,M)$ can be endowed with the structure of an associative K-algebra using the Yoneda product (see [10]). The algebra $\mathcal{E}(M)$ is called *the* Ext-*algebra of* M.

If Λ is a basic finite dimensional K-algebra, we set $\overline{\Lambda} = \Lambda/J(\Lambda)$ where $J(\Lambda)$ is the Jacobson radical of Λ. The Ext-algebra $\mathcal{E}(\overline{\Lambda})$ of $\overline{\Lambda}$ is called *the Yoneda algebra of* Λ and is denoted by $\mathcal{Y}(\Lambda)$.

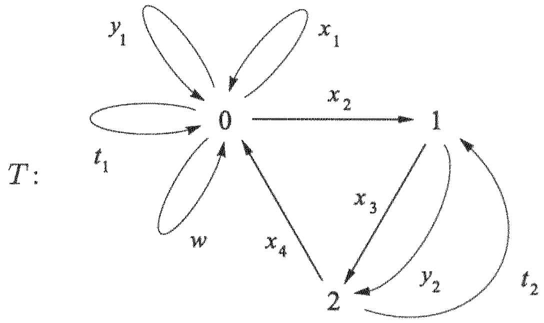
Let k, s be integers $\geqslant 2$. We define the K-algebra $R^{(\mathcal{L})}_{k,s}$ by the following quiver with relations (we write down a composition from the right to the left):

Q:



$$\beta\alpha = 0 = \alpha\lambda, \quad \delta(\beta\lambda\delta)^k = 0, \quad \alpha^s = (\lambda\delta\beta)^k. \tag{1.1}$$

The algebras $R^{(\mathcal{L})}_{k,s}$, k, $s \geqslant 2$, compose an infinite family of dihedral algebras, which is denoted in [1] by $D(3\mathcal{L})$. Every $R^{(\mathcal{L})}_{k,s}$ is a symmetric algebra (and therefore a QF-algebra). To describe the Yoneda algebra $\mathcal{Y}(R_{k,s}^{(\mathcal{L})})$ of $R_{k,s}^{(\mathcal{L})}$, let us consider the quiver T:



Let $K[T]$ be the path algebra of T. We define the following grading on $K[T]$:

$$\deg(x_i) = 1,\ i = 1,2,3,4;\ \deg(y_j) = 2,\ j = 1,2;$$
$$\deg(t_l) = 4,\ l = 1,2;\ \deg(w) = 5.$$

Consider the following relations on the quiver T:

$$\left.\begin{array}{c} x_3x_2 = x_4x_3 = x_2x_4 = x_2y_1 = y_1x_4 = y_1^2 = 0, \\ x_4y_2x_2 = x_2t_1 = t_1x_4 = wx_4 = x_2w = 0, \\ y_1x_1 = x_1y_1, t_1x_1 = x_1t_1, t_1y_1 = y_1t_1, wy_1 = y_1w, t_1w = wt_1, \end{array}\right\} \tag{1.2}$$

$$\left.\begin{array}{c} y_2x_2x_1x_4 = x_3t_2,\ x_2x_1x_4y_2 = t_2x_3,\ x_3t_2x_3 = 0, \\ t_2y_2x_2 = -x_2x_1w,\ x_4y_2t_2 = -wx_1x_4, \end{array}\right\} \tag{1.3}$$

$$x_1^2 = \begin{cases} 0, & \text{if } s > 2, \\ y_1, & \text{if } s = 2, \end{cases} \qquad w^2 = \begin{cases} 0, & \text{if } k > 2, \\ -y_1t_1^2, & \text{if } k = 2. \end{cases} \tag{1.4}$$

Let $\mathcal{E}_{k,s}^{(\mathcal{L})}$ be the K-algebra defined by the quiver T with the relations (1.2)–(1.4). As all these relations are homogeneous, the algebra $\mathcal{E}_{k,s}^{(\mathcal{L})}$ inherits a grading from $K[T]$.

Theorem 1.1. *The Yoneda algebra $\mathcal{Y}(R_{k,s}^{(\mathcal{L})})$ is isomorphic, as a graded algebra, to $\mathcal{E}_{k,s}^{(\mathcal{L})}$.*

The proof of Theorem 1.1 occupies the rest of the paper. We also give a description of the Ext-algebras of simple $R_{k,s}^{(\mathcal{L})}$-modules in Corollary 3.5.

2. Resolutions

To simplify notation, we set $R = R^{(\mathcal{L})}_{k,s}$ and $\mathcal{E} = \mathcal{E}^{(\mathcal{L})}_{k,s}$. For an R-module M, let

$$\ldots \xrightarrow{d_1(M)} Q_1(M) \xrightarrow{d_0(M)} Q_0(M) \xrightarrow{d_{-1}(M)} M \to 0$$

denote the minimal projective resolution of M. We will write $\Omega^n(M)$ for its n-th syzygy $\mathrm{Im}\,(d_{n-1}(M))$, $n \geqslant 0$. We denote by e_i the idempotents of R corresponding to the vertices $i = 0, 1, 2$ of Q. The corresponding indecomposable projective and simple R-modules are denoted $P_i = Re_i$ and $S_i = P_i/(J(R)P_i)$, respectively. The multiplication on the right by an element $x \in e_i Re_j$ induces a homomorphism from P_i into P_j, we denote this homomorphism by the same letter x.

We refer the reader to [2] for the main notions of the diagrammatic method of Benson and Carlson (see also [3, 5]). If M is a uniserial module of length t with the diagram

$$S_{i_t} \longleftarrow \cdots \longleftarrow S_{i_1},$$

we write $M = \mathrm{U}(S_{i_1}, \ldots, S_{i_t})$.

It is easily seen that the indecomposable projective modules $P_i = Re_i$, $i = 0, 1, 2$, have the following diagrams (cf. [1]):

$$
\begin{array}{ccccccccc}
 & & S_0 & & & S_1 & & S_2 \\
 & \swarrow_{\alpha} & & {}_{\beta}\searrow & & |\,\delta & & |\,\lambda \\
 & S_0 & & S_1 & & S_2 & & S_0 \\
 & |\,\alpha & & |\,\delta & & |\,\lambda & & |\,\beta \\
 & S_0 & & S_2 & & S_0 & & S_1 \\
 & |\,\alpha & & |\,\lambda & & |\,\beta & & |\,\delta \\
P_0\colon & S_0 & & S_0 & P_1\colon & S_1 & P_2\colon & S_2 \\
 & |\,\alpha & & |\,\beta & & |\,\delta & & |\,\lambda \\
 & \vdots & & \vdots & & \vdots & & \vdots \\
 & |\,\alpha & & |\,\delta & & |\,\lambda & & |\,\beta \\
 & S_0 & & S_2 & & S_0 & & S_1 \\
 & {}_{\alpha}\searrow & & \swarrow_{\lambda} & & |\,\beta & & |\,\delta \\
 & & S_0 & , & & S_1 & & S_2\,.
\end{array}
\tag{2.1}
$$

Set $b = (\beta\lambda\delta)^{k-1}$, $d = (\delta\beta\lambda)^{k-1}$, $l = (\lambda\delta\beta)^{k-1}$. By abuse of notation, we will use the same letters for the elements of the path algebra $K[Q]$ as well as for their images in R. For abbreviation, we denote a sequence of edges in a diagram by one edge and write the composition of the original edges nearby. For example, the diagrams (2.1) can be written in this notation as

$$\begin{array}{ccc} & S_0 & \\ \diagup{\scriptstyle\alpha} & & {\scriptstyle\beta}\diagdown \\ S_0 & & S_1 \\ {\scriptstyle\alpha^{s-1}}\diagdown & & \diagup{\scriptstyle\lambda\delta b} \\ & S_0 & \end{array}, \qquad \begin{array}{c} S_1 \\ \big|{\scriptstyle\beta\lambda\delta b} \\ S_1 \end{array}, \qquad \begin{array}{c} S_2 \\ \big|{\scriptstyle\delta\beta\lambda d} \\ S_2 \end{array}.$$

Proposition 2.1. a) *The diagrams of* $\Omega^0(S_0)$ *and* $\Omega^1(S_0)$ *are, respectively,*

$$S_0 \quad \text{and} \quad \begin{array}{ccc} S_0 & & S_1 \\ {\scriptstyle\alpha^{s-1}}\diagdown & & \diagup{\scriptstyle\lambda\delta b} \\ & S_0 & \end{array}.$$

b) *Let* $m \geqslant 2$ *be an integer. Suppose* $m \equiv r \pmod 6$ *with* $0 \leqslant r \leqslant 5$. *Let* D *be the diagram of* $\Omega^{m-2}(S_0)$. *Then the diagram of the module* $\Omega^m(S_0)$ *can be obtained from* D *by adjoining or omitting some subdiagrams (depending on* r*) on both sides of* D. *The following table shows the subdiagrams to adjoin (+) and to omit (−) on the left and on the right side of* D:

$r=0$	$\begin{array}{cc} S_2 & \\ {\scriptstyle\lambda}\diagdown & \\ & S_0 \end{array} - D + \begin{array}{ccc} S_2 & & S_0 \\ {\scriptstyle\lambda}\diagdown & & \diagup{\scriptstyle\alpha^{s-1}} \\ & S_0 & \end{array}$
$r=1$	$\begin{array}{ccc} S_0 & & S_2 \\ {\scriptstyle\alpha^{s-1}}\diagdown & & \diagup{\scriptstyle\lambda} \\ & S_0 & \end{array} + D + \begin{array}{ccc} S_0 & & S_1 \\ {\scriptstyle\alpha}\diagdown & & \diagup{\scriptstyle\lambda\delta b} \\ & S_0 & \end{array}$
$r=2$	$\begin{array}{ccc} S_1 & & S_0 \\ {\scriptstyle\lambda\delta b}\diagdown & & \diagup{\scriptstyle\alpha} \\ & S_0 & \end{array} + D + \begin{array}{cc} S_0 & \\ {\scriptstyle\beta}\diagdown & \\ & S_1 \end{array}$
$r=3$	$\begin{array}{cc} & S_0 \\ \diagup{\scriptstyle\beta} & \\ S_1 & \end{array} + D - \begin{array}{cc} & S_1 \\ \diagup{\scriptstyle\delta} & \\ S_2 & \end{array}$
$r=4$	$\begin{array}{cc} S_1 & \\ {\scriptstyle\delta}\diagdown & \\ & S_2 \end{array} - D + \begin{array}{cc} S_1 & \\ {\scriptstyle\delta}\diagdown & \\ & S_2 \end{array}$
$r=5$	$\begin{array}{cc} & S_1 \\ \diagup{\scriptstyle\delta} & \\ S_2 & \end{array} + D - \begin{array}{cc} & S_2 \\ \diagup{\scriptstyle\lambda} & \\ S_0 & \end{array}$

For example, the diagram of $\Omega^3(S_0)$ is obtained from that of $\Omega^1(S_0)$ as follows (we have $r=3$, $\lambda\delta b=\lambda d\delta$):

$$\begin{array}{cccccccccc}
 & S_0 & & S_0 & & S_1 & & & S_1 & \\
\diagup_{\beta} & & + & {}_{\alpha^{s-1}}\diagdown & & \diagup_{\lambda\delta b} & - & \diagup_{\delta} & & = \\
S_1 & & & & S_0 & & & S_2 & &
\end{array}$$

$$\begin{array}{cccccc}
 & & S_0 & & S_2 & \\
= \quad \diagup_{\beta} & {}_{\alpha^{s-1}}\diagdown & & \diagup_{\lambda d} & & . \\
S_1 & & S_0 & & &
\end{array}$$

The proof of Proposition 2.1 is similar to [2] (see also [3, 4]) and is left to the reader. Note that the diagrams of syzygies $\Omega^n(S_0)$ have the property called D-uniqueness (see [2, p. 68]). This fact can be established as in [2, Lemma 11.1].

Consider the following bicomplex $B_{\bullet\bullet}$ lying in the first quadrant of the plane (i.e., its rows and columns are numbered by $0,1,2,\dots$):

$$\begin{array}{ccccccccccccc}
 & & & & \cdots & & \cdots & & \cdots & & \cdots & & \\
 & & & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\lambda\delta\beta} & & \downarrow{\scriptstyle\alpha^{s-1}} & & \downarrow{\scriptstyle l} & & \\
 & & & & P_0 & \xleftarrow{-l} & P_0 & \xleftarrow{-\alpha} & P_0 & \xleftarrow{-\lambda\delta\beta} & P_0 & \xleftarrow{-\alpha} & \cdots \\
 & & & & \downarrow{\scriptstyle\lambda\delta\beta} & & \downarrow{\scriptstyle\alpha^{s-1}} & & \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\alpha^{s-1}} & & \\
 & & P_2 & \xleftarrow{\lambda d} & P_0 & \xleftarrow{\alpha} & P_0 & \xleftarrow{\lambda\delta\beta} & P_0 & \xleftarrow{\alpha} & P_0 & \xleftarrow{\beta} & \cdots \\
 & & \downarrow{\scriptstyle\delta\beta} & & \downarrow{\scriptstyle\alpha^{s-1}} & & \downarrow{\scriptstyle\lambda d} & & & & & & \\
P_1 & \xleftarrow{-\lambda\delta b} & P_0 & \xleftarrow{-\alpha} & P_0 & \xleftarrow{-\delta\beta} & P_2 & & & & & & \\
{\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\alpha^{s-1}} & & \downarrow{\scriptstyle\lambda\delta b} & & & & & & & & \\
P_0 & \xleftarrow{\alpha} & P_0 & \xleftarrow{\beta} & P_1\ . & & & & & & & &
\end{array} \tag{2.2}$$

The nonzero horizontal differentials $\Delta^{(h)}_{ij}\colon B_{ij}\to B_{i-1,j}$ of the bicomplex are defined by the following:

$$\Delta^{(h)}_{ij}=\begin{cases}
(-1)^j\alpha, & \text{if } i-j\equiv 1 \pmod 2,\ j<i\leqslant 2j+1,\\
(-1)^j\lambda\delta\beta, & \text{if } i-j\equiv 0 \pmod 2,\ j+1<i\leqslant 2j,\\
(-1)^j\alpha^{s-1}, & \text{if } i-j\equiv 1 \pmod 2,\ i<j<2i-1,\\
(-1)^j l, & \text{if } i-j\equiv 0 \pmod 2,\ i\leqslant j<2i-2,\\
\beta, & \text{if } j\equiv 0 \pmod 2,\ i=2j+2,\\
-\delta\beta, & \text{if } j\equiv 1 \pmod 2,\ i=2j+1,\\
-\lambda\delta\cdot b, & \text{if } \ j=2i-1,\\
\lambda\cdot d, & \text{if } \ j=2i-2.
\end{cases}$$

The vertical differentials are determined by anti-commutativity of squares and the relations (1.1). For example, the nonzero part of the the $4i$-th and $(4i+1)$-th rows $(i > 0)$ is contained between the columns $2i$ and $8i+3$ and has the form:

$$\begin{tikzcd}
B_{2i,4i+1} \arrow[d, equal] & & & & \\
P_1 \arrow[d, "\beta"'] & P_0 \arrow[l, "-\lambda\delta b"'] \arrow[d, "\alpha"'] & P_0 \arrow[l, "-\alpha^{s-1}"'] \arrow[d, "\lambda\delta\beta"'] & P_0 \arrow[l, "-l"'] \arrow[d, "\alpha"'] & \cdots \arrow[l, "-\alpha^{s-1}"'] \\
P_0 & P_0 \arrow[l, "\alpha^{s-1}"'] & P_0 \arrow[l, "l"'] & P_0 \arrow[l, "\alpha^{s-1}"'] & \cdots \arrow[l, "l"']
\end{tikzcd}$$

$$\begin{tikzcd}
 & B_{4i,4i+1} \arrow[d, equal] & & & \\
\cdots & P_0 \arrow[l, "-\alpha^{s-1}"'] \arrow[d, "\lambda\delta\beta"'] & P_0 \arrow[l, "-l"'] \arrow[d, "\alpha^{s-1}"'] & P_0 \arrow[l, "-\alpha"'] \arrow[d, "l"'] & \cdots \arrow[l, "-\lambda\delta\beta"'] \\
\cdots & P_0 \arrow[l, "l"'] & P_0 \arrow[l, "\alpha"'] & P_0 \arrow[l, "\lambda\delta\beta"'] & \cdots \arrow[l, "\alpha"']
\end{tikzcd}$$

$$\begin{tikzcd}
 & & & & B_{8i+3,4i+1} \arrow[d, equal] \\
\cdots & P_0 \arrow[l, "-\alpha"'] \arrow[d, "l"'] & P_0 \arrow[l, "-\lambda\delta\beta"'] \arrow[d, "\alpha^{s-1}"'] & P_0 \arrow[l, "-\alpha"'] \arrow[d, "\lambda\delta b"'] & P_2 \arrow[l, "-\delta\beta"'] \\
\cdots & P_0 \arrow[l, "\lambda\delta\beta"'] & P_0 \arrow[l, "\alpha"'] & P_1 . \arrow[l, "\beta"'] &
\end{tikzcd}$$

The proof of Proposition 2.2 is similar to [4, Theorem 2] (see also [2, 3]) and is left to the reader. A straightforward verification shows the exactness of the periodic sequences of Proposition 2.3.

Proposition 2.2. *The minimal projective resolution of the R-module S_0 coincides with the totalization of the bicomplex* (2.2).

Proposition 2.3. *The minimal projective resolutions of S_1 and S_2 are the following:*

$$\ldots \xrightarrow{\delta} P_1 \xrightarrow{(\beta\lambda\delta)^k} P_1 \xrightarrow{\beta} P_0 \xrightarrow{\alpha} P_0 \xrightarrow{\lambda} P_2 \xrightarrow{(\delta\beta\lambda)^k} P_2 \xrightarrow{\delta} P_1 \longrightarrow S_1 \longrightarrow 0,$$

$$\ldots \xrightarrow{\lambda} P_2 \xrightarrow{(\delta\beta\lambda)^k} P_2 \xrightarrow{\delta} P_1 \xrightarrow{(\beta\lambda\delta)^k} P_1 \xrightarrow{\beta} P_0 \xrightarrow{\alpha} P_0 \xrightarrow{\lambda} P_2 \longrightarrow S_2 \longrightarrow 0.$$

Corollary 2.4.

a) $$\dim_K \operatorname{Ext}_R^m(S_0,S_0)=\begin{cases} 2\left\lfloor \dfrac{m}{6}\right\rfloor+2, & \textit{if } m\equiv 5 \pmod 6,\\[2ex] 2\left\lfloor \dfrac{m}{6}\right\rfloor+1, & \textit{otherwise;}\end{cases}$$

b) $\dim_k \operatorname{Ext}_R^m(S_0,S_1)=\dim_k \operatorname{Ext}_R^m(S_2,S_0)=\dim_k \operatorname{Ext}_R^m(S_1,S_2)$

$$=\begin{cases} 1, & \textit{if } m\equiv 1 \textit{ or } 2 \pmod 6,\\ 0, & \textit{otherwise;}\end{cases}$$

c) $\dim_k \operatorname{Ext}_R^m(S_0,S_2)=\dim_k \operatorname{Ext}_R^m(S_1,S_0)=\dim_k \operatorname{Ext}_R^m(S_2,S_1)$

$$=\begin{cases} 1, & \textit{if } m\equiv 3 \textit{ or } 4 \pmod 6,\\ 0, & \textit{otherwise;}\end{cases}$$

d) $\dim_k \operatorname{Ext}_R^m(S_1,S_1)=\dim_k \operatorname{Ext}_R^m(S_2,S_2)$

$$=\begin{cases} 1, & \textit{if } m\equiv 0 \textit{ or } 5 \pmod 6,\\ 0, & \textit{otherwise.}\end{cases}$$

Remark 2.5. By Proposition 2.2, we have $Q_m(S_0)=\bigoplus_{i+j=m} B_{ij}$ where $B_{\bullet\bullet}$ is the bicomplex (2.2). The modules in this direct sum will be always ordered with respect to the second index, for example, we write $Q_4(S_0)=B_{31}\oplus B_{22}=P_2\oplus P_0$. The simple direct summands of $\operatorname{top}\Omega^m(S_k)\simeq \operatorname{top} Q_m(S_k)$ will be ordered in the same way: $\operatorname{top} Q_4(S_0)=S_2\oplus S_0$ (where $\operatorname{top} M$ stands for $M/\operatorname{Rad}(M)$). We call such decompositions of $Q_m(S_k)$ and $\operatorname{top} Q_m(S_k)$ the *canonical* decompositions. The differentials $d_m(S_k)$ in the minimal projective resolution of S_k will be denoted in the sequel by $d_m^{(k)}$, $m\geqslant -1$.

3. Generators

In this section, we indicate a finite set of generators for the Yoneda algebra of R:

$$\mathcal{Y}(R)=\mathcal{E}(R/J(R))=\bigoplus_{m\geqslant 0}\bigoplus_{i,j=0}^{2}\operatorname{Ext}_R^m(S_i,S_j).$$

Let us recall some facts and notation related to the Yoneda algebra (see also [3, 5]). As the R-module S_j is simple, we have $\operatorname{Ext}_R^m(S_i,S_j)\simeq$

$\mathrm{Hom}_R(\Omega^m(S_i), S_j)$. Let ψ be an element of $\mathrm{Ext}_R^m(S_i, S_j)$. Its image $\widehat{\psi}$ in $\mathrm{Hom}_R(\Omega^m(S_i), S_j)$ induces a morphism of projective resolutions f_l : $Q_{m+l-1}(S_i) \to Q_{l-1}(S_j)$, $l \geqslant 1$, and a homomorphism $f_0 : Q_{m-1}(S_i) \to P_j$. We have a commutative diagram:

$$\begin{array}{ccccc} Q_m(S_i) & \longrightarrow & \Omega^m(S_i) & \subset & Q_{m-1}(S_i) \\ \Big\downarrow{\scriptstyle f_1} & & \Big\downarrow{\scriptstyle \widehat{\psi}} & & \Big\downarrow{\scriptstyle f_0} \\ Q_0(S_j) & \xrightarrow{d_{-1}^{(j)}} & S_j & \subset & P_j \end{array} . \tag{3.1}$$

We see that $\widehat{\psi}$ can be represented by the commutative square:

$$\begin{array}{ccc} Q_m(S_i) & \xrightarrow{d_{m-1}^{(i)}} & Q_{m-1}(S_i) \\ \Big\downarrow{\scriptstyle f_1} & & \Big\downarrow{\scriptstyle f_0} \\ Q_0(S_j) & \longrightarrow & P_j \end{array}$$

because this commutative square uniquely defines the map $\widehat{\psi}$ in (3.1). Moreover, $\widehat{\psi}$ is uniquely defined by providing only a homomorphism f_1 : $Q_m(S_i) \to Q_0(S_j)$ such that $d_{-1}^{(j)} f_1$ annihilates $\mathrm{Ker}\, d_{m-1}^{(i)}$. In this case we write $\widehat{\psi} = \mathrm{sq}\Big(Q_m(S_i) \xrightarrow{f_1} Q_0(S_j)\Big)$. The homomorphisms

$$\Omega^l(\widehat{\psi}) : \Omega^{m+l}(S_i) \to \Omega^l(S_j), \quad \Omega^l(\widehat{\psi}) = f_l|_{\Omega^{m+l}(S_i)},$$

are called *the Ω-translates of* $\widehat{\psi}$. If $\varphi \in \mathrm{Ext}_R^n(S_j, S_e) \simeq \mathrm{Hom}_R(\Omega^n(S_j), S_e)$, the Yoneda product $\varphi\psi \in \mathrm{Ext}_R^{m+n}(S_i, S_e)$ has the image $\widehat{\varphi\psi} = \widehat{\varphi} \cdot \Omega^n(\widehat{\psi})$ in $\mathrm{Hom}_R(\Omega^{m+n}(S_i), S_e)$. Although the maps f_l and the Ω-translates are not uniquely defined by $\widehat{\psi}$, it is easily seen that the resulting map into a simple module does not depend on their choice.

Consider the homogeneous elements of $\mathcal{Y}(R)$:

$$x_1 \in \mathrm{Ext}_R^1(S_0, S_0), \quad x_2 \in \mathrm{Ext}_R^1(S_0, S_1),$$

$$x_3 \in \mathrm{Ext}_R^1(S_1, S_2), \quad x_4 \in \mathrm{Ext}_R^1(S_2, S_0),$$

$$y_1 \in \mathrm{Ext}_R^2(S_0, S_0), \quad y_2 \in \mathrm{Ext}_R^2(S_1, S_2),$$

$$t_1 \in \mathrm{Ext}_R^4(S_0, S_0), \quad t_2 \in \mathrm{Ext}_R^4(S_2, S_1),$$

$$w \in \mathrm{Ext}_R^5(S_0, S_0)$$

defined as follows:

$$\widehat{x}_1 = \mathrm{sq}(Q_1(S_0) \xrightarrow{(1,0)} P_0), \quad \widehat{x}_2 = \mathrm{sq}(Q_1(S_0) \xrightarrow{(0,1)} P_1),$$

$$\widehat{x}_3 = \mathrm{sq}(Q_1(S_1) \xrightarrow{1} P_2), \quad \widehat{x}_4 = \mathrm{sq}(Q_1(S_2) \xrightarrow{1} P_0),$$

$$\widehat{y}_1 = \mathrm{sq}(Q_2(S_0) \xrightarrow{(0,1)} P_0), \quad \widehat{y}_2 = \mathrm{sq}(Q_2(S_1) \xrightarrow{1} P_2),$$

$$\widehat{t}_1 = \mathrm{sq}(Q_4(S_0) \xrightarrow{(0,1)} P_0), \quad \widehat{t}_2 = \mathrm{sq}(Q_4(S_2) \xrightarrow{1} P_1),$$

$$\widehat{w} = \mathrm{sq}(Q_5(S_0) \xrightarrow{(0,1)} P_0).$$

Proposition 3.1. *The extension groups presented below have the following K-bases:*

$$\begin{array}{ll}
\mathrm{Ext}^2_R(S_0,S_1) = \langle x_2x_1\rangle, & \mathrm{Ext}^2_R(S_2,S_0) = \langle x_1x_4\rangle,\\
\mathrm{Ext}^2_R(S_1,S_2) = \langle y_2\rangle, & \mathrm{Ext}^3_R(S_0,S_0) = \langle x_1y_1\rangle,\\
\mathrm{Ext}^3_R(S_0,S_2) = \langle y_2x_2\rangle, & \mathrm{Ext}^3_R(S_2,S_1) = \langle x_2x_1x_4\rangle,\\
\mathrm{Ext}^4_R(S_0,S_2) = \langle y_2x_2x_1\rangle, & \mathrm{Ext}^4_R(S_0,S_0) = \langle t_1\rangle,\\
\mathrm{Ext}^5_R(S_0,S_0) = \langle x_1t_1, w\rangle, & \mathrm{Ext}^5_R(S_1,S_1) = \langle t_2x_3\rangle,
\end{array}$$

$$\begin{array}{l}
\mathrm{Ext}^2_R(S_0,S_0) = \langle y_1\rangle,\\
\mathrm{Ext}^3_R(S_1,S_0) = \langle x_4y_2\rangle,\\
\mathrm{Ext}^4_R(S_1,S_0) = \langle x_1x_4y_2\rangle,\\
\mathrm{Ext}^4_R(S_2,S_1) = \langle t_2\rangle,\\
\mathrm{Ext}^5_R(S_2,S_2) = \langle x_3t_2\rangle.
\end{array}$$

Proof. We consider only $\mathrm{Ext}^5_R(S_0,S_0)$. For the other groups, $\dim_K \mathrm{Ext}^m_R(S_i,S_j) = 1$ by Corollary 2.4, therefore it is sufficient to verify that the given elements are nonzero in the corresponding Ext-groups, which is done in the same manner.

We have $\widehat{x_1 t_1} = \widehat{x}_1 \cdot \Omega^1(\widehat{t}_1)$. The Ω-translate $\Omega^1(\widehat{t}_1)$ is defined by the left square of the commutative diagram

$$\begin{tikzcd}[column sep=large]
 & \Omega^1(\widehat{t}_1): & & \widehat{t}_1: & \\
Q_5(S_0) \arrow[rr, "d_4^{(0)}"] \arrow[d, "\begin{pmatrix} 1 & 0 \\ 0 & \lambda\delta \end{pmatrix}"'] & & Q_4(S_0) \arrow[rr, "d_3^{(0)}"] \arrow[d, "{(1,0)}"'] & & Q_3(S_0) \arrow[d, "{(\alpha,0)}"] \\
Q_1(S_0) \arrow[rr, "d_0^{(0)}"'] & & Q_0(S_0) \arrow[rr, "(\lambda\delta\beta)^k"'] & & P_0 .
\end{tikzcd}$$

The map $\widehat{x}_1 \cdot \Omega^1(\widehat{t}_1)$ is described by the outer square of the following commutative diagram:

$$\begin{tikzcd}
 & Q_5(S_0) \arrow[r, "d_4^{(0)}"] \arrow[d, "\begin{pmatrix} 1 & 0 \\ 0 & \lambda\delta \end{pmatrix}"'] & Q_4(S_0) \arrow[d, "{(0,1)}"] \\
\Omega^1(\widehat{t}_1): & Q_1(S_0) \arrow[r, "d_0^{(0)}"] \arrow[d, "{(1,0)}"'] & Q_0(S_0) \arrow[d, "\alpha^{s-1}"] \\
\widehat{x}_1: & Q_0(S_2) \arrow[r, "(\lambda\delta\beta)^k"] & P_0 .
\end{tikzcd}$$

Comparing the composition $\widehat{x}_1 \cdot \Omega^1(\widehat{t}_1) = \mathrm{sq}(Q_5(S_0) \xrightarrow{(1,0)} P_0)$ and $\widehat{w} = \mathrm{sq}(Q_5(S_0) \xrightarrow{(0,1)} P_0)$, we see that $\widehat{x_1 t_1}$ and $\widehat{w}$ are linearly independent. It remains to note that $\dim_K \mathrm{Ext}_R^2(S_0, S_0) = 2$ by Corollary 2.4. $\square$

Proposition 3.2. *The set*

$$\mathcal{X}_{\mathcal{L}} = \{x_1, x_2, x_3, x_4, y_1, y_2, t_1, t_2, w\} \tag{3.2}$$

generates the Yoneda algebra $\mathcal{Y}(R)$ as a K-algebra.

Proof. We prove by induction on m that the groups $\mathrm{Ext}_R^m(S_i, S_j)$ are generated by some products of elements of $\mathcal{X}_{\mathcal{L}}$. For $m \leqslant 5$, this follows directly from Proposition 3.1, Corollary 2.4 and the definition of the elements of $\mathcal{X}_{\mathcal{L}}$. Assume that $m \geqslant 6$ and that our statement holds for all $\mathrm{Ext}_R^{m'}(S_i, S_j)$ with $m' < m$, we will prove it for m.

If $j \in \{1, 2\}$, we notice that the multiplication on the left by y_2 and by t_2 induces the isomorphisms

$$\mathrm{Ext}_R^{m-2}(S_i, S_1) \xrightarrow{\simeq} \mathrm{Ext}_R^m(S_i, S_2),$$

$$\mathrm{Ext}_R^{m-4}(S_i, S_2) \xrightarrow{\simeq} \mathrm{Ext}_R^m(S_i, S_1),$$

respectively, and our statement for $\mathrm{Ext}_R^m(S_i, S_j)$ follows from the induction hypothesis. Similar arguments apply to the case $i \in \{1, 2\}$. It remains to prove our statement for $i = j = 0$.

Using the isomorphism $\mathrm{Ext}_R^m(S_0, S_0) \simeq \mathrm{Hom}_R(\Omega^m(S_0), S_0)$, we represent an element of the group $\mathrm{Ext}_R^m(S_0, S_0)$ by the corresponding map $f: \Omega^m(S_0) \to S_0$. Without loss of generality we can assume that f induces a nonzero map on at most one simple direct summand in the canonical decomposition of $\operatorname{top} \Omega^m(S_0)$ (see Remark 2.5).

Case 1: $m \equiv 1 \pmod 6$. a) Assume that $f: \Omega^m(S_0) \to S_0$ induces zero maps on the extreme (the left and the right) simple direct summands of the module $\operatorname{top} \Omega^m(S_0)$. It follows from Proposition 2.1 that the diagram of $\Omega^{m-2}(S_0)$ is a closed subdiagram in that of $\Omega^m(S_0)$, hence $\Omega^{m-2}(S_0)$ is a quotient of $\Omega^m(S_0)$. Let $\rho : \Omega^m(S_0) \to \Omega^{m-2}(S_0)$ be the canonical epimorphism. Then there exists a homomorphism $f' : \Omega^{m-2}(S_0) \to S_0$ such that $f = f'\rho$. Since $\rho = \Omega^{m-2}(\tilde{\rho})$ for some homomorphism $\tilde{\rho} : \Omega^2(S_0) \to S_0$, the desired statement follows from $f = f' \cdot \Omega^{m-2}(\tilde{\rho})$ and the induction hypothesis.

b) Assume now that f induces a nonzero map on an extreme direct summand of $\operatorname{top} \Omega^m(S_0)$. As the right one is S_1, f can induce a nonzero map only on the left one. By Proposition 2.1, the diagram of $\Omega^m(S_0)$ contains on the left side the closed subdiagram

$$\begin{array}{ccccc} S_0 & & & & S_0 \\ & {\scriptstyle \alpha^{s-1}}\diagdown & & \diagup{\scriptstyle \lambda\delta\beta} & \\ & & S_0 & & \end{array} \tag{3.3}$$

(the edges δ and β being added while constructing $\Omega^{m-2}(S_0)$ from $\Omega^{m-4}(S_0)$ and $\Omega^{m-4}(S_0)$ from $\Omega^{m-6}(S_0)$). Let M be the quotient module of $\Omega^m(S_0)$ having the diagram (3.3), and let $\pi: \Omega^m(S_0) \to M$ denote the corresponding epimorphism. As the diagram of $\Omega^1(S_0)$ also contains (3.3) on the left, we have a monomorphism $i: M \to \Omega^1(S_0)$ and can identify M with its image $i(M) \subset \Omega^1(S_0)$.

The map f factors through π, so $f = f'\pi$ for some homomorphism $f' : M \to S_0$. We claim that there exists a homomorphism $f'': \Omega^1(S_0) \to S_0$ such that $f''i = f'$. Since

$$\operatorname{Soc} M \simeq S_0 \simeq \operatorname{Soc} \Omega^1(S_0),$$

factoring out the socles gives the embedding $M/S_0 \hookrightarrow \Omega^1(S_0)/S_0$. Moreover, we have $\Omega^1(S_0)/S_0 = X \oplus Y$ with

$$X = \mathrm{U}(S_0, S_0, \ldots, S_0) \quad \text{and} \quad Y = \mathrm{U}(S_1, S_2, S_0, \ldots, S_2),$$

and $M/S_0 = X \oplus Y_1$ with $Y_1 = \mathrm{U}(S_0, S_1, S_2) \subset Y$. Since $\mathrm{Soc}\, M = S_0$ is contained in $\mathrm{Ker} f'$, f' factors through M/S_0, and the restriction of the induced map $g: M/S_0 \to S_0$ onto Y_1 is zero. Hence g can be extended (by zero on Y) to a homomorphism $g': \Omega^1(S_0)/S_0 \to S_0$. Let f'' be the composition of g' and the canonical epimorphism $\Omega^1(S_0) \to \Omega^1(S_0)/S_0$. It is easy to see that $f' = f''i$, hence $f = f''i\pi$. The map $\rho = i\pi$: $\Omega^m(S_0) \to \Omega^1(S_0)$ has the form $\rho = \Omega^1(\tilde{\rho})$ for some $\tilde{\rho}$: $\Omega^{m-1}(S_0) \to S_0$. Our statement follows now from $f = f'' \cdot \Omega^1(\tilde{\rho})$ and the induction hypothesis.

Case 2: $m \equiv 2 \pmod 6$. In this case $\Omega^{m-2}(S_0)$ is again a quotient of $\Omega^m(S_0)$. As the extreme left simple direct summand of $\mathrm{top}\, \Omega^m(S_0)$ is isomorphic to S_1, $f: \Omega^m(S_0) \to S_0$ induces a zero map on it, and we can proceed analogously to the case 1a).

Case 3: $m \equiv 3 \pmod 6$. Let D be the diagram of $\Omega^{m-4}(S_0)$. By Proposition 2.1, the diagram of $\Omega^m(S_0)$ contains D as a closed subdiagram and can be obtained as follows:

$$\begin{array}{ccccccccccc} & S_0 & & & S_2 & & & & S_0 & & S_2 \\ \diagup_{\beta} & & {}^{\alpha^{s-1}}\diagdown & & \diagup_{\lambda} & + & D & + & {}^{\alpha}\diagdown & & \diagup_{\lambda d} \\ S_1 & & & S_0 & & & & & & S_0 & \end{array} .$$

As the extreme right simple direct summand of $\mathrm{top}\, \Omega^m(S_0)$ is isomorphic to S_2, f induces a zero map on it. If f induces a zero map on the extreme left one, we can proceed again as in the case 1a).

Assume now that f induces a nonzero map on the extreme left direct summand of $\mathrm{top}\, \Omega^m(S_0)$. Let M be a module having the diagram

$$\begin{array}{ccc} & S_0 & \\ \diagup_{\beta} & & {}^{\alpha}\diagdown \\ S_1 & & S_0 . \end{array}$$

We can consider M as a quotient of $\Omega^m(S_0)$ and a submodule of $\Omega^2(S_0)$. As in the proof of the case 1b), we find a factorization $f = \tilde{f} \cdot \rho$, where $\rho: \Omega^m(S_0) \to \Omega^2(S_0)$ is a map such that $\mathrm{Im}\, \rho = M$. It implies the required statement.

Case 4: $m \equiv 4 \pmod 6$. As in the case 3, we see that $\Omega^{m-4}(S_0)$ is a quotient of $\Omega^m(S_0)$. Since the extreme left simple direct summand of top $\Omega^m(S_0)$ is isomorphic to S_2, f induces a zero map on it. The proof is completed as in the case 1a).

Case 5: $m \equiv 5 \pmod 6$. Let D be the diagram of $\Omega^{m-6}(S_0)$. In this case, D is a closed subdiagram of the diagram of $\Omega^m(S_0)$, which has the following form:

$$\begin{array}{ccccccccccccc} & S_0 & & & S_2 & & & & & S_0 & & & S_0 \\ \diagup_{\delta\beta} & & {\scriptstyle\alpha^{s-1}}\diagdown & & \diagup_{\lambda} & & + \; D \; + & & & {\scriptstyle\alpha}\diagdown & & \diagup_{l} & . \\ S_2 & & & S_0 & & & & & & & S_0 & & \end{array}$$

If the given map f induces zero maps on the extreme simple direct summands of top $\Omega^m(S_0)$, we apply the argument of the case 1a).

If f induces a nonzero map on the extreme right direct summand of the top $\Omega^m(S_0)$, then f goes through the module M defined by the diagram

$$\begin{array}{ccccc} S_0 & & & S_0 & \\ {\scriptstyle\alpha}\diagdown & & \diagup_{l} & & . \\ & S_0 & & & \end{array}$$

As this module can be embedded into $\Omega^1(S_0)$, we can find a factorization $f = \tilde{f}\rho$, where $\rho\colon \Omega^m(S_0) \to \Omega^1(S_0)$ is a map such that $\operatorname{Im}\rho = M$, and our statement follows.

Let us now assume that f induces a nonzero map on the extreme left direct summand of top $\Omega^m(S_0)$. In this case f goes through the module M defined by the diagram

$$\begin{array}{cccc} & S_0 & & \\ \diagup_{\beta} & & {\scriptstyle\alpha}\diagdown & . \\ S_1 & & & S_0 \end{array}$$

The module M can be embedded into $\Omega^2(S_0)$. Hence, we obtain a factorization $f = \tilde{f}\rho$, where $\rho\colon \Omega^m(S_0) \to \Omega^2(S_0)$ is a map such that $\operatorname{Im}\rho = M$, which implies our statement.

Case 6: $m \equiv 0 \pmod 6$. This case is also proved by reduction to the syzygy $\Omega^{m-6}(S_0)$. If D is the diagram of $\Omega^{m-6}(S_0)$, then that of $\Omega^m(S_0)$ is obtained as follows:

$$\begin{array}{ccccccccc} S_0 & & S_0 & & & S_0 & & S_0 & \\ {\scriptstyle l}\diagdown & & \diagup_{\alpha} & + \; D \; + & & {\scriptstyle\lambda\delta\beta}\diagdown & & \diagup_{\alpha^{s-1}} & . \\ & S_0 & & & & & S_0 & & \end{array}$$

If the given map f induces zero maps on the extreme simple direct summands of $\operatorname{top}\Omega^m(S_0)$, we can apply the argument of the case 1a). If f induces a nonzero map on the extreme right direct summand of $\operatorname{top}\Omega^m(S_0)$, we can repeat the arguments of the case 1b).

Finitely, assume that f induces a nonzero map on the extreme left direct summand of $\operatorname{top}\Omega^m(S_0)$. We proceed as in the case 1b) and use the module M defined by the diagram

$$\begin{tikzcd}[column sep=small] S_0 \arrow[dr, dash, "l"'] & & S_0 \arrow[dl, dash, "\alpha"] \\ & S_0 & \end{tikzcd}$$

and considered as a quotient of $\Omega^m(S_0)$ and a submodule of $\Omega^1(S_0)$. $\square$

Proposition 3.3. *The generators* (3.2) *of the algebra* $\mathcal{Y}(R)$ *satisfy the relations* (1.2)–(1.4).

Proof. Let us prove only the last relation in (1.4). The verification of the other relations is similar and is left to the reader.

Since $(w^2)\widehat{\ } = \widehat{w}\cdot\Omega^5(\widehat{w})$, we have to compute the Ω-translate $\Omega^5(\widehat{w})$ of the map

$$\widehat{w}\colon Q_5(S_0) \to Q_0(S_0).$$

Lifting $\widehat{w}$ to a morphism of minimal projective resolutions of $\Omega^5(S_0)$ and S_0 step by step, we obtain that $\Omega^5(\widehat{w})$ can be represented by the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -(\lambda\delta\beta)^{k-2} & 0 \end{pmatrix} : Q_{10}(S_0) \to Q_5(S_0),$$

which is written, as usually, with respect to the canonical decompositions of $Q_{10}(S_0)$ and $Q_5(S_0)$ (see Remark 2.5). Hence the composition $\widehat{w}\cdot\Omega^5(\widehat{w})$ can be represented by

$$\begin{pmatrix} 0 & 0 & -(\lambda\delta\beta)^{k-2} & 0 \end{pmatrix} : Q_{10}(S_0) \to Q_0(S_0).$$

If $k > 2$, this map obviously induces a zero map $\Omega^{10}(S_0) \to S_0$, which implies $w^2 = 0$.

On the other hand,

$$(y_1 t_1^2)\widehat{\ } = \widehat{y}_1 \cdot \Omega^2(\widehat{t}_1)\cdot\Omega^6(\widehat{t}_1).$$

Computing the Ω-translates of the map $\widehat{t}_1$, we obtain

$$\Omega^2(\widehat{t}_1) = \begin{pmatrix} \lambda\delta & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Omega^6(\widehat{t}_1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and hence, $(y_1t_1^2)^\frown = (\,0 \quad 0 \quad 1 \quad 0\,)$. We conclude for $k = 2$ that $y_1t_1^2 = -w^2$. $\square$

Let $\mathcal{E} = \mathcal{E}_{k,s}^{(\mathcal{L})} = K[T]/I$ be the graded K-algebra defined in Section 1, where $I \subset K[T]$ is the ideal defined by the relations (1.2)–(1.4). By Propositions 3.2 and 3.3, there exists an epimorphism of graded K-algebras $\mathcal{E} \longrightarrow \mathcal{Y}(R)$ which takes the canonical generators of the algebra $\mathcal{E}$ (represented by the edges of T) to the corresponding elements of $\mathcal{X}_{\mathcal{L}} \subset \mathcal{Y}(R)$.

Let $\mathcal{E} = \bigoplus_{m\geqslant 0} \mathcal{E}^m$ be the decomposition of $\mathcal{E}$ into homogeneous direct summands. Let ε_i denote the idempotents of $K[T]$ corresponding to the vertices $i = 0, 1, 2$ of T; we use the same notation for their images in $\mathcal{E}$. Theorem 1.1 now follows from the following result.

Proposition 3.4. *For any $i, j \in \{0, 1, 2\}$ and $m \geqslant 0$, we have*

$$\dim_K(\varepsilon_i\mathcal{E}^m\varepsilon_j) = \dim_K \operatorname{Ext}_R^m(S_j, S_i). \tag{3.4}$$

Proof. For $m \leqslant 5$, the relations (3.4) are verified directly. Let us assume that $m > 5$. We suppose additionally that $k > 2$ and $s > 2$. The cases where $k = 2$ or $s = 2$ are proved in a similar way, and we leave their proof to the reader.

a) First we consider the case $i = j = 0$. It follows from the relations (1.2)–(1.4) that the K-algebra $\varepsilon_1\mathcal{E}\varepsilon_1$ is generated by elements x_1, y_1, t_1, w, and any nonzero monomial in $\varepsilon_1\mathcal{E}^m\varepsilon_1$ (i.e., the image of a path in $K[T]$) is equal to one of the following:

$$y_1^\eta t_1^r(x_1w)^t, \quad y_1^\eta t_1^r(wx_1)^t, \quad y_1^\eta t_1^r x_1(wx_1)^t, \quad y_1^\eta t_1^r w(x_1w)^t, \tag{3.5}$$

with $\eta \in \{0, 1\}$, $r, t \geqslant 0$.

Put $d_m = \dim_K \varepsilon_0\mathcal{E}^m\varepsilon_0$. We claim that $d_m - d_{m-4}$ is equal to the number of monomials in (3.5) for which $r = 0$. Indeed, the monomials of degree m in (3.5) form a K-basis of $\varepsilon_0\mathcal{E}^m\varepsilon_0$. Replacing r by $r + 1$ in the elements of the similar basis of $\varepsilon_0\mathcal{E}^{m-4}\varepsilon_0$ gives those basis elements of $\varepsilon_0\mathcal{E}^m\varepsilon_0$ for which $r > 0$. It shows that the basis elements of $\varepsilon_0\mathcal{E}^m\varepsilon_0$ for which $r > 0$, are in one-to-one correspondence with the basis elements of $\varepsilon_0\mathcal{E}^{m-4}\varepsilon_0$, which implies our claim. If $r = 0$, then we have for the monomials in (3.5)

$$m = 2\eta + 6t, \quad \text{or} \quad m = 2\eta + 6t + 1, \quad \text{or} \quad m = 2\eta + 6t + 5,$$

whence we obtain that

$$d_m - d_{m-4} = \begin{cases} 1, & \text{if } m \equiv 3 \text{ or } 5 \pmod 6, \\ 0, & \text{if } m \equiv 4 \pmod 6, \\ 2, & \text{otherwise.} \end{cases}$$

Corollary 2.4 a) implies that the sequence $\{\dim_K \operatorname{Ext}_R^m(S_0, S_0)\}$ also satisfies a similar recursive relation. The assertion (3.4) can be now established by induction on m.

b) Assume now that $i = 1, j = 0$. It is clear that any nonzero monomial in $\varepsilon_1 \mathcal{E}^m \varepsilon_0$ is equal to $x_2 \cdot u$ for some monomial $u \in \varepsilon_0 \mathcal{E}^{m-1} \varepsilon_0$. As x_2 annihilates y_1, t_1 and w, the monomial u have to be of the form

$$u = (x_1 w)^r \quad \text{or} \quad u = x_1 (w x_1)^r, \tag{3.6}$$

where $r \in \mathbb{N}$. Consequently, $m - 1 = 1 + 6r$ or $m - 1 = 6r$, hence

$$\dim_K \varepsilon_1 \mathcal{E}^m \varepsilon_0 = \begin{cases} 1, & \text{if } m \equiv 1 \text{ or } 2 \pmod 6, \\ 0, & \text{otherwise.} \end{cases}$$

The relation (3.4) is now obtained from Corollary 2.4 b).

c) Let $i = 2$, $j = 0$. In this case any nonzero monomial in $\varepsilon_2 \mathcal{E}^m \varepsilon_0$ is equal to $y_2 x_2 \cdot u$ for some monomial $u \in \varepsilon_2 \mathcal{E}^{m-3} \varepsilon_0$. As in the case b), u has the form (3.6), and we see that $m - 3 \equiv 0$ or $1 \pmod 6$. Corollary 2.4 c) gives the relation (3.4).

d) Suppose that $i = j = 1$. A nonzero monomial f in $\varepsilon_1 \mathcal{E} \varepsilon_1$ is either $f = x_2 \cdot f'$ or $f = t_2 \cdot f'$ for some monomial f'. In the first case, f has to be of the form $f = x_2 \cdot u \cdot x_4 \cdot v$, where $u \in \varepsilon_0 \mathcal{E} \varepsilon_0$, $v \in \varepsilon_2 \mathcal{E}^m \varepsilon_1$. Moreover, it follows from (1.2) that $u = x_2 (w x_1)^r$, $r \in \mathbb{N}$, and $v = y_2 \cdot v'$, $v' \in \varepsilon_1 \mathcal{E} \varepsilon_1$. Hence, we have $f = x_2 x_1 (w x_1)^r x_4 y_2 \cdot v'$. The relations (1.3) imply that

$$(w x_1)^r x_4 = (-1)^r x_4 (y_2 t_2)^r, \; (x_2 x_1 x_4)(y_2 t_2)^r = (t_2 y_2)^r (x_2 x_1 x_4).$$

Consequently, in any case we can represent the given monomial f in the form $f = t_2 x_3 \cdot g$ or $f = t_2 y_2 \cdot g$ for some $g \in \varepsilon_1 \mathcal{E} \varepsilon_1$. From this, it follows that $\varepsilon_1 \mathcal{E} \varepsilon_1$ is generated as a K-algebra by the elements $\xi = t_2 x_3$ and $\zeta = t_2 y_2$ which satisfy relations $\zeta^2 = 0$, $\zeta \xi = \xi \zeta$. It is easily seen that

$$\dim_K \varepsilon_1 \mathcal{E}^m \varepsilon_1 = \begin{cases} 1, & \text{if } m \equiv 0 \text{ or } 5 \pmod 6, \\ 0, & \text{otherwise.} \end{cases}$$

Using Corollary 2.4 b), we deduce (3.4).

e) Let $i = 2, j = 1$. We begin by proving that any nonzero monomial $f \in \varepsilon_2\mathcal{E}\varepsilon_1$ can be represented in the form $f = y_2 \cdot f'$, where $f' \in \varepsilon_1\mathcal{E}\varepsilon_1$. Indeed, if $f = x_3 \cdot u$ with $u \in \varepsilon_1\mathcal{E}\varepsilon_1$, the relations (1.2) and (1.3) imply that $f = x_3t_2 \cdot u' = y_2x_2x_1x_4 \cdot u'$ for some $u' \in \varepsilon_2\mathcal{E}\varepsilon_1$. Consequently, we have

$$\begin{aligned}
&\dim_K \varepsilon_2\mathcal{E}^m\varepsilon_1 = \dim_K \varepsilon_1\mathcal{E}^{m-2}\varepsilon_1 \\
&= \begin{cases} 1, & \text{if } m-2 \equiv 0 \text{ or } 5 \pmod 6, \\ 0, & \text{otherwise,} \end{cases} = \begin{cases} 1, & \text{if } m \equiv 1 \text{ or } 2 \pmod 6, \\ 0, & \text{otherwise,} \end{cases} \\
&= \dim_K \operatorname{Ext}^m_R(S_1, S_2)
\end{aligned}$$

by Corollary 2.4 b).

f) Assume $i = j = 2$. As in the case d), we observe that K-algebra $\varepsilon_2\mathcal{E}\varepsilon_2$ is generated by the elements $\eta = x_3t_2, \kappa = y_2t_2$ which satisfy the relations $\eta^2 = 0$, $\eta\kappa = \kappa\eta$, and we obtain (3.4) for this case.

g) Assume $i = 2$, $j = 0$. Any nonzero monomial $f \in \varepsilon_0\mathcal{E}\varepsilon_2$ has the form $f = u \cdot x_4 \cdot v$, where $u \in \varepsilon_0\mathcal{E}\varepsilon_0$, $v \in \varepsilon_2\mathcal{E}\varepsilon_2$. Since $y_1x_4 = 0 = t_1x_4$, $x_4x_3 = 0$, we see that elements u, v have the form $u = x_1(wx_1)^r$ or $u = (wx_1)^r$, $r \geqslant 0$, and $v = (y_2t_2)^t$, $t \geqslant 0$ (see the proof of a) and f) above). Using the relation

$$(wx_1)^r x_4 = (-1)^r x_4(y_2t_2)^r,$$

which is a consequence of a relation in (1.3), we see that $f = x_1x_4(y_2t_2)^l$ or $f = x_4(x_2t_2)^l$, $l \in \mathbb{N}$. Hence, we have

$$\begin{aligned}
\dim_K \varepsilon_0\mathcal{E}^m\varepsilon_2 &= \begin{cases} 1, & \text{if } m \equiv 1 \text{ or } 2 \pmod 6, \\ 0, & \text{otherwise,} \end{cases} \\
&= \dim \operatorname{Ext}^m_R(S_2, S_0)
\end{aligned}$$

(see Corollary 2.4 b)).

h) Finally, let $i = 1, j = 2$. Any nonzero monomial $f \in \varepsilon_1\mathcal{E}\varepsilon_2$ has the form $f = u \cdot t_2 \cdot v$ or $f = u \cdot x_2x_1x_4 \cdot v$, where $u \in \varepsilon_1\mathcal{E}\varepsilon_1$, $v \in \varepsilon_2\mathcal{E}\varepsilon_2$. As in the case g), using the relations (1.2) and (1.3), we obtain

$$f = (t_2y_2)^r \cdot t_2 \cdot (x_3t_2)^\sigma \cdot (y_2t_2)^t,$$

where $\sigma \in \{0, 1\}$, $r, t \geqslant 0$. Hence,

$$\deg f = 4 + 5\sigma + 6(r+t) \equiv 3 \quad \text{or} \quad 4 \pmod 6,$$

and the required statement follows from Corollary 2.4 c). □

Consider the graded K-algebra $K\langle x, y, t, w\rangle$ with

$$\deg x = 1, \quad \deg y = 2, \quad \deg t = 4, \quad \deg w = 5,$$

and the subset

$$M = \{xy - yx, xt - tx, yt - ty, yw - wy, tw - wt, y^2\} \subset K\langle x, y, t, w\rangle.$$

Let

$$M_{k,s} = \begin{cases} M \cup \{x^2, w^2\}, & \text{if } k > 2 \text{ and } s > 2, \\ M \cup \{x^2 - y, w^2\}, & \text{if } k > 2 \text{ and } s = 2, \\ M \cup \{x^2, w^2 + yt^2\}, & \text{if } k = 2 \text{ and } s > 2, \\ M \cup \{x^2 - y, w^2 + yt^2\}, & \text{if } k = s = 2. \end{cases}$$

Let $I_{k,s} \subset K\langle x, y, t, w\rangle$ be the ideal generated by the set $M_{k,s}$, and $\mathcal{B}_{k,s}$ be the quotient algebra $K\langle x, y, t, w\rangle / I_{k,s}$. Then $\mathcal{B}_{k,s}$ inherits a grading from $K\langle x, y, t, w\rangle$.

Corollary 3.5. *There are the following isomorphisms of graded K-algebras:*

a) $\mathcal{E}(S_0) \simeq \mathcal{B}_{k,s}$;

b) $\mathcal{E}(S_1) \simeq \mathcal{E}(S_2) \simeq K[\zeta, \xi]/(\zeta^2)$, *where* $\deg \zeta = 5$, $\deg \xi = 6$.

Proof. If $k > 2$ and $s > 2$, then the above statements follow immediately from the proof of Proposition 3.4. If $k = 2$ or $s = 2$, the argument is similar. □

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