Записки научных
семинаров ПОМИ
Том 306, 2003 г.

S. A. Nazarov, M. Specovius-Neugebauer

THE PRESSURE STABILIZATION METHOD FOR STEADY VISCOUS FLOWS IN A SYSTEM OF PIPES

Abstract. This paper deals with a singular perturbation of the stationary Stokes and Navier–Stokes system. Thereby the term $\varepsilon^2\Delta p$ is added to the continuity equation, where ε is small parameter. For a domain with cylindrical outlets to infinity and exponentially decaying data, existence and uniqueness of solutions under flux conditions at infinity are shown for the linear problem, and for the nonlinear problem in the case of small data. Asymptotically precise estimates are proved, as ε tends to zero. For sufficiently regular data, they lead to convergence in $H^{5/2-\delta}_{\rm loc}$ for the velocity parts and in $H^{3/2-\delta}_{\rm loc}$ for the pressure parts, respectively.

**Dedicated to V. A. Solonnikov
on occasion of his 70th birthday**

1. Introduction

We consider the following elliptic boundary value problems which appear in the context of viscous flow problems. In the first one we look for a solution $u^\varepsilon = (v_1^\varepsilon, v_2^\varepsilon, v_3^\varepsilon, p^\varepsilon)$ to

$$\begin{aligned} S_\varepsilon u^\varepsilon &= \begin{pmatrix} -\Delta v^\varepsilon + \nabla p^\varepsilon \\ -\varepsilon^2\Delta p^\varepsilon + \operatorname{div} v^\varepsilon \end{pmatrix} = \begin{pmatrix} f' \\ f_4 \end{pmatrix} \quad \text{in} \quad \Omega, \\ B_\varepsilon u^\varepsilon &= (v^\varepsilon, \partial_n p^\varepsilon) = g \qquad\qquad\qquad \text{on} \quad \partial\Omega, \end{aligned} \tag{S$_\varepsilon$}$$

where $f = (f_1, \dots, f_4) = (f', f_4)$ is a given vector field on a three-dimensional domain Ω, $g = (g_1, \dots, g_4) = (g', g_4)$ is a prescribed field on the boundary $\partial\Omega$, and $\varepsilon \in (0,1]$ is a small parameter. The system (S_ε) is strongly elliptic and can be accounted as a singular perturbation of the Stokes system, which appears if we formally set $\varepsilon = 0$ and cancel the last boundary condition. Thus, our second system, which is elliptic in the general sense of Agmon–Douglis–Nirenberg only, reads as follows:

$$\begin{aligned} S_0 u &= \begin{pmatrix} -\Delta v + \nabla p \\ \operatorname{div} v \end{pmatrix} = \begin{pmatrix} f' \\ f_4 \end{pmatrix} \quad \text{in} \quad \Omega, \\ B_0 u &= v = g' \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{S$_0$}$$

We are interested what happens if $\varepsilon \searrow 0$, and, for $g_4 = 0$, to compare u^ε and u^0. Perturbations of this type are used in the framework of so called quasi-compressibility methods both for the theory of numerical schemes (see, e.g. [11]) and existence proofs for weak solutions for fluid models with shear dependent viscosities (cf. [2]). Usually energy methods are used to obtain a convergence result as $\varepsilon \searrow 0$, which cannot give more than H^1- convergence of the velocity part v^ε and L^2-convergence of the pressure part p^ε. Moreover, up to now no convergence results is known for unbounded domains. These facts motivate an investigation of the problem (S_ε) in function spaces with weighted norms dependent on both the small parameter ε and the "distance" to infinity.

In [9] asymptotically precise estimates with $\varepsilon \searrow 0$ were derived in the case of bounded smoothly surrounded domains, in particular it was shown there, that for sufficiently smooth data $\|v^\varepsilon - v^0; H^{5/2-\delta}(\Omega)\| + \|p^\varepsilon - p^0; H^{3/2-\delta}(\Omega)\| \to 0$ as $\varepsilon \searrow 0$, for arbitrarily small $\delta > 0$. Here we consider domains with cylindrical outlets to infinity, where even the boundary is noncompact. For simplicity we treat the case of a pipe here, this means a domain with two cylindrical outlets to infinity. In [8], e.g., for exponentially decaying data, existence and uniqueness of solutions to the problem (S_0) in spaces with separated asymptotics were shown under various conditions at infinity. We will extend these results to the problem (S_ε), it turns out that the main asymptotic terms both for u^ε and u^0, are the same (Theorem 3.5). Moreover, we estimate the difference $u^\varepsilon - u^0$, like for bounded domains we obtain convergence of the velocity part in H^2_{loc}, and of the pressure part in H^1_{loc}, as $\varepsilon \searrow 0$ (see Theorem 3.6). Although we carry out the proofs for the case of prescribed flux through the outlets, we emphazise that it is possible to treat in the same manner further conditions at infinity as they were analyzed in [8], and obtain completely analogous results for (S_ε) together with error estimates.

We also treat the nonlinear problem

$$\begin{aligned} &S_\varepsilon u^\varepsilon + \begin{pmatrix} B(v^\varepsilon, v^\varepsilon) \\ 0 \end{pmatrix} = \begin{pmatrix} f' \\ f_4 \end{pmatrix} \quad \text{in} \quad \Omega, \\ &B_\varepsilon u^\varepsilon = (v^\varepsilon, \partial_n p^\varepsilon) = g \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{NS_ε}$$

with $B(v,w) = B^1(v,w) := (v \cdot \nabla)w$ or $B(v,w) = B^2(v,w) = (v\cdot\nabla)v + 2^{-1}(\operatorname{div} v)w$. We refer to the problem (NS_0) if $\varepsilon = 0$ (including cancelling the Neumann boundary condition). Note that usually the second variant for the nonlinear operator appears in the literature (see [17, 11], e.g.). If Ω is a bounded domain, $f_4 = 0$ and $g = 0$, the bilinear operator B^2

permits to use similar arguments as for the unperturbed problem, namely to derive a priori estimates and use Schauders fix point theorem to obtain the existence of a weak solution (see [4, 13], e.g.). However, in the case of unbounded domains, it is not obvious how the arguments used for the existence of solutions to (NS_0) (see, [5, 15], e.g.) carry over to (NS_ε). Since we are interested in error estimates, which are not available by energy methods, moreover, we admit inhomogeneities in the divergence equation and in the boundary data, we prove existence and uniqueness results, together with error estimates for strong solutions, which are available under smallness conditions for the data (Theorem 4.2).

The plan of the paper is as follows. In Sec. 2 we introduce some basic notations and recall the results for bounded domains. In Sec. 3 we investigate the linear problems for exponentially decaying data under flux conditions at infinity. Section 4 is devoted to the nonlinear problems (NS_0) and (NS_ε).

2. General notations and the optimal estimate for bounded domains

For $G \subset \mathbb{R}^3$, the closure of G is denoted by $\overline{G}$, the boundary by ∂G, and for $x \in \partial G$ the exterior unit normal vector by $n(x)$, if it exists. For any $t \in \mathbb{R}$, we call $[t]$ the integer part of t, i.e., $[t] = \max\{j \in \mathbb{Z} : j \leq t\}$, while the number $t_+ = (t + |t|)/2$ means the positive part of t.

Expressing partial derivatives we use the common multi-index terminology: $\partial^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1}\frac{\partial^{\alpha_2}}{\partial x_2}\frac{\partial^{\alpha_3}}{\partial x_3}$, with $\alpha \in \mathbb{N}_0^3$. If $k \in \mathbb{N}$, then ∇^k indicates the collection of all partial derivatives of order k.

Depending on our problems, we need different kinds of function spaces, most of them are introduced in the context. For any function space X, we indicate the norm in X by $\|\cdot\,; X\|$. We recall the notations for some standard spaces. $C_0^\infty(G)$, $C_0^\infty(\overline{G})$ are the spaces of all smooth functions with compact support in G and $\overline{G}$, respectively.

For $l \in \mathbb{N}$, the Sobolev space $H^l(G)$, ($G \subset \mathbb{R}^3$ open) consists of all $\varphi \in L^2(G)$ with distributional derivatives $\partial^\alpha\varphi \in L^2(G)$, $|\alpha| \leq l$, supplied with the usual norm. If $s > 0$, $s \notin \mathbb{N}$, then, for $G \subset \mathbb{R}^n$, the Sobolev-Slobodetskii space $H^s(G)$ coincides with the set $\{\varphi \in H^{[s]}(G) : \|\varphi; H^s(G)\| < \infty\}$, while

$$\|\varphi; H^s(G)\|^2 = \|v; H^{[s]}(G)\|^2 + \sum_{|\alpha|=[s]} \int_G \int_G \frac{|\partial^\alpha\varphi(x) - \partial^\alpha\varphi(\xi)|^2}{|x-\xi|^{n+2(s-[s])}} d\xi\, dx.$$

Furthermore, $H^{-s}(G)$ is the dual space of $\overset{\circ}{H}{}^s(G)$, here the supscript $\circ$ indicates the closure of $C_0^\infty(G)$ in $H^s(G)$. If ∂G is compact and sufficiently smooth, i.e., of class $C^{[s]+2}$, for example, then $H^s(\partial G)$ is defined, by using local coordinates and a partition of unity on ∂G, via the definition of $H^s(\mathbb{R}^2)$, while $H^{-s}(\partial G)$ is the dual space of $H^s(\partial G)$ (see [6] for details.)

Now let $G \subset \mathbb{R}^3$ be a bounded domain with $\partial G \in C^{l+2}$, $l \in \mathbb{N}_0$. To obtain asymptotically precise estimates for solutions u^ε to problem (S_ε), we supply the spaces $H^l(G)$ with equivalent norms, where the small parameter $\varepsilon \in (0,1]$ is included (see [7, 9]). For $\varkappa \le l$ and $\varphi \in H^l(G)$ we set

$$\|\varphi; H^l_\varkappa(G;\varepsilon)\|^2 = \|\varphi; H^\varkappa(G)\|^2 + \sum_{k=0}^{l} \varepsilon^{2(k-\varkappa)_+} \|\nabla\varphi; L^2(G)\|^2. \tag{2.1}$$

Note that the exponents in the powers of ε depend step-like on order k of the derivatives. Then for any differential operator ∂^α with $|\alpha| \le l$, and k with $\varkappa + k \le l$, we have

$$\|\partial^\alpha\varphi; H^{l-|\alpha|}_{\varkappa-|\alpha|}(G;\varepsilon)\| \le C\|\varphi; H^l_\varkappa(G;\varepsilon)\|, \tag{2.2}$$

$$\|\varepsilon^k\partial^\alpha\varphi; H^{l-|\alpha|}_{\varkappa-|\alpha|+k}(G;\varepsilon)\| \le C\|\varphi; H^l_\varkappa(G;\varepsilon)\|. \tag{2.3}$$

We process the trace spaces $H^{l-1/2}(\partial G)$ in the following way. We define $\|v; \mathcal{H}^{l-1/2}_{\varkappa-1/2}(\partial G;\varepsilon)\|$ as

$$\begin{cases} \Big(\|v; H^{\varkappa-1/2}(\partial G)\|^2 + \varepsilon^{2(l-\varkappa)}\|v; H^{l-1/2}(\partial G)\|^2\Big)^{1/2} \\ \text{for} \quad \varkappa > 1/2; \\ \varepsilon^{-\varkappa+1/2}\Big(\|v; L^2(\partial G)\|^2 + \varepsilon^{2l-1}\|v; H^{l-1/2}(\partial G)\|^2\Big)^{1/2} \\ \text{for} \quad \varkappa < 1/2. \end{cases} \tag{2.4}$$

The norms of the trace operators $\partial_n^h : H^l_\varkappa(G;\varepsilon) \to H^{l-h-1/2}_{\varkappa-h-1/2}(\partial G;\varepsilon)$, where $h = 0$ and $h = 1$, can be bounded independent on $\varepsilon \in (0,1]$, the converse result on extensions is also true.

Theorem 2.1. *Let $G \subset \mathbb{R}^3$ be a bounded domain with ∂G of class C^{l+2}, $l \in \mathbb{N}$ and let $\varkappa \in [0, 3/2)$, $\varkappa \le l-1$. Then for any $f \in H^{l-1}(G)^4$ and $g = (g', g_4) \in H^{l+1/2}(\partial G)^3 \times H^{l-1/2}(\partial G)$ subject to the compatibiliy* *condition*

$$\int_G f_4 - \int_{\partial G} g' \cdot n do + \varepsilon^2 \int_{\partial G} g_4 do = 0, \tag{2.5}$$

there exists a solution $u^\varepsilon = (v^\varepsilon, p^\varepsilon) \in H^{l+1}(G)$ *to problem* (S_ε)*, which is unique under the orthogonality condition* $\int_G p^\varepsilon = 0$*. This solution satisfies the estimate*

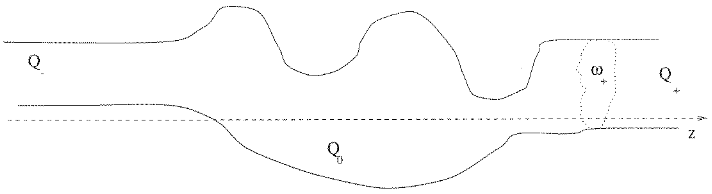
$$\begin{aligned} &\|v^\varepsilon; H^{l+1}_{\varkappa+1}(G;\varepsilon)\| + \|p^\varepsilon; H^{l+1}_{\varkappa}(G;\varepsilon)_\perp\| \\ &\leq C \Big(\|f'; H^{l-1}_{\varkappa-1}(G;\varepsilon)\| + \|f_4; H^{l-1}_{\varkappa}(G;\varepsilon)\| \\ &+\|g'; H^{l+1/2}_{\varkappa+1/2}(\partial G;\varepsilon)\| + \|g_4; H^{l-1/2}_{\varkappa-3/2}(\partial G;\varepsilon)\| \Big), \end{aligned} \tag{2.6}$$

where C *is a constant depending on* $\varkappa, \partial G$ *and* l*, but neither on* $\varepsilon \in (0,1]$ *nor on the data* (f,g)*. The subscript* $\perp$ *indicates the subspace of mean-value-free functions.* □

Remark 2.2. In [9] even $\varkappa \leq l$ was admissible, this uses $H^m_\varkappa(G;\varepsilon)$-norms, where $\varkappa > m$.

3. The geometry of the pipe, function spaces and basic results for the Stokes problem

In the sequel we fix a domain $\Omega \subset \mathbb{R}^3$ with boundary $\partial\Omega$ of class C^{l+2} for some $l \in \mathbb{N}_0$ depending on the context, and with two cylindrical outlets to infinity, both with a common z-axis for simplicity, as indicated in the figure.



This means $\Omega = Q_- \cup Q_0 \cup Q_+$, where $Q_- = \omega_- \times (-\infty, -R_-)$, $Q_+ = \omega_+ \times (R_+, \infty)$, and $\omega_\pm \subset \mathbb{R}^2$ are bounded, smoothly surrounded domains, $R_\pm > 0$. The connecting part Q_0 is of the form $Q_0 = \{(y,z) : y \in \omega(z), z \in (-R_0, R_0)\}$, where now the cross section $\omega(z) \subset \mathbb{R}^2$ is varying, but still uniformly bounded, and $R_0 > \max\{R_-, R_+\}$. The canonical function spaces for the problem (S_0) as for the problem (S_ε) with fixed $\varepsilon > 0$, are weighted Sobolev spaces with exponential weights. We introduce the smooth positive function $\mathfrak{z}(y, z) = (1 + z^2)^{1/2}$ and, for $s, \beta \in \mathbb{R}$, the function space $W^s_\beta(\Omega)$ as the closure of $C_0^\infty(\overline{\Omega})$ (the smooth functions with compact support in $\overline{\Omega}$) with respect to the norm

$$\|\varphi; W^s_\beta(\Omega)\| = \|e^{\beta\mathfrak{z}}\varphi; H^s(\Omega)\|.$$

For $s > 1/2$, the space of traces to $W^s_\beta(\Omega)$-functions is denoted by $W^{s-1/2}_\beta(\partial\Omega)$ and equipped with its natural norm

$$\|\varphi; W^{s-1/2}_\beta(\partial\Omega)\| = \|\varphi e^{\beta\mathfrak{z}}; H^{s-1/2}(\partial\Omega)\| = \inf\{\|\Phi e^{\beta\mathfrak{z}}; H^s(\Omega)\|\},$$

where the infimum is taken over all $\Phi \in W^s_\beta(\Omega)$ such that $\Phi|_{\partial\Omega} = \varphi$. For $s > 0$, we introduce the natural spaces for the solutions and the data of the problem (S_0) :

$$\mathrm{D}^s_\beta W(\Omega) = W^{s+1}_\beta(\Omega)^3 \times W^s_\beta(\Omega), \tag{3.7}$$

$$\mathrm{R}^s_\beta W(\Omega) = W^{s-1}_\beta(\Omega)^3 \times W^s_\beta(\Omega) \times W^{s+1/2}_\beta(\partial\Omega)^3. \tag{3.8}$$

Then the mapping

$$(S_0, B_0) : \mathrm{D}^s_\beta W(\Omega) \to \mathrm{R}^s_\beta W(\Omega) \tag{3.9}$$

is continuous, and moreover, possesses the Fredholm property, provided β does not meet a countable discrete set $\mathcal{I}_0$ of forbidden indices related to solutions of exponential type (see, e.g., [8]). The latter ones are solutions to the homogeneous problems (S_0) in the cylinders $\Pi_\pm = \omega_\pm \times \mathbb{R}$ of the form

$$U(x, \lambda) = e^{\lambda z} \sum_{q=0}^{h} z^q \psi_q(y). \tag{3.10}$$

The weight index β is forbidden, if there exists a solution of the form (3.10) in Π_+ or Π_- with $\operatorname{Im}\lambda = \beta$, the associated operator has no closed range then, while the kernel is always finite dimensional (see e.g. [8] for the details). More precisely, (3.9) is injective for any $\beta \geqslant 0$, and the mapping is an epimorphism if $\beta \in (-\infty, 0) \setminus \mathcal{I}_0$, while $0 \in \mathcal{I}_0$. Following [8], we deal with spaces with separated asymptotics here. In each cylinder $\Pi_\pm$ we have two solutions to the homogeneous system of type (3.10) with

$\operatorname{Im}\lambda = 0$. These are the constant pressure solution and the Poisseuille flow

$$u^{1,\pm}(y,z) = (0,0,0,1)^\top, \quad u^{2,\pm}(y,z) = (0,0,\psi^\pm(y), -c_p^\pm z)^\top,$$

where $\psi^\pm$ solves the Poisson problem $-\Delta\psi^\pm = c_p^\pm$ in $\omega_\pm$, $\psi^\pm = 0$ on $\partial\omega_\pm$, and the constant $c_P^\pm$ is chosen in such a way that $\int_{\omega_\pm}\psi^\pm dy = 1$, i.e., $u^{2,\pm}$ carries the unit flux through $\Pi_\pm$. We choose a C^∞-partition of unity $\{\chi_-, \chi_0, \chi_+\}$, depending only on the axis variable z, subordinated to the covering $\{Q_-, Q_0, Q_+\}$ of Ω and put $U^{h,\pm} = \chi_\pm u^{h,\pm}$, $h = 1,2$, where $\chi_\pm u^{h,\pm}$ is extended by 0 outside of $Q_\pm$. Clearly $U^{h,\pm} \in \mathrm{D}^s_\beta W(\Omega)$ for any $\beta < 0$, but not for $\beta \geqslant 0$. Since the set of forbidden values for the weight indices β is discrete and countable, we have a $\beta_0 > 0$ such that any β with $0 < |\beta| < \beta_0$ is admissible for the Fredholm property of (3.9). With these notations we have the following result on the problem (S_0).

Theorem 3.1. *Let $s \geqslant 0$ and $\beta \in (0,\beta_0)$ be fixed. To any set of data $(f,g') \in \mathrm{R}^s_\beta W(\Omega)$, there exists a solution $u^0 \in \mathrm{D}^s_{-\beta}W(\Omega)$ to the problem (S_0). Any solution $u^0 \in \mathrm{D}^s_{-\beta}W(\Omega)$ admits the asymptotic representation*

$$u^0 = a_+U^{1,+} + b_+U^{2,+} + a_-U^{1,-} + b_-U^{2,-} + \tilde{u}^0, \tag{3.11}$$

with $\tilde{u} \in \mathrm{D}^s_\beta W(\Omega)$. The solution becomes unique under the asymptotic condition

$$(a_+, b_+) = (\Phi_1, \Phi_2) \tag{3.12}$$

with a prescribed $\Phi \in \mathbb{R}^2$, and the following estimate holds true with a constant independent on u^0:

$$|a_-| + |b_-| + \|\tilde{u}^0; \mathrm{D}^s_\beta W(\Omega)\| \leq C\,\Big(\,\|(f,g'); \mathrm{R}^s_\beta W(\Omega)\| + |\Phi|\,\Big) \tag{3.13}$$

Proof. This result was proved in [8] for integer $s > 0$ and, by literally repeating arguments, holds true for $s = 0$ as well. It can be extended to the case of non integer s by using the method of increasing the local smoothness. We cover the domain Ω by two systems of bounded domains G_k, G'_k, $k \in \mathbb{Z}$. Here we choose $G_0 = Q_0$, and $G_{\pm k} = \{x \in Q_\pm : |z \mp (k \pm (R_\pm + 1))| < 1\}$ for $k \in \mathbb{N}$, while $G'_k \supset G_k$, namely $G'_k = G_{k-1} \cup G_k \cup G_{k+1}$. Then, for each $k \in \mathbb{Z}$,

$$m\, e^{\beta|k|} \leq e^{\beta\mathfrak{z}} \leq M\, e^{\beta|k|} \tag{3.14}$$

on G_k and G'_k, with positive constants m, M independent on k. We use the a priori estimate

$$\|\tilde{v}^0; H^{s+1}(G_k)\|^2 + \|\tilde{p}^0; H^s(G_k)\|^2$$

$$\leq C \Big(\|S_0 \tilde{u}^0; H^{s-1}(G'_k)^3 \times H^s(G'_k)\|^2 + \|\tilde{v}^0; H^{s+1/2}(\partial G'_k \cap \partial\Omega)\|$$

$$+\|\tilde{v}^0; L^2(G'_k)\|^2 + \|\tilde{p}^0; H^{-1}(G'_k)\|^2 \Big), \tag{3.15}$$

(see Theorem 10.1.1. in [18], Theorem 6.4.8 in [3] for $s > 1/2$, for $s \in [0, 1/2)$ the inequality (3.15) follows from the results of [1] and interpolation arguments). Note that the constant C is independent on k here. Multiplication by $e^{\beta|k|}$ and then summing up leads to the inequality

$$\|\tilde{u}^0; \mathrm{D}^s_\beta W(\Omega)\|^2 \leq 3C\Big(\|(f, g'); \mathrm{R}^s_\beta W(\Omega)\|^2 + \|\tilde{v}^0 e^{\beta\mathfrak{z}}; L^2(\Omega)\|^2$$

$$+\|\tilde{p}^0 e^{\beta\mathfrak{z}}; H^{-1}(\Omega)\|^2 + |a_+|^2 + |a_-|^2 + |b_+|^2 + |a_-|^2\Big),$$

which provides the assertion by means of (3.13) for $s = 0$. □

Remark 3.2. The solution of Theorem 3.1 is what is usually called a solution with prescribed flux. The second flux b_- can be calculated from the data. Since the total flux must vanish, we have (see [8])

$$b_+ - b_- + \int_\Omega f_4 dx - \int_{\partial\Omega} g' \cdot n do = 0. \tag{3.16}$$

The solution with prescribed flux is unique up to a constant in pressure, to fix this constant is just to choose $a_+ = \Phi_1$.

As in (2.1) and (2.4), we introduce norms including the small parameter ε. For $\varkappa \leq l$ and $\beta \in \mathbb{R}$ we put

$$\|\varphi; W^l_{\varkappa,\beta}(\Omega; \varepsilon)\|^2 = \|e^{\beta\mathfrak{z}}\varphi; H^l_\varkappa(\Omega; \varepsilon)\|^2$$

$$\sim \sum_{k=0}^{l} \varepsilon^{2(k-\varkappa)_+} \|e^{\beta\mathfrak{z}} \nabla^k \varphi; L^2(\Omega)\|^2 + \|e^{\beta\mathfrak{z}}\varphi; H^\varkappa(\Omega)\|^2. \tag{3.17}$$

The trace space $\mathcal{W}^{l-1/2}_{\varkappa-1/2,\beta}(\partial\Omega;\varepsilon)$, $\varkappa \neq 1/2$, is the space $W^{l-1/2}_{\beta}(\partial\Omega)$ equipped with the norm $\|\varphi;\mathcal{W}^{l-1/2}_{\varkappa-1/2,\beta}(\partial\Omega;\varepsilon)\|$ which is equal to

$$\begin{cases} \|\varphi; W^{\varkappa-1/2}_{\beta}(\partial\Omega)\| + \varepsilon^{l-\varkappa}\|\varphi; W^{l-1/2}_{\beta}(\partial\Omega)\| \\ \text{for} \quad \varkappa > 1/2, \\ \varepsilon^{1/2-\varkappa}\left(\|\varphi; W^{0}_{\beta}(\partial\Omega)\| + \varepsilon^{l-1/2}\|\varphi; W^{l-1/2}_{\beta}(\partial\Omega)\| \right) \\ \text{for} \quad \varkappa < 1/2. \end{cases} \tag{3.18}$$

To find estimates for the solutions to problem (S_0) in the $W^{l}_{\varkappa,\beta}(\Omega;\varepsilon)$-spaces, we introduce

$$\mathbf{D}^{l}_{\varkappa,\beta}W(\Omega;\varepsilon) = W^{l+1}_{\varkappa+1,\beta}(\Omega;\varepsilon)^3 \times W^{l}_{\varkappa,\beta}(\Omega;\varepsilon), \tag{3.19}$$

$$\mathbf{R}^{l}_{\varkappa,\beta}W(\Omega;\varepsilon) = W^{l-1}_{\varkappa-1,\beta}(\Omega;\varepsilon)^3 \times W^{l}_{\varkappa,\beta}(\Omega;\varepsilon) \times \mathcal{W}^{l+1/2}_{\varkappa+1/2,\beta}(\partial\Omega;\varepsilon)^3, \tag{3.20}$$

here $l \in \mathbb{N}$, $l \geqslant \varkappa$, observe that again $\mathbf{D}^{l}_{\varkappa,\beta}W(\Omega;\varepsilon)$ and $\mathbf{R}^{l}_{\varkappa,\beta}W(\Omega;\varepsilon)$ coincide with $\mathrm{D}^{l}_{\beta}W(\Omega)$ and $\mathrm{R}^{l}_{\beta}W(\Omega)$, respectively, with equivalent norms for any $\varepsilon > 0$, where the embedding constants depend on ε, of course.

Theorem 3.3. *Assume $\beta \in (0,\beta_0)$, $\varkappa \geqslant 0$, $l \in \mathbb{N}$ with $l \geqslant \varkappa$. For given $(f,g') \in \mathrm{R}^{l}_{\beta}W(\Omega)$, $\Phi \in \mathbb{R}^2$, let u^0 be the unique solution to the Dirichlet problem (S_0) with prescribed flux* (3.12). *Then u^0 obeys the estimate*

$$\begin{aligned} &|a_-| + |b_-| + \|\tilde{u}^0; \mathbf{D}^{l}_{\varkappa,\beta}W(\Omega;\varepsilon)\| \\ &\leq C\left(\|(f,g'); \mathbf{R}^{l}_{\varkappa,\beta}W(\Omega;\varepsilon)\| + |\Phi| \right) \end{aligned} \tag{3.21}$$

with a constant C independent on $\varepsilon \in (0,1]$ and on the data f, g', and Φ.

Proof. The inequality (3.13) gives already the estimation of the stable parts in (3.21) (i.e., the terms without powers of ε):

$$|a_-| + |b_-| + \|\tilde{u}^0; \mathrm{D}^{\varkappa}_{\beta}W(\Omega)\| \leq C\left(\|(f,g'); \mathrm{R}^{\varkappa}_{\beta}W(\Omega)\| + |\Phi| \right).$$

It remains to check up the terms

$$\varepsilon^{k-\varkappa-1}\|e^{\beta\mathfrak{z}}\nabla^k\tilde{v}^0; L^2(\omega)\|,\ [\varkappa+1] < k \leq l+1,$$
$$\varepsilon^{k-\varkappa}\|e^{\beta\mathfrak{z}}\nabla^k\tilde{p}^0; L^2(\Omega)\|,\ [\varkappa] < k \leq l.$$

Since $(S_0,B_0)U^{h,\pm} = 0$ outside of supp $\nabla\chi_\pm$, we obtain $(S_0,B_0)\tilde{u}^0 = (f,g') + (f^1,0)$, where $f_1 = b_\pm[S_0,\chi_\pm]u^{2,\pm} + a_\pm[S_0,\chi_\pm]u^{1,\pm}$, hence

$$\|(f^1,0); \mathbf{R}^{l}_{\varkappa,\beta}W(\Omega;\varepsilon)\| \leq C\left(|a_+| + |b_+| + |a_-| + |b_-| \right)$$

$$\leq C\ \left(\ \|(f,g');\mathrm{R}^{\varkappa}_{\beta}W(\Omega)\| + |\Phi|\ \right),$$

for the first inequality we used the bound $\varepsilon \leq 1$. Now we employ the a priori estimate

$$\|\tilde{u}^0;\mathrm{D}^k_{\beta}W(\Omega)\| \leq C\ \Big(\ \|(S_0,B_0)\tilde{u}^0;\mathrm{R}^k_{\beta}W(\Omega)\| + \|\tilde{u}^0;W^0_{\beta}(\Omega)^3\times W^0_{\beta}(\Omega)\|\ \Big)$$

with $k=[\varkappa]+1,\ldots,l$. For $k>\varkappa$, this leads to

$$\begin{aligned}
&\varepsilon^{k-\varkappa}\Big(\|\nabla^{k+1}\tilde{v}^0;W^0_{\beta}(\Omega)\| + \|\nabla^k\tilde{p}^0;W^0_{\beta}(\Omega)\|\Big)\\
\leq &C\varepsilon^{k-\varkappa}\Big(\|(S_0,B_0)\tilde{u}^0;\mathrm{R}^k_{\beta}W(\Omega)\| + \|\tilde{u}^0;W^0_{\beta}(\Omega)\|\Big)\\
\leq &C\varepsilon^{k-\varkappa}\Big(\|(f,g');\mathrm{R}^k_{\beta}W(\Omega)\| + |\Phi| + \|\tilde{u}^0;W^0_{\beta}(\Omega)\|\Big)\\
\leq &C\Big(\|(f,g');\mathbf{R}^k_{\varkappa,\beta}W(\Omega;\varepsilon)\|\\
+&\varepsilon^{k-\varkappa}\Big(\|(f,g');\mathrm{R}^{\varkappa}_{\beta}W(\Omega)\| + |\Phi| + \|\tilde{u}^0;W^0_{\beta}(\Omega)^4\|\Big)\Big)\\
\leq &C\Big(\|(f,g');\mathbf{R}^k_{\varkappa,\beta}W(\Omega;\varepsilon)\| + |\Phi|\Big),
\end{aligned}$$

here we again used the bound $\varepsilon^{k-\varkappa}\leq 1$ and also inequality (3.13).

3.1. The problem (S_ε): Construction of a quasiparametrix

In contrast to the case of exterior domains (– the subject of a forthcoming paper), here the usual Kondratiev theory in domains with cylindrical outlets to infinity is available for the problem (S_ε), too. Following the different structure of the problem (S_ε), it is clear that we have to choose the data $f\in W^{l-1}_{\beta}(\Omega)$, $g\in W^{l+1/2}_{\beta}(\partial\Omega)^3\times W^{l-1/2}_{\beta}(\partial\Omega)$, while we hope to find solutions $u^\varepsilon\in W^{l+1}_{\beta}(\Omega)^4$. The investigation of the model problem in the cylinders $\Pi_\pm$ leads to the Fredholm property of (S_ε) in $W^l_{\beta}(\Omega)$-spaces, again provided $\beta\in\mathbb{R}\setminus\mathcal{I}$, and $\mathcal{I}$ is a countable discrete set of forbidden values, where the associate operator fails to have a closed range due to the existence of solutions of exponential type (see (3.10)) to the problem (S_ε) in $\Pi_\pm$. Although a straightforward calculation following the scheme of Proposition 3.3, p. 153 in [8], shows that the only solution of type (3.10) with $\lambda=0$ to the problem (S_ε) in $\Pi_\pm$ are the constant pressure solution and the Poiseuille flow again, in general the forbidden values depend on ε. Thus we present a different technique here. We will use the results of [9] not only for deriving a priori estimates in $W^l_{\varkappa,\beta}(\Omega;\varepsilon)$, but also for constructing solutions in spaces with separated asymptotics. This will lead to the unique solvability of (S_ε) in a frame comparable to Theorem 3.1 and to optimal error estimates. We introduce the spaces $\mathcal{D}^l_{\varkappa,\beta}W(\Omega;\varepsilon)$ and $\mathcal{R}^l_{\varkappa,\beta}W(\Omega;\varepsilon)$ with ε dependent norms related to the problem (S_ε), namely

$$\begin{aligned}\mathcal{D}^l_{\varkappa,\beta}W(\Omega;\varepsilon) &= W^{l+1}_{\varkappa+1,\beta}(\Omega;\varepsilon)^3 \times W^{l+1}_{\varkappa,\beta}(\Omega;\varepsilon)\\ \mathcal{R}^l_{\varkappa,\beta}W(\Omega;\varepsilon) &= W^{l-1}_{\varkappa-1,\beta}(\Omega;\varepsilon)^3 \times W^{l-1}_{\varkappa,\beta}\\ &\qquad \times \mathcal{W}^{l+1/2}_{\varkappa+1/2,\beta}(\partial\Omega;\varepsilon)^3 \times \mathcal{W}^{l-1/2}_{\varkappa-3/2,\beta}(\partial\Omega,\varepsilon).\end{aligned}$$

Lemma 3.4 Existence of a quasiparametrix. *Let $\varkappa \in [0,3/2)$, $l \in \mathbb{N}$, $l \geqslant \varkappa + 1$, and $\beta \in \mathbb{R}$. For any $\varepsilon \in (0,1]$ there exists a continuous linear operator $P_\varepsilon : \mathcal{R}^l_{\varkappa,\beta}W(\Omega;\varepsilon) \to \mathcal{D}^l_{\varkappa,\beta}W(\Omega;\varepsilon)$ with the following properties: The norm of P_ε is bounded independent on ε. If $\mathbb{I}$ denotes the identity operator on $\mathcal{D}^l_{\varkappa,\beta}W(\Omega;\varepsilon)$ and $T_\varepsilon = (S_\varepsilon, B_\varepsilon)P_\varepsilon - \mathbb{I}$, then it holds*

$$\|T_\varepsilon(f,g);\mathcal{R}^{l+1}_{\varkappa+1,\beta}W(\Omega;\varepsilon)\| \leq C\|(f,g);\mathcal{R}^l_{\varkappa,\beta}W(\Omega;\varepsilon)\|, \tag{3.22}$$

and C is again independent of ε.

Proof. Similar as in the proof of Theorem 3.1 we choose on open covering $\{G_k\}_{k\in\mathbb{Z}}$ of Ω, but now with smooth boundaries. Each G_k is bounded with $\partial G_k \in C^{l+2}$ at least, $G_0 \supset Q_0$. For $k \neq 0$, the domain $G_{\pm k} \supset \{x \in \Omega : |z \mp (k \pm R_\pm| < 1\}$, moreover, we may again assume that the domain $G_{\pm k}$ is obtained by shifting $G_{\pm 1}$ along the z-axis. Clearly, $\overline{\Omega} \subset \bigcup_{k\in\mathbb{Z}} \overline{G}_k$. Next we fix two C_0^∞-partitions of unity on $\mathbb{R}$, $\{\eta_k, k \in \mathbb{Z}\}$, $\{\zeta_k, k \in \mathbb{Z}\}$ subordinated to this covering, with the properties $\zeta_k\eta_k = \eta_k$, the supports of η_0 and ζ_0 are contained in the interval $(-R_-, R_+)$. Further for $k \geqslant 2$, we assume $\eta_{\pm k}(x) = \eta_{\pm k}(z)$ and $\eta_k(z) = \eta(z - k - R_+)$, $\eta_{-k}(z) = \eta(z + k + R_-)$ with supp $\eta \subset (-1,1)$. Observe, that (3.14) is valid for G_k again. Let $(f,g) \in \mathcal{R}^l_{\varkappa,\beta}W(\Omega;\varepsilon)$ be given, we divide

$$(f,g)(x) = \sum_{k=-\infty}^{\infty} \eta_k(z)(f(x),g(x)) =: \sum_{k=-\infty}^{\infty} (f^k,g^k)(x).$$

then it holds (g^k is extended by 0 to $\partial G_k \cap \Omega$)

$$\sum_{k=-\infty}^{\infty} e^{2\beta|k|}\|(f^k,g^k);\mathcal{R}^l_\varkappa H(G_k;\varepsilon)\|^2 \leq C\|(f,g);\mathcal{R}^l_{\varkappa,\beta}W(\Omega;\varepsilon)\|^2, \tag{3.23}$$

here $\mathcal{R}^l_\varkappa H(G_k;\varepsilon)$ means the natural combination of $H^l_\varkappa(G_k;\varepsilon)$ spaces as in (2.6). Since $\partial G_k \cap \Omega \neq \emptyset$ for all k, we can change g^k to $\tilde g^k$ on ∂G_k such that $\tilde g^k$ fulfils the compatibility condition (2.5) on G_k and

$$\|\tilde g^k; \mathcal{H}^{l+1/2}_{\varkappa+1/2}(\partial G_k;\varepsilon)^3 \times \mathcal{H}^{l-1/2}_{\varkappa-3/2}(\partial G_k;\varepsilon)\|$$

$$\leq C\|g^k; \mathcal{H}^{l+1/2}_{\varkappa+1/2}(\partial G_k;\varepsilon)^3 \times \mathcal{H}^{l-1/2}_{\varkappa-3/2}(\partial G_k;\varepsilon)\|$$

holds with a constant independent of ε. To this end, we just fix $h' \in C^{l+2}_0(\partial G_{\pm 1} \cap \Pi_\pm)^3$ with $\int_{\partial G_{\pm 1}} h' \cdot n = 1$, then we extend g^k_4 by 0 on $\partial G_k \cap \Omega$ for all k, but the first three components $g^{k\prime}$ we modify in the following way

$$\tilde g^{k\prime} = \begin{cases} g^{k\prime} & \text{on} \quad \partial G_k \cap \partial\Omega, \\ -c_k h'(y, z \mp (k + R_\pm)) & \text{on} \quad \partial G_{\pm k} \cap \Omega, \end{cases}$$

with

$$c_k = \int\limits_{G_k} f^k_4\, dx - \int\limits_{\partial G_k \cap \partial\Omega} g^{k\prime} \cdot n do + \varepsilon^2 \int\limits_{\partial G_k} \tilde g^k_4 do.$$

In a similar way we proceed with G_0. Then $(f^k, \tilde g^k)$ fulfills the compatibility condition (2.5) on each G_k. Let $u^k = (v^k, p^k) \in H^{l+1}(G_k)^4$ be the unique solution to the problem $S_\varepsilon u^k = f^k$ in G_k, $B_\varepsilon u^k = \tilde g^k$ on ∂G_k and $\int_{G_k} p^k = 0$, according to Theorem 2.1 then with $\mathcal{D}^l_\varkappa H(G_k;\varepsilon) = H^{l+1}_{\varkappa+1}(G_k;\varepsilon)^3 \times H^{l+1}_\varkappa(G;\varepsilon)$, we get

$$\|u^k; \mathcal{D}^l_\varkappa H(G_k;\varepsilon)\|^2 \leq C\|(f^k, g^k); \mathcal{R}^l_\varkappa H(G_k;\varepsilon)\|^2,$$

where $C = C(l, \varkappa, \Xi_\pm, G_0)$, thus independent on k and ε. We put $P_\varepsilon(f,g) = \sum_{k=-\infty}^{\infty} \zeta_k u_k$, observe that $\tilde g^k|_\Omega$ now is cut off again. Then

$$\|P_\varepsilon(f,g); \mathcal{D}^l_{\varkappa,\beta} W(\Omega;\varepsilon)\|^2 \leq C \sum_{k=-\infty}^{\infty} e^{2\beta|k|}\|u^k; \mathcal{D}^l_\varkappa H(G_k;\varepsilon)\|^2$$

$$\leq C \sum_{k=-\infty}^{\infty} e^{2\beta|k|}\|(f^k, g^k); \mathcal{R}^l_\varkappa H(G_k;\varepsilon)\|^2 \leq C\,\|(f,g); \mathcal{R}^l_{\varkappa,\beta} W(\Omega;\varepsilon)\|^2,$$

with constants independent of ε, this proves already the first assertion. On G_k, we have

$$T_\varepsilon(f,g) = ([S_\varepsilon, \zeta_k]u^k, g', [\partial_n, \zeta_k]u^k) =: (\mathfrak{f}^k, \mathfrak{g}^k),$$

where the brackets [,] denote the commutator. We use (2.2) and (2.3) and find

$$\|\mathfrak{f}^{k\prime}; H^l_\varkappa(G_k;\varepsilon)\|^2 \le C\ \Big(\ \|\nabla\zeta_k\nabla v^k; H^l_\varkappa(G_k;\varepsilon)\|^2$$

$$+\|(\Delta\zeta_k)v^k; H^l_\varkappa(G_k;\varepsilon)\|^2 + \|(\nabla\zeta_k)p^k; H^l_\varkappa(G_k;\varepsilon)\|^2\ \Big)$$

$$\le C\ \Big(\ \|v^k; H^{l+1}_{\varkappa+1}(G_k;\varepsilon)\|^2 + \|p^k; H^{l+1}_\varkappa(G_k;\varepsilon)\|^2\ \Big)\,. \tag{3.24}$$

Furthermore, we have

$$\|\mathfrak{f}^k_4; H^l_{\varkappa+1}\|^2 \le C\ \Big(\ \|(\nabla\zeta_k)u^k; H^l_{\varkappa+1}(G_k;\varepsilon)\|^2$$

$$+\|(\Delta\zeta_k)\varepsilon^2p^k; H^l_{\varkappa+1}(G_k;\varepsilon)\|^2 + \|(\nabla\zeta_k)\varepsilon^2\nabla p^k; H^l_{\varkappa+1}(G_k;\varepsilon)\|^2\ \Big)$$

$$\le C\ \Big(\ \|u^k; H^{l+1}_{\varkappa+1}(G_k;\varepsilon)\|^2 + \|p^k; H^{l+1}_\varkappa(G_k;\varepsilon)\|^2\ \Big)\,. \tag{3.25}$$

In the boundary values, non vanishing expressions appear only for $k = -1, 0, 1$, because we have $[\partial_n, \zeta_k] = 0$ for $|k| > 2$, due to the choice $\zeta_k = \zeta_k(z)$. Thus

$$\|\mathfrak{g}_4; W^{l+1/2}_{\varkappa-1/2}(\partial\Omega;\varepsilon)\| \le C\|\sum_{k=-1}^{1}(\partial_n\zeta_k)p^k; H^{l+1}_\varkappa(G_{-1}\cup G_0\cup G_1;\varepsilon)\|$$

$$\le C\|(f,g); \mathcal{R}^l_{\varkappa,\beta}W(\Omega;\varepsilon)\|.$$

We multiply (3.24) and (3.25) by $e^{2\beta|k|}$ and use (3.23), which gives (3.22). □

3.2. Existence and uniqueness results for (S_ε) in spaces with separated asymptotics

Based on Theorem 3.3, we introduce weighted spaces with separated asymptotics adopted to the system (S_0). Let $\beta \in (0, \beta_0)$ be fixed, we put

$$\mathfrak{D}^{l,0}_{\varkappa,\beta}W(\Omega;\varepsilon) = \{u \in \mathbf{D}^l_{\varkappa,-\beta}W(\Omega;\varepsilon),\, u \ \text{ fulfils (3.11)}\},$$

provided with the norm

$$\|u; \mathfrak{D}^{l,0}_{\varkappa,\beta}W(\Omega;\varepsilon)\|$$

$$= \left(\|\tilde{u}; \mathbf{D}^l_{\varkappa,\beta}W(\Omega;\varepsilon)\| + |a_+|^2 + |b_+|^2 + |a_-|^2 + |b_-|^2\right)^{1/2}. \tag{3.26}$$

Then Theorem 3.3 asserts that

$$\mathbf{A}^0 : \mathfrak{D}^{l,0}_{\varkappa,\beta}W(\Omega;\varepsilon) \to \mathbf{R}^l_{\varkappa,\beta}W(\Omega;\varepsilon) \times \mathbb{R}^2, u \mapsto (S_0u, B_0u, a_+, b_+),$$

defines an isomorphism, where the norms of $\mathbf{A}^0$ as well as of $(\mathbf{A}^0)^{-1}$ are bounded independent of ε. To treat (S_ε), we compose the space $\mathfrak{D}^{l,\varepsilon}_{\varkappa,\beta}W(\Omega;\varepsilon)$ from functions

$$u = a_+U^{1,+} + b_+U^{2,+} + a_-U^{1,-} + b_-U^{2,-} + \tilde{u},$$

where now $\tilde{u} \in \mathcal{D}^l_{\varkappa,\beta}W(\Omega;\varepsilon)$, the space provided with the norm (3.26), but with $\|\cdot;\mathbf{D}^l_{\varkappa,\beta}W(\Omega;\varepsilon)\|$ replaced by $\|\cdot;\mathcal{D}^l_{\varkappa,\beta}W(\Omega;\varepsilon)\|$.

Theorem 3.5. *Let $\varkappa \in [0,3/2)$, $\beta \in (0,\beta_0)$ and $l \in \mathbb{N}$ with $l \geqslant \varkappa + 1$ be fixed, where β_0 is the same as in Theorem 3.1. Then there exists $\varepsilon_0 > 0$ such that the operator*

$$\mathbf{A}^\varepsilon : \mathfrak{D}^{l,\varepsilon}_{\varkappa,\beta}W(\Omega;\varepsilon) \to \mathcal{R}^l_{\varkappa,\beta}W(\Omega;\varepsilon) \times \mathbb{R}^2, u \mapsto (S_\varepsilon u, B_\varepsilon u, a_+, b_+),$$

defines an isomorphism, while the norms of $\mathbf{A}^\varepsilon$ as well as of $(\mathbf{A}^\varepsilon)^{-1}$ are bounded independent of $\varepsilon \in (0,\varepsilon_0]$.

Proof. The existence of ε-independent bounds for $\mathbf{A}^\varepsilon$ easily follows from the definition of the spaces. Let $(f,g) \in \mathcal{R}^l_{\varkappa,\beta}W(\Omega;\varepsilon)$ and $\Phi \in \mathbb{R}^2$ be given. We want to show: There exists a unique solution $u^\varepsilon \in \mathfrak{D}^{l,\varepsilon}_{\varkappa,\beta}W(\Omega;\varepsilon)$ to the problem (S_ε) , with $(a_+,b_+) = \Phi$, and

$$\|u^\varepsilon;\mathfrak{D}^{l,\varepsilon}_{\varkappa,\beta}W(\Omega;\varepsilon)\| \le C\Big(\|(f,g);\mathcal{R}^l_{\varkappa,\beta}W(\Omega;\varepsilon)\| + |\Phi|\Big), \qquad (3.27)$$

with a constant C independent of ε and the data. To start with the uniqueness result, we fix a solution to the homogeneous problem (S_ε) with $\Phi = 0$, i.e.,

$$u^\varepsilon = a_-U^{1,-} + b_-U^{2,-} + \tilde{u}^\varepsilon, \quad \tilde{u}^\varepsilon \in \mathcal{D}^l_{\varkappa,\beta}W(\Omega;\varepsilon).$$

We first prove $b_- = 0$ by exploiting the divergence-equation and the homogeneous boundary conditions. To this end, we fix some $R \in \mathbb{R}$ and integrate over the part of Ω between $z = R$ and $z = T \geqslant R$, where we first assume $T > R_+$. From the Gauss theorem we obtain

$$\begin{aligned} 0 &= \int_R^T \int_{\omega(z)} (-\varepsilon^2\Delta p^\varepsilon + \operatorname{div} v^\varepsilon)(x)dx \\ &= \int_{\omega(R)} (\varepsilon^2\partial_z p^\varepsilon - v_3^\varepsilon)(y,R)dy - \int_{\omega(T)} (\varepsilon^2\partial_z p^\varepsilon - v_3^\varepsilon)(y,T)dy. \end{aligned}$$

Recall that $\omega(T) = \omega_+$ for $T \geqslant R_+$, thus, it follows

$$\int\limits_{\omega(R)} (\varepsilon^2 \partial_z p^\varepsilon - v_3^\varepsilon)(y,R)dy = \int\limits_{\omega_+} (\varepsilon^2 \partial_z p^\varepsilon - v_3^\varepsilon)(y,T)dy$$

$$= \lim_{T\to\infty} \int\limits_{\omega_+} (\varepsilon^2 \partial_z p^\varepsilon - v_3^\varepsilon)(y,T)dy = 0,$$

the last equality holds, since $u^\varepsilon = \tilde{u}^\varepsilon$ for $z > R_0$ and $\tilde{u}^\varepsilon \in W_\beta^{l+1}(\Omega)^4$ with $\beta > 0$. Now we apply a similar argument to the integral $\int_{-T}^R \dots$ and obtain for all $T > R_-$:

$$0 = \int\limits_{\omega(R)} (\varepsilon^2 \partial_z p^\varepsilon - v_3^\varepsilon)(y,R)dy = \int\limits_{\omega_-} (\varepsilon^2 \partial_z p^\varepsilon - v_3^\varepsilon)(y,-T)dy$$

$$= b_- \int\limits_{\omega_-} (-\varepsilon^2 c_p^- - \psi_-(y))dy + \int\limits_{\omega_-} (\varepsilon^2 \partial_z \tilde{p}^\varepsilon - \tilde{v}_3^\varepsilon)(y,-T)dy.$$

Again we may pass to the limit $T \to \infty$. The limit of the last integral vanishes due to the same argument as above, and, since $c_p^- > 0$ and $\int_{\omega_-} \psi_- = 1$, the equality $b_- = 0$ must hold. Now we can consider a solution with finite Dirichlet integral. We multiply the homogeneous differential equation system by u^ε and integrate by parts on $\Omega \cap \{|z| < T\}$

$$0 = \int\limits_{-T}^{T} \int\limits_{\omega(z)} \nabla v^\varepsilon : \nabla v^\varepsilon + \varepsilon^2 |\nabla p^\varepsilon|^2 dy\, dz$$

$$+ \int\limits_{-T}^{T} \int\limits_{\omega(z)} \operatorname{div} v^\varepsilon p^\varepsilon - p^\varepsilon \operatorname{div} v^\varepsilon \, dy\, dz$$

$$+ \int\limits_{\omega_-} (\partial_z v^\varepsilon \cdot v^\varepsilon + \varepsilon^2 \partial_z p^\varepsilon p^\varepsilon + p^\varepsilon v_3^\varepsilon)(y,-T)dy$$

$$- \int\limits_{\omega_+} (\partial_z v^\varepsilon \cdot v^\varepsilon + \varepsilon^2 \partial_z p^\varepsilon p^\varepsilon + p^\varepsilon v_3^\varepsilon)(y,T)dy.$$

Passing to the limit $T \to \infty$, the first integral converges, while the second one obviously vanishes for all T and the integrals over the cross sections tend to 0 with the arguments used above. This finally leads to

$$0 = \int_\Omega |\nabla v^\varepsilon|^2 + \varepsilon^2 |\nabla p^\varepsilon|^2 dx,$$

which implies $v^\varepsilon = 0$ due to $v^\varepsilon|_{\partial\Omega} = 0$ and $p^\varepsilon = 0$, since $p^\varepsilon|_{Q_+} \in W^{l+1}_\beta(Q_+)$ with $\beta > 0$. To see the existence, we use Theorem 3.3 to construct an *almost inverse operator*, i.e., we define

$$B^\varepsilon_{ap} : \mathcal{R}^l_{\varkappa,\beta} W(\Omega;\varepsilon) \times \mathbb{R}^2 \to \mathfrak{D}^{l,\varepsilon}_{\varkappa,\beta} W(\Omega;\varepsilon)$$

with

$$\mathbf{A}^\varepsilon B^\varepsilon_{ap}(f, g, a_+, b_+) = (f, g, a_+, b_+) + R(f, g, a_+, b_+),$$

where R is a small operator. Then we use the Neumann series to find the solution $u^\varepsilon = B^\varepsilon_{ap}(\mathbb{I} + R)^{-1}(f, g, a_+, b_+)$, here $\mathbb{I}$ denotes the identity operator on $\mathcal{R}^l_{\varkappa,\beta} W(\Omega;\varepsilon) \times \mathbb{R}^2$. Due to the different structures of (S_0) and (S_ε), we cannot directly employ $B^\varepsilon_{ap} = (\mathbf{A}^0)^{-1}$. Instead of this we operate with the quasiparametrix P_ε in order to reconstruct the necessary smoothness properties of $u^\varepsilon_{ap} = B^\varepsilon_{ap}(f, g, a_+, b_+)$. Let $(f, g) \in \mathcal{R}^l_{\varkappa,\beta} W(\Omega;\varepsilon)$ and $\Phi = (a_+, b_+) \in \mathbb{R}^2$ be fixed. We put

$$u^\varepsilon_{ap} = u^{1,\varepsilon} + u^{2,\varepsilon} + u^{3,\varepsilon},$$

where, with the notations of Lemma 3.4,

$$u^{1,\varepsilon} = P_\varepsilon(f, g), \quad u^{2,\varepsilon} = P_\varepsilon T_\varepsilon(f, g).$$

The embedding properties of the $W^l_{\varkappa,\beta}$-spaces and Lemma 3.4 lead to the following estimates independent of $\varepsilon \in (0, 1]$:

$$\|u^{1,\varepsilon}; \mathcal{D}^l_{\varkappa,\beta} W(\Omega;\varepsilon)\| \le C\, \|(f, g); \mathcal{R}^l_{\varkappa,\beta} W(\Omega;\varepsilon)\|, \tag{3.28}$$

$$\begin{aligned} \|T_\varepsilon(f, g); \mathcal{R}^{l+1}_{\varkappa,\beta} W(\Omega;\varepsilon)\| &\le C\, \|T_\varepsilon(f, g); \mathcal{R}^{l+1}_{\varkappa+1,\beta} W(\Omega;\varepsilon)\| \\ &\le C\, \|(f, g); \mathcal{R}^l_{\varkappa,\beta} W(\Omega;\varepsilon)\|, \end{aligned} \tag{3.29}$$

$$\begin{aligned} &\|u^{2,\varepsilon}; \mathcal{D}^l_{\varkappa,\beta} W(\Omega;\varepsilon)\| \le C\, \|u^{2,\varepsilon}; \mathcal{D}^{l+1}_{\varkappa,\beta} W(\Omega;\varepsilon)\| \\ &\le C\, \|T_\varepsilon(f, g); \mathcal{R}^{l+1}_{\varkappa,\beta} W(\Omega;\varepsilon)\| \le C\, \|(f, g); \mathcal{R}^l_{\varkappa,\beta} W(\Omega;\varepsilon)\|, \end{aligned} \tag{3.30}$$

$$\|T_\varepsilon T_\varepsilon(f,g);\mathcal{R}^{l+2}_{\varkappa+1,\beta}W(\Omega;\varepsilon)\| \le C\,\|T_\varepsilon(f,g);\mathcal{R}^{l+1}_{\varkappa,\beta}W(\Omega;\varepsilon)\|$$

$$\le C\,\|(f,g);\mathcal{R}^{l}_{\varkappa,\beta}W(\Omega;\varepsilon)\|. \qquad (3.31)$$

To define $u^{3,\varepsilon}$, let $\mathbb{P}':\mathbb{R}^8\to\mathbb{R}^7$ denote the canonical projection, i.e., $\mathbb{P}'(\xi_1,\dots,\xi_8)=(\xi_1,\dots,\xi_7)$. We put $T'_\varepsilon(f,g)(x)=\mathbb{P}'T_\varepsilon(f,g)(x)$ (this means we omit the last component in the boundary values). Recalling the definitions of $\mathcal{R}^{l+2}_{\varkappa+1,\beta}W(\Omega;\varepsilon)$ (see (3.8)) and of $\mathbf{R}^{l+1}_{\varkappa+1,\beta}W(\Omega;\varepsilon)$ (see (3.20)) we find

$$\|T'_\varepsilon T_\varepsilon(f,g);\mathbf{R}^{l+1}_{\varkappa+1,\beta}W(\Omega;\varepsilon)\le\|T_\varepsilon T_\varepsilon(f,g)\mathcal{R}^{l+2}_{\varkappa+1,\beta}W(\Omega;\varepsilon)\|. \qquad (3.32)$$

We put $u^{3,\varepsilon}=(\mathbf{A}^0)^{-1}\{T'_\varepsilon T_\varepsilon(f,g),\Phi\}$, then $u^{3,\varepsilon}$ is of the form (3.11), and from (3.21) it follows now

$$|a_-|+|b_-|+\|\tilde{u}^{3,\varepsilon};\mathbf{D}^{l+1}_{\varkappa+1,\beta}W(\Omega;\varepsilon)\|$$

$$\le C\,\Big(\,\|T'_\varepsilon T_\varepsilon(f,g);\mathbf{R}^{l+1}_{\varkappa+1,\beta}W(\Omega;\varepsilon)\|+|\Phi|\,\Big), \qquad (3.33)$$

here we observe that there was no upper bound for $\varkappa$ required in Theorem 3.5. On the other hand, with

$$\mathbf{D}^{l+1}_{\varkappa+1,\beta}W(\Omega;\varepsilon)\subset\mathcal{D}^{l}_{\varkappa+1,\beta}W(\Omega;\varepsilon)\subset\mathbf{D}^{l}_{\varkappa,\beta}W(\Omega;\varepsilon)$$

we obtain with (3.33), (3.32) and (3.31):

$$\|u^{3,\varepsilon};\mathfrak{D}^{l,\varepsilon}_{\varkappa,\beta}W(\Omega;\varepsilon)\|\le C\|u^{3,\varepsilon};\mathfrak{D}^{l+1,0}_{\varkappa+1,\beta}W(\Omega;\varepsilon)\|$$

$$\le C\,\Big(\,\|(f,g);\mathcal{R}^{l}_{\varkappa,\beta}W(\Omega;\varepsilon)\|+|\Phi|\,\Big) \qquad (3.34)$$

independent of $\varepsilon\in(0,1]$. Collecting (3.28), (3.30) and (3.34) we get

$$\|u^\varepsilon_{ap};\mathfrak{D}^{l,\varepsilon}_{\varkappa,\beta}W(\Omega;\varepsilon)\|\le C\,\Big(\,\|(f,g);\mathcal{R}^{l}_{\varkappa,\beta}W(\Omega;\varepsilon)\|+|\Phi|\,\Big)\,.$$

Now we process the discrepancy $(f,g,\Phi)-\mathbf{A}^\varepsilon u^\varepsilon_{ap}$. Since the equations $(\mathrm{S}_\varepsilon)_{1,2,3}$ and the boundary condition $v^\varepsilon=g'$ as well as the flux condition are completely served by $u^{3,\varepsilon}$, the residuum appears only in the last equations:

$$f-S_\varepsilon u^\varepsilon_{ap}=(0,\varepsilon^2\Delta p^{3,\varepsilon})\quad\text{in}\quad\Omega,$$

$$g-B_\varepsilon u^\varepsilon_{ap}=(0,T_{\varepsilon,8}T_\varepsilon(f,g)-\partial_n p^{3,\varepsilon})\quad\text{on}\quad\partial\Omega,$$

where $T_{\varepsilon,8}(\mathfrak{f},\mathfrak{g}) = \mathfrak{g}_4$. Here we have (compare with (2.2), (2.3)

$$\|\varepsilon^2 \Delta p^{3,\varepsilon}; W^{l-1}_{\varkappa,\beta}(\Omega;\varepsilon)\| \le C\,\varepsilon \|p^{3,\varepsilon}; W^{l+1}_{\varkappa+1,\beta}(\Omega;\varepsilon)\|,$$

while in virtue of (2.4),

$$\|\partial_n p^{3,\varepsilon}; \mathcal{W}^{l-1/2}_{\varkappa-3/2,\beta}(\partial\Omega;\varepsilon)\| = \varepsilon^{3/2-\varkappa}\|\partial_n p^{3,\varepsilon}; \mathcal{W}^{l-1/2}_{0,\beta}(\partial\Omega;\varepsilon)\|$$

$$\le C\,\varepsilon\,\|\partial_n p^{3,\varepsilon}; \mathcal{W}^{l-1/2}_{\varkappa-1/2,\beta}(\partial\Omega;\varepsilon)\| \le C\,\varepsilon\,\|p^{3,\varepsilon}; W^{l+1}_{\varkappa+1,\beta}(\Omega;\varepsilon)\|,$$

$$\|T_{\varepsilon,8}T_\varepsilon(f,g); \mathcal{W}^{l-1/2}_{\varkappa-3/2,\beta}(\partial\Omega;\varepsilon)\| \le C\,\varepsilon\,\|T_{\varepsilon,8}T_\varepsilon(f,g); \mathcal{W}^{l-1/2}_{\varkappa-1/2,\beta}(\partial\Omega;\varepsilon)\|$$

$$\le C\,\varepsilon\,\|T_\varepsilon T_\varepsilon(f,g); \mathcal{R}^{l+2}_{\varkappa+1,\beta}W(\Omega;\varepsilon)\|.$$

We outline again that the constants C are independent of $\varepsilon \in (0,1]$ and the data in all the preceding inequalities, collecting the last three estimates and (3.31), we find $\varepsilon_0 > 0$ such that the residuum $(f,g,\Phi) - \mathbf{A}^\varepsilon u^\varepsilon_{ap}$ is small independent of $\varepsilon \in (0,\varepsilon_0]$, which finishes the proof of the theorem. □

Now we have prepared the frame to compare the solutions to the linear problems.

Theorem 3.6. *Let $\varkappa, l, \beta, \beta_0$ and ε_0 be fixed as in Theorem 3.5. For $(f,g') \in \mathbf{R}^l_{\varkappa,\beta}W(\Omega;\varepsilon)$ and $\Phi \in \mathbb{R}^2$, let $u^\varepsilon \in \mathfrak{D}^{l,\varepsilon}_{\varkappa,\beta}W(\Omega;\varepsilon)$, $u^0 \in \mathfrak{D}^{l,0}_{\varkappa,\beta}W(\Omega;\varepsilon)$ be the unique solution to $S_\varepsilon u^\varepsilon = f$, $B_\varepsilon u^\varepsilon = (g',0)$ and $S_0 u^0 = f$, $B_0 u^0 = g'$, respectively, together with the asymptotic conditions $(a^\varepsilon_+, b^\varepsilon_+) = \Phi$ for $\varepsilon \in [0,\varepsilon_0]$. Then the following error estimate holds true:*

$$|a^0_- - a^\varepsilon_-| + |b^0_- - b^\varepsilon_-| + \|\tilde u^0 - \tilde u^\varepsilon; \mathbf{D}^{l-1}_{0,\beta}W(\Omega;\varepsilon)\|$$

$$\le C\varepsilon^\varkappa \left(\|(f,g'); \mathbf{R}^l_{\varkappa,\beta}W(\Omega;\varepsilon)\| + |\Phi| \right). \tag{3.35}$$

Remark. With some additional efforts, the results can be extended to $\varkappa \le l$, which requires less regularity properties of the data.

Proof. Since $W^l_{\varkappa,\beta}(\Omega;\varepsilon) \subset W^{l-1}_{\varkappa,\beta}(\Omega;\varepsilon)$, we have $(f,(g',0)) \in \mathcal{R}^l_{\varkappa,\beta}W(\Omega;$ ε. The difference $u^0 - u^\varepsilon \in \mathfrak{D}^{l,0}_{\varkappa,\beta}W(\Omega;\varepsilon) \subset \mathfrak{D}^{l,0}_{0,\beta}W(\Omega;\varepsilon)$ solves the Stokes problem

$$S_0(u^0 - u^\varepsilon) = (0, \varepsilon^2 \Delta p^\varepsilon), \quad B_0(u^0 - u^\varepsilon) = 0,$$

while the coefficients in front of $U^{1,+}$, $U^{2,+}$ vanish. Since the main asymptotic parts $P^{h,-}$, $h=1,2$, of the pressure are harmonic functions for $|z|>R_0$ with a suitable $R_0>\max\{R_-,R_+\}$, it follows $\Delta p^\varepsilon=\Delta(a_-^\varepsilon P^{1,-}+b_-^\varepsilon P^{2,-})+\Delta\tilde{p}^\varepsilon\in W_\beta^{l-1}(\Omega)$. We estimate

$$\|\varepsilon^2\Delta\tilde{p}^\varepsilon;W_{0,\beta}^{l-1}(\Omega;\varepsilon)\|\le\varepsilon^2\sum_{j=0}^{l-1}\varepsilon^j\|\nabla^j\Delta\tilde{p}^\varepsilon;W_\beta^0(\Omega)\|$$

$$\le\varepsilon^\varkappa\sum_{j=2}^{l+1}\varepsilon^{j-\varkappa}\|\nabla^j\tilde{p}^\varepsilon;W_\beta^0(\Omega)\|\le C\varepsilon^\varkappa\|\tilde{p}^\varepsilon;W_{\varkappa,\beta}^{l+1}(\Omega;\varepsilon)\|$$

$$\le C\varepsilon^\varkappa(\|(f,(g',0));\mathcal{R}_{\varkappa,\beta}^l W(\Omega;\varepsilon)\|+|\Phi|)$$

$$\le C\varepsilon^\varkappa(\|(f,g');\mathbf{R}_{\varkappa,\beta}^l W(\Omega;\varepsilon)\|+|\Phi|),\tag{3.36}$$

here we used Theorem 3.5. Even simpler arguments prove

$$\|\varepsilon^2\Delta(a_-^\varepsilon P^{1,-}+b_-^\varepsilon P^{2,-});W_{0,\beta}^{l-1}(\Omega;\varepsilon)\|\le\varepsilon^\varkappa(|a_-^\varepsilon|+|b_-^\varepsilon|)$$

$$\le C\varepsilon^\varkappa(\|(f,g');\mathbf{R}_{\varkappa,\beta}^l W(\Omega;\varepsilon)\|+\Phi).$$

Now Theorem 3.1 gives the assertion. □

Remark. If we look at the stable parts of the norm $\|\,.\,;\mathbf{D}_{0,\beta}^{l-1}W(\Omega;\varepsilon)\|$, we always have

$$|a_-^0-a_-^\varepsilon|+|b_-^0-b_-^\varepsilon|+\|(\tilde{v}^0-\tilde{v}^\varepsilon)e^{\beta\mathfrak{z}};H^1(\Omega)\|$$

$$+\|(\tilde{p}^0-\tilde{p}^\varepsilon)e^{\beta\mathfrak{z}};L^2(\Omega)\|=O(\varepsilon^\varkappa),$$

where $\varkappa$ is arbitrarily close to $3/2$, if the data are regular enough. If $1<\varkappa<3/2$, then even

$$\|\nabla^2(\tilde{v}^0-\tilde{v}^\varepsilon)e^{\beta\mathfrak{z}};L^2(\Omega)\|+\|\nabla(\tilde{p}^0-\tilde{p}^\varepsilon)e^{\beta\mathfrak{z}};L^2(\Omega)\|=O(\varepsilon^{\varkappa-1})$$

is valid.

4. The nonlinear stationary problem

The nonlinear Navier–Stokes problem (NS_0), as well as the perturbed problem (NS_ε) have the same structure, they can be written in the form

$$\mathbf{A}\mathbf{x}+\mathbf{B}(\mathbf{x},\mathbf{x})=\mathbf{y},\tag{4.1}$$

where $\mathbf{A}:\mathcal{D}\to\mathcal{R}$ is an isomorphism between two Banach spaces $\mathcal{D}$ and $\mathcal{R}$, and $\|\mathbf{A}^{-1}\|\le a$, while $\mathbf{B}:\mathcal{D}\times\mathcal{D}\to\mathcal{R}$ is bilinear and bicontinuous, i.e., $\|\mathbf{B}(\mathbf{x},\tilde{\mathbf{x}});\mathcal{R}\|\le b\|\mathbf{x};\mathcal{D}\|\,\|\tilde{\mathbf{x}};\mathcal{D}\|$. Then for any $\mathbf{y}\in\mathcal{R}$ with $\|\mathbf{y};\mathcal{R}\|<2a^2b^{-1}$ there exists a unique solution $\mathbf{x}$ to (4.1) in the ball $\|\mathbf{x};\mathcal{D}\|<2ab^{-1}$ and this solution fulfills $\|\mathbf{x};\mathcal{D}\|<2a\|\mathbf{y};\mathcal{R}\|$. In [8, p. 209] it was shown, that for u, $\hat{u}$ of the asymptotic form (3.11), the term $(v\cdot\nabla)\hat{v}\in W^{l-1}_\beta(\Omega)$ and

$$\|(v\cdot\nabla)\hat{v};W^{l-1}_\beta(\Omega)\|$$

$$\le C\Big(\|\tilde{v};W^{l+1}_\beta(\Omega)\|+|b_+|+|b_-|\Big)\Big(\|\tilde{\hat{v}};W^{l+1}_\beta(\Omega)\|+|\hat{b}_+|+|\hat{b}_-|\Big).$$

The arguments can be repeated word by word, if $(v\cdot\nabla)\hat{v}$ is replaced by the bilinear operator $B^2(v,\hat{v})$ defined after formula (NS$_\varepsilon$). For fixed $\varepsilon>0$, the abstract scheme from above gives the existence of a unique (small) strong solution to (NS$_\varepsilon$) for small data. However, we want to have (NS$_0$), and (NS$_\varepsilon$) simultaneously solvable for $0<\varepsilon<\varepsilon_0$ at least. This means we apply the abstract scheme to $\mathcal{D}=\mathfrak{D}^{l,0}_{\varkappa,\beta}(\Omega;\varepsilon)$, $\mathcal{R}=\mathbf{R}^l_{\varkappa,\beta}(\Omega;\varepsilon)\times\mathbb{R}^2$ and $\mathcal{D}=\mathfrak{D}^{l,\varepsilon}_{\varkappa,\beta}(\Omega;\varepsilon)$, $\mathcal{R}=\mathcal{R}^l_{\varkappa,\beta}(\Omega;\varepsilon)\times\mathbb{R}^2$, respectively, the corresponding isomorphisms are $\mathbf{A}^0$ and $\mathbf{A}^\varepsilon$, respectively, and Theorem 3.3 and 3.5 imply $\max\{\|\mathbf{A}^\varepsilon\|,\|\mathbf{A}^0\|\}\le a$ independent of $\varepsilon\in(0,1]$. It remains to estimate the operator $\mathbf{B}(u,\hat{u})=((B(v,\hat{v},0),0,0)$, the zeros indicate that neither the divergence equation nor the boundary values or the conditions at infinity are influenced by the nonlinear term.

Lemma 4.1. *Let $1\le\varkappa<3/2$, $l\geqslant\varkappa+1$, $0<\beta<\beta_0$, $B=B^1$ or $B=B^2$, then for $u,\hat{u}\in\mathfrak{D}^{l,\varepsilon}_{\varkappa,\beta}(\Omega;\varepsilon)$ (or $\mathfrak{D}^{l,0}_{\varkappa,\beta}(\Omega;\varepsilon)$*

$$\|B(v,\hat{v});W^{l-1}_{\varkappa-1,\beta}(\Omega;\varepsilon)\|\le b\|u;\mathfrak{D}^{l,\varepsilon}_{\varkappa,\beta}(\Omega;\varepsilon)\|\,\|\hat{u};\mathfrak{D}^{l,\varepsilon}_{\varkappa,\beta}(\Omega;\varepsilon)\| \qquad (4.2)$$

with a constant b independent of u, $\hat{u}$, and $\varepsilon\in(0,1]$.

Proof. Clearly, we have

$$\|\,.\,;W^{l-1}_{\varkappa-1,\beta}W(\Omega;\varepsilon)\|\simeq\|\,.\,;H^{l-1}_{\varkappa-1}(Q_0;\varepsilon)\|$$

$$+\|\,.\,;W^{l-1}_{\varkappa-1,\beta}(Q_+;\varepsilon)\|+\|\,.\,;W^{l-1}_{\varkappa-1,\beta}(Q_-;\varepsilon)\|.$$

In [9] the inequality

$$\|B(v,\hat{v});H^{l-1}_{\varkappa-1}(Q_0;\varepsilon)\|\le b_0\|u;H^{l+1}_{\varkappa+1}(Q_0;\varepsilon)\|\|\hat{u};H^{l+1}_{\varkappa+1}(Q_0;\varepsilon)\|$$

was already shown. We extend the arguments to the weighted norms in the outlets, and undoubtedly it is enough to consider Q_+, e.g., For $z > R_0$, we have $v = b_+ v^{2,+} + \tilde{v}$, and $\hat{v} = \hat{b}_+ v^{2,+} + \tilde{\hat{v}}$, and due to the special structure of the Poisseuille flow:

$$B(v, \hat{v}) = B(\tilde{v}, \tilde{\hat{v}}) + \hat{b}_+ B(\tilde{v}, v^{2,+}) + b_+ B(v^{2,+}, \tilde{\hat{v}})$$

The task is now to estimate

$$\sum_{k=0}^{l-1} \varepsilon^{(k-(\varkappa-1))_+} \|\nabla^k B(v, \hat{v}) e^{\beta \mathfrak{z}}; L^2(Q_+)\| + \|B(v, \hat{v}) e^{\beta \mathfrak{z}}; H^{\varkappa-1}(Q_+)\|.$$

The stable part was already treated in [8, Lemma 11.1] for integer $\varkappa$, it can be extended to non integer $\varkappa$ by standard interpolation arguments, thus, we have already

$$\|B(v, \hat{v}); W_\beta^{\varkappa-1}(Q_+)\|$$
$$\leq b_1 \Big(|b_+| + \|\tilde{v}; W_\beta^{\varkappa+1}(Q_+)\| \Big) \Big(|\hat{b}_+| + \|\tilde{\hat{v}}; W_\beta^{\varkappa+1}(Q_+)\| \Big). \tag{4.3}$$

To estimate the other terms, we apply embedding theorems and obtain for $k \leq l-1$:

$$\sup_{x \in Q_+} |\nabla^k \tilde{v}(x)| \leq c \|\nabla^k \tilde{v}; H^2(Q_+)\| \leq c \|\tilde{v}; W_\beta^{k+2}(Q_+)\|. \tag{4.4}$$

Furthermore, we have even for $k \leq l+1$:

$$\sup_{x \in Q_+} |\nabla^k v^{2,+}(x)| \leq c(\omega_+). \tag{4.5}$$

Leibnitz' rule for differentiating products leads to

$$\nabla^k B(v, \hat{v})(x)| \leq c \sum_{\nu=0}^{k} \Big(|\nabla^\nu v(x)| \, |\nabla^{k-\nu+1} \hat{v}| + |\nabla^{\nu+1} v(x)| \, |\nabla^{k-\nu} \hat{v}(x)| \Big),$$

where c is independent on v, $\hat{v}$ and x, and the second term appears only if $B = B^2$. We carry out the details for the first summand, the arguments for the second run in an analogous way. By (4.4) and (4.5), we get

$$\| |\nabla^\nu v| \, |\nabla^{k-\nu+1} \hat{v}|; W_\beta^0(Q_+)\|$$
$$\leq c(|b_+| + \|\tilde{v}; W_\beta^{\nu+2}(Q_+)\|)(|\hat{b}_+| + \|\tilde{\hat{v}}; W^{k-\nu+1}(Q_+)\|).$$

We multiply this inequality by $\varepsilon^{(k-(\varkappa-1))_+}$ and obtain

$$\varepsilon^{(k-(\varkappa-1))_+}\|\nabla^\nu v(x)|\,|\nabla^{k-\nu+1}\hat v|;W^0_\beta(Q_+)\|$$

$$\le C\varepsilon^{(k-(\varkappa-1))_+}(|b_+|+\varepsilon^{-(\nu+2-(\varkappa+1))_+}\varepsilon^{(\nu+2-(\varkappa+1))_+}\|\tilde v;W^{\nu+2}_\beta(Q_+)\|)$$

$$\times(|\hat b_+|+\varepsilon^{-(k-\nu+1-(\varkappa+1))_+}\varepsilon^{(k-\nu+1-(\varkappa+1))_+}\|\tilde v;W^{k-\nu+1}_\beta(Q_+)\|). \quad (4.6)$$

We are ready if $\varepsilon^{(k-(\varkappa-1))_+}$ compensates for the negative powers of ε, i.e., if

$$\sigma := (k-(\varkappa-1))_+ - (\nu+1-\varkappa)_+ - (k-\nu-\varkappa)_+ =:$$
$$=: \sigma_1-\sigma_2-\sigma_3 \geqslant 0.$$

Recalling $\nu\le k$, the following implications are easy to check

$$\begin{aligned}
&\sigma_i>0 \quad \text{for all } i \Rightarrow \sigma=\varkappa>0,\\
&\sigma_1=0 \qquad\qquad\quad \Rightarrow \sigma_2=\sigma_3=0,\\
&\sigma_1>0 \ \text{ and } \ \sigma_2=0 \Rightarrow \nu\in\{0,1\}\Rightarrow\sigma\geqslant 0,\\
&\sigma_1>0 \ \text{ and } \ \sigma_3=0 \Rightarrow \sigma\geqslant 0.
\end{aligned}$$

Collecting inequalities (4.3) and (4.6) gives (4.2). □

We use this result to prove our final conclusion for the nonlinear problem. For

$$(f,g')\in W^{l-1}_\beta(\Omega)^3\times W^l_\beta(\Omega)\times W^{l+1/2}_\beta(\Omega)^3,$$

we recall the following inequalities

$$\|(f,(g',0));\mathcal{R}^l_{\varkappa,\beta}(\Omega;\varepsilon)\|\le\|(f,g');\mathbf{R}^l_{\varkappa,\beta}(\Omega;\varepsilon)\|$$

$$\le\|(f,g');W^{l-1}_\beta(\Omega)^3\times W^l_\beta(\Omega)\times W^{l+1/2}_\beta(\partial\Omega)^3\|. \quad (4.7)$$

We take (f,g') as data for (NS_0), and $(f,(g',0))$ for $(\mathrm{NS}_\varepsilon)$, respectively, and we look for a solution $u^0\in\mathfrak{D}^{l,0}_{\varkappa,\beta}W(\Omega;\varepsilon)$ to (NS_0) and for a solution $u^\varepsilon\in\mathfrak{D}^{l,\varepsilon}_{\varkappa,\beta}W(\Omega;\varepsilon)$ to $(\mathrm{NS}_\varepsilon)$, both problems we supply with the condition at infinity

$$(a_+,b_+)=\Phi, \quad \text{with} \quad \Phi\in\mathbb{R}^2. \quad (4.8)$$

Theorem 4.2. *Let $\varkappa$, β and l as in Lemma 4.1, ε_0 the constant of Theorem 3.5. Then there exist ρ_0 and ρ_1 with the following properties: If (f, g', Φ) fulfils the smallness condition*

$$\|(f, g'); W_\beta^{l-1}(\Omega)^3 \times W_\beta^l(\Omega) \times W_\beta^{l+1/2}(\Omega)^3\| + |\Phi| \le \rho_0, \tag{4.9}$$

then there exists a unique solution u^0 with $\|u^0; \mathfrak{D}_{\varkappa,\beta}^{l,0}(\Omega;\varepsilon)\| \le \rho_1$. Simultanuously, for all $\varepsilon \in (0, \varepsilon_0]$ there exists a unique solution u^ε to $(\mathrm{NS}_\varepsilon)$ with $\|u^\varepsilon; \mathfrak{D}_{\varkappa,\beta}^{l,\varepsilon}(\Omega;\varepsilon)\| \le \rho_1$. Moreover, the solutions fulfil

$$\|u^0; \mathfrak{D}_{\varkappa,\beta}^{l,0}(\Omega;\varepsilon)\| \le C\|(f, g'); \mathbf{R}_{\varkappa,\beta}^l(\Omega;\varepsilon)\| + |\Phi|), \tag{4.10}$$

$$\|u^\varepsilon; \mathfrak{D}_{\varkappa,\beta}^{l,\varepsilon}(\Omega;\varepsilon)\| \le C\|(f, (g', 0); \mathcal{R}_{\varkappa,\beta}^l(\Omega;\varepsilon)\| + |\Phi|), \tag{4.11}$$

with C independent of ε and the data. The difference fulfils the error estimate of Theorem 3.6, and in particular

$$|a_-^\varepsilon - a_-^0| + |b_-^\varepsilon - b_-^\varepsilon| + \|\tilde{v}^\varepsilon - \tilde{v}^0; W_\beta^2(\Omega)\| + \|\tilde{p}^\varepsilon - \tilde{p}^0; W_\beta^1(\Omega)\|$$

$$\le C\varepsilon^{\varkappa-1}\Big(\|f'; W_\beta^2(\Omega)^3\| + \|f_4; W_\beta^3(\Omega)\| + \|g'; W_\beta^{5/2}(\partial\Omega)\| + |\Phi|\Big), \tag{4.12}$$

Remark 4.3. The smoothness indices for the right hand sides in (4.12) result from the assumption $l \geqslant \varkappa + 1$, and $\varkappa \geqslant 1$. With additional technical efforts, each of them can be diminished by one.

Proof. The existence of unique small solutions for small data, together with the estimates (4.10) and (4.11), follows with Lemma 4.1 by using the abstract scheme outlined at the beginning of this section. Due to (4.7) this reasonings can be performed independent of ε. It remains to prove the error estimate (3.35). Following [9], we put $r^\varepsilon = v^\varepsilon - v^0$, $q^\varepsilon = p^\varepsilon - p^0$. The error $(r^\varepsilon, q^\varepsilon)$ solves the following perturbed Stokes problem in $\mathfrak{D}_{\varkappa,\beta}^{l,0}(\Omega;\varepsilon)$:

$$-\Delta r^\varepsilon + \nabla q^\varepsilon + B(v^\varepsilon, r^\varepsilon) - B(r^\varepsilon, v^0) = 0, \quad \operatorname{div} r^\varepsilon = \varepsilon^2 \Delta p^\varepsilon \text{ in } \Omega,$$

$$r^\varepsilon = 0 \text{ on } \partial\Omega, \qquad (a_+(r^\varepsilon, q^\varepsilon), b_+(r^\varepsilon, q^\varepsilon)) = (0, 0). \tag{4.13}$$

Analyzing the proof of the error estimate in the linear case, we used two main arguments to estimate the error (r, q):

$$\|(r, q); \mathbf{D}(\varepsilon)\| \le C\|\mathbf{A}^0(r, q), \mathbf{R}(\varepsilon)\|, \tag{4.14}$$

where $\mathbf{A}^0 : \mathbf{D}(\varepsilon) \to \mathbf{R}(\varepsilon)$ is an isomorphism with $\|\mathbf{A}^0\|, \|(\mathbf{A}^0)^{-1}\| \le a$ independent of ε. Then we proved

$$\|\mathbf{A}^0(r, q); \mathbf{R}(\varepsilon)\| \le C\varepsilon^\varkappa \|(f, g'); \widetilde{\mathbf{R}}(\varepsilon)\| \tag{4.15}$$

with

$$\mathbf{D}(\varepsilon) = \mathfrak{D}^{l-1,0}_{0,\beta} W(\Omega;\varepsilon), \quad \mathbf{R}(\varepsilon) = \mathbf{R}^{l-1}_{0,\beta} W(\Omega;\varepsilon) \times \mathbb{R}^2,$$

and

$$\widetilde{\mathbf{R}}(\varepsilon) = \mathbf{R}^{l}_{\varkappa,\beta} W(\Omega;\varepsilon) \times \mathbb{R}^2.$$

Here we combine these arguments with a classical perturbation argument. To this end, we show that the norm of the perturbation operator

$$\mathbf{P} : \mathbf{D}(\varepsilon) \to \mathbf{R}(\varepsilon); (r,q) \mapsto ((B(v^\varepsilon, r) - B(r, v^0), 0), 0, 0)$$

can be controlled by the data, i.e.,

$$\|\mathbf{P} : \mathbf{D}(\varepsilon) \to \mathbf{R}(\varepsilon)\| \le C\Big(\|(f,g'); \mathbf{R}^l_{\varkappa,\beta}(\Omega;\varepsilon)\| + |\Phi|\Big) := C\mathfrak{F}. \tag{4.16}$$

The operator $\mathbf{A}^0 + \mathbf{P} : \mathbf{D}(\varepsilon) \to \mathbf{R}(\varepsilon)$ keeps the properties of $\mathbf{A}^0$ if $\mathfrak{F}$ is small enough, which can be achieved by eventually diminishing the constant ρ_0. To show (4.16) we have to estimate $\|B(v^\varepsilon, r) - B(r, v^0); W^{l-2}_{-1,\beta}(\Omega;\varepsilon)\|$. We notice first that, for any $w \in W^{l-2}_\beta(\Omega)$,

$$\begin{aligned}\|w; W^{l-2}_{-1,\beta}(\Omega;\varepsilon)\| &= \|w; W^{-1}_\beta(\Omega)\| + \varepsilon\|w; W^{l-2}_{0,\beta}(\Omega;\varepsilon)\| \\ &\le C\Big(\|w; W^0_\beta(\Omega)\| + \varepsilon\|w; W^{l-2}_{0,\beta}(\Omega;\varepsilon)\|\Big),\end{aligned}$$

here we utilized $l \geqslant 2$ and the embedding $W^0_\beta(\Omega) \hookrightarrow W^{-1}_\beta(\Omega)$. Now we follow similar reasonings as in the proof of Lemma 4.1. Recall again the asymptotic representation (3.11), which reads for the velocity parts

$$\begin{gathered} v^\varepsilon = b^\varepsilon_+ V^{2,+} + b^\varepsilon_- V^{2,-} + \tilde{v}^\varepsilon, \quad \varepsilon \in [0, \varepsilon_0], \\ r = b_+(r) V^{2,+} + b_-(r) V^{2,-} + \tilde{r} \quad \text{with} \quad \tilde{r} \in W^l_{\varkappa+1,\beta}(\Omega), \\ |b^\varepsilon_+| + |b^\varepsilon_-| + \|\tilde{v}^\varepsilon; W^{l+1}_{\varkappa+1,\beta}(\Omega;\varepsilon)\| \le C\mathfrak{F}, \end{gathered} \tag{4.17}$$

the last estimate holds true for v^ε, $\varepsilon > 0$ as well as for v^0. Since $\varkappa \geqslant 1$, we have

$$\|\tilde{v}^\varepsilon; W^2_\beta(\Omega)\| \le \|\tilde{v}^\varepsilon; W^{l+1}_{\varkappa+1,\beta}(\Omega;\varepsilon)\|,$$

thus, we may apply (4.4) and (4.5) to see that

$$\|B(v^\varepsilon, r); W^0_\beta(\Omega)\| \le C\Big(\sup_{x\in\Omega} |v^\varepsilon(x)| \|\tilde{r}; W^1_\beta(\Omega)\|$$

$$+\|\tilde{v}^\varepsilon; W^1_\beta(\Omega)(|b_-(r)| + |b_+(r)| + \|\tilde{r}; W^1_\beta(\Omega)\|) + (|b^\varepsilon_+|$$

$$+|b_-^\varepsilon|)\|\tilde r; W^0_\beta(\Omega)\|\Big) \tag{4.18}$$

$$\leq C\Big(|b_+^\varepsilon| + |b_-^\varepsilon| + \|\tilde v^\varepsilon; W^{l+1}_{\varkappa+1,\beta}W(\Omega;\varepsilon)\|\Big)\|(r,0); \mathfrak{D}^{l-1,0}_{0,\beta}(\Omega;\varepsilon)\|.$$

In a similar manner it follows

$$\|B(r,v^0); W^0_\beta(\Omega)\|$$

$$\leq C\Big(|b_+^0| + |b_-^0| + \|\tilde v^0; W^{l+1}_{\varkappa+1,\beta}W(\Omega;\varepsilon)\|\Big)\|(r,0); \mathfrak{D}^{l-1,0}_{0,\beta}(\Omega;\varepsilon)\|. \tag{4.19}$$

It remains to investigate

$$\varepsilon^{k+2}\|\nabla^k(B(v^\varepsilon,r) - B(r,v^0)); W^0_\beta(\Omega)\| \quad \text{for} \quad k \leq l-2.$$

Like in the proof of Lemma 4.1 we end up with

$$\|\nabla^k B(v^\varepsilon,r)W^0_\beta(\Omega)\| \leq C\sum_{\nu=0}^{k}\Big(|b_+^\varepsilon| + |b_-^\varepsilon| + \|\tilde v^\varepsilon; W^{\nu+2}_\beta W(\Omega)\|\Big)$$

$$\times\Big(|b_-(r)| + |b_+(r)| + \|\tilde r; W^{k+1-\nu}_\beta(\Omega)\|\Big).$$

Now we multiply this inequality by ε^{k+2} and calculate

$$\varepsilon^{k+2}\|\nabla^k B(v^\varepsilon,r)W^0_\beta(\Omega)\| \leq$$

$$C\varepsilon^{k+2}\sum_{\nu=0}^{k}\Big(|b_+^\varepsilon|+|b_-^\varepsilon|+\varepsilon^{-(\nu+2-(\varkappa+1))_+}\varepsilon^{(\nu+2-(\varkappa+1))_+}\|\tilde v^\varepsilon; W^{\nu+2}_\beta W(\Omega)\|\Big)$$

$$\times\Big(|b_-(r)| + |b_+(r)| + \varepsilon^{-(k+1-\nu)}\varepsilon^{k+1-\nu}\|\tilde r; W^{k+1-\nu}_\beta(\Omega)\|\Big).$$

Again an elementary calculation shows that ε^{k+2} compensates for the negative powers of ε, whence we finally obtain with (4.17) and (4.18):

$$\|B(v^\varepsilon,r); W^{l-2}_{-1,\beta}(\Omega;\varepsilon)\|$$

$$\leq C\Big(|b_+^\varepsilon| + |b_-^\varepsilon| + \|\tilde v^\varepsilon; W^{l+1}_{\varkappa+1,\beta}(\Omega;\varepsilon)\|\Big)\|r; \mathfrak{D}^{l-1,0}_{\varkappa,\beta}(\Omega;\varepsilon)\|$$

$$\leq C\mathfrak{F}\|r; \mathfrak{D}^{l-1,0}_{\varkappa,\beta}(\Omega;\varepsilon)\|, \tag{4.20}$$

for the last inequality we used (4.11) again. With similar calculations we derive, now with (4.10),

$$\|B(r,v^0); W^{l-2}_{-1,\beta}(\Omega;\varepsilon)\| \leq C\mathfrak{F}\|r; \mathfrak{D}^{l-1,0}_{\varkappa,\beta}(\Omega;\varepsilon)\|. \tag{4.21}$$

Collecting now (4.18)–(4.21) leads to (4.16). Finally the estimate (3.36) follows with the same arguments as used in (3.36), which finishes the proof of the error estimate (3.35) also for the nonlinear problem. The inequality (4.12) follows from the particular choice $l = 3$, here we multiply the inequality $\varepsilon\|\nabla^2(v^\varepsilon - v^0); W^0_\beta(\Omega)\| \leq C\varepsilon^\varkappa\mathfrak{F}$ by ε^{-1}, and use $\varepsilon^\alpha \leq 1$, as long as $0 \leq \varepsilon \leq 1$ and $\alpha \geqslant 0$. □

Remark 4.4. The same arguments as applied to derive (4.12) can be applied to the $\nabla^3(\tilde{v}^\varepsilon - \tilde{v}^0)$ and $\nabla^2(\tilde{p}^\varepsilon - \tilde{p}^0)$, (of course there is no convergence then) together with interpolation inequalities, one can show for arbitrarily small $\delta > 0$ with $\varkappa - 3/2 + \delta > 0$:

$$\|\tilde{v}^\varepsilon - \tilde{v}^0; W^{5/2-\delta}_\beta(\Omega)\| + \|\tilde{p}^\varepsilon - \tilde{p}^0; W^{3/2-\delta}_\beta(\Omega)\|$$
$$\leq C\varepsilon^{\varkappa-3/2+\delta}\Big(\|f'; W^2_\beta(\Omega)^3\| + \|f_4; W^3_\beta(\Omega)\| + \|g'; W^{5/2}_\beta(\partial\Omega)\| + |\Phi|\Big).$$

References

1 L. Cattabriga, *Su un problema al contorno relativo al sistema di equazioni di Stokes*, Sem. Mat. Univ. Padova **31**, (1964) 308–340.
2. M. Franta, J. Malek, and K. R. Rajagopal, *On steady flows of fluids with pressure and shear dependent viscosities*, Preprint, Prague (2003).
3. C. B. jr. Morrey, *Multiple integrals in the calculus of variations*, Springer Verlag, Berlin (1966).
4. O. A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, Gordon & Breach, New York (1966).
5. O. A. Ladyzhenskaya and V. A. Solonnikov, *Determination of the solutions of boundary value problems for stationary Stokes and Navier–Stokes equations having an unbounded Dirichlet integral*, J. Sov. Math. **21**, (1983) 728–761.
6. J. L. Lions and E. Magenes, *Nonhomogeneous boundary value problems and applications*, Vol. I, Springer Verlag, Berlin (1972).
7. S. A. Nazarov, *Vishik–Lyusternik method for elliptic boundary-value problems in regions with conical points. 1 The problem in a cone*, Sibirsk. Mat. Zh. **22**(4) (1981), 142–163.
8. S. A. Nazarov and K. Pileckas, *Asymptotic conditions at infinity for the Stokes and Navier–Stokes problems in domains with cylindrical outlets to infinity*, Quaderni di matematica **4** (1999), 141–243.
9. S. A. Nazarov and M. Specovius-Neugebauer, *Optimal results for the Brezzi–Pitkäranta approximation of viscous flow problems* Preprint, Kassel (2003).
10. K. I. Piletskas and V. A. Solonnikov, *Stationary Stokes and Navier-Stokes systems in an infinite open channel* I, Lith. Math. J. **29**(1) (1989), 34–47.
11. A. Prohl, *Projection and Quasi-Compressibility Methods for Solving the Incompressible Navier–Stokes Equations*, Teubner Verlag, Stuttgart (1997).
12. H. Sohr, *The Navier-Stokes Equations. An Elementary Functional Analytic Approach*, Birkhäuser V., Basel (2001).

13. V. A. Solonnikov, *On solutions of the stationary system of Navier–Stokes equations which have an infinite Dirichlet integral*, Zap. Nauchn. Semin. LOMI, **115** (1982), 251–263.
14. V. A. Solonnikov, *Boundary and initial-boundary value problems for the Navier–Stokes equations in domains with noncompact boundaries* In: Rodrigues, J. F. (ed.) et al., Mathematical topics in fluid mechanics. Proceedings of the summer course, held in Lisbon, Portugal, September 9–13 (1991). Harlow, Essex: Longman Scientific & Technical. Pitman Res. Notes Math. Ser. 274, (1993), pp. 117-162.
15. V. A. Solonnikov and V. E. Scadilov, *On a boundary value problem for a stationary system of Navier–Stokes equations*, Proc. Steklov Inst. Math. **125** (1973), 186–199.
16. R. Temam, *Navier Stokes Equations*, North Holland PC, Amsterdam (1977).
17. Ya. A. Roĭtberg, *Elliptic boundary value problems in the spaces of distributions*, Kluwer Academic publishers, Dordrecht (1996).

Institute of Mechanical
Engineering Problems,
St. Petersburg, Russia,
Fachbereich 17, Mathematik/Informatik,
Universität Kassel,
D-34109 Kassel, Germany

Поступило 25 сентября 2003