Записки научных
семинаров ПОМИ
Том 306, 2003 г.

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CAPILLARY/GRAVITY FILM FLOWS ON THE SURFACE OF A ROTATING CYLINDER

Abstract. We derive the equations describing the motion of a viscous incompressible capillary film on the surface of a rotating cylinder in the transversal gravity field. As a result, we obtain the equation for the film thickness, which has a fourth order in two space variables and a first order in time. We study both space periodic solutions in the axial coordinate and localized solutions of this equation in the stationary case. The stability of stationary solutions is discussed also. Analysis of the one-dimensional problem shows that its solution strongly depends on Galileo number and it does not exist if this number is large.

**To dear colleague and friend
with all best wishes**

1. Problem statement

The motion of a viscous incompressible capillary liquid on the surface of a circular cylinder rotating around its horizontal axis in a transversal gravitational field is considered. Let us accept the following notations: u, v, w are projections of the velocity vector on the axes r, θ, z of a cylindrical coordinate system; p is the difference between liquid pressure and the hydrostatic one; ρ is the density of liquid; ν is its kinematic viscosity coefficient; σ is a surface tension coefficient; a is the radius of a cylinder; ω is a constant angular velocity of its rotation around the axis z; g is the gravity acceleration. Therefore, the problem formulated below contains three dimensionless governing parameters: $R = a^2\omega/\nu$ (Reynolds number), $G = ga/\omega\nu$ (Galileo number) and $C = \sigma/\rho a\omega\nu$ (inverse capillary number). The mathematical statement of the problem

This work was fulfilled under financial support of Japan Society for the Promotion of Science (Grant No RC2/00106), President Grant for the Russian Federation Leading Scientific Schools Support (No NSh-902.2003.1) and London Mathematical Society.

is to find a nonnegative function $h(\theta, z, t)$ $(\theta \in (0, 2\pi),\ z \in \Re,\ t \in (0, T))$ and solution u, v, w, p of the Navier–Stokes equations,

$$u_t + uu_r + \frac{v}{r}u_\theta + wu_z - \frac{v^2}{r} =$$

$$= -\frac{1}{\rho}p_r + \nu\left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} + u_{zz} - \frac{u}{r^2} - \frac{2}{r^2}v_\theta\right),$$

$$v_t + uv_r + \frac{v}{r}v_\theta + wv_z + \frac{uv}{r} = \tag{1.1}$$

$$= -\frac{1}{\rho r}p_\theta + \nu\left(v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta} + v_{zz} - \frac{v}{r^2} + \frac{2}{r^2}u_\theta\right),$$

$$w_t + uw_r + \frac{v}{r}w_\theta + ww_z = -\frac{1}{\rho}p_z + \nu\left(w_{rr} + \frac{1}{r}w_r + \frac{1}{r^2}w_{\theta\theta} + w_{zz}\right),$$

$$u_r + \frac{u}{r} + \frac{1}{r}v_\theta + w_z = 0$$

in the domain $\Omega_t = \{r, \theta, z, t : a < r < a + h(\theta, z, t),\ \theta \in (0, 2\pi),\ z \in \Re,\ 0 < t < T\}$ under the following conditions:

$$u = 0, \quad v = a\omega, \quad w = 0 \quad \text{if} \quad r = a \tag{1.2}$$

(no-slip condition on the surface of rotating cylinder),

$$h_t + \frac{v}{r}h_\theta + wh_z = u \quad \text{if} \quad r = a + h \tag{1.3}$$

(kinematic condition on a free surface $r = a + h$),

$$\left(1 - \frac{1}{r^2}h_\theta^2\right)\left(v_r - \frac{v}{r} + \frac{1}{r}u_\theta\right) - \frac{1}{r}h_\theta h_z (u_z + w_r) - h_z\left(v_z + \frac{1}{r}w_\theta\right) +$$

$$+ \frac{2}{r}h_\theta\left(u_r - \frac{u}{r} - \frac{1}{r}v_\theta\right) = 0 \quad \text{if} \quad r = a + h, \tag{1.4}$$

$$\left(1 - h_z^2\right)(w_r + u_z) - \frac{1}{r}h_\theta h_z\left(v_r - \frac{v}{r} + \frac{1}{r}u_z\right) - \frac{1}{r}h_\theta\left(\frac{1}{r}w_\theta + v_z\right) +$$

$$+ 2h_z(u_r - w_z) = 0 \quad \text{if} \quad r = a + h \tag{1.5}$$

(vanishing of tangential stress on a free surface),

$$p - 2\rho\nu\left(1 + \frac{1}{r^2}h_\theta^2 + h_z^2\right)^{-1}\left[u_r + \frac{1}{r^2}h_\theta^2\left(\frac{1}{r}v_\theta + \frac{u}{r}\right) + h_z^2 w_z -\right.$$

$$\frac{1}{r}h_\theta\left(v_r-\frac{v}{r}+\frac{1}{r}u_\theta\right)-h_z\left(w_r+u_z\right)-\frac{1}{r}h_\theta h_z\left(v_z+\frac{1}{r}w_\theta\right)\Big]=\rho g r\sin\theta+$$

$$+\left(h^2+h_\theta^2+h^2h_z^2\right)^{-3/2}\left(h^2+2h_\theta^2-hh_{\theta\theta}+h^2h_z^2-\right.$$

$$\left.hh_z^2h_{\theta\theta}-h^3h_{zz}-hh_{zz}h_\theta^2+2hh_zh_\theta h_{z\theta}\right)\qquad(1.6)$$

(equality of normal stress to a capillary pressure),

$$h=h_0\left(\theta,z\right),\quad u=u_0\left(r,\theta,z\right),\quad v=v_0\left(r,\theta,z\right),$$

$$w=w_0\left(r,\theta,z\right)\quad\text{if}\quad t=0\qquad(1.7)$$

(initial conditions), periodically condition for functions h, u, v, w, p in θ with the period 2π and some boundary conditions in z coordinate. For the simplicity, we suppose that all the sought functions are periodic in z with the period $2\pi l$ and, moreover, functions h, u, v, p are even in z while function w is odd in z. Besides, for the stationary case, we consider solutions of the system (1.1) localized in z coordinate.

Free boundary problem (1.1)–(1.7) is interesting for many reasons. Firstly, the flow geometry is rather simple. Further, the motion under consideration is governed by four factors: viscosity, gravity, capillary and centrifugal forces. Their concerted action leads to a multiplicity of the flow regimes appearing. Besides, some of the above-mentioned factors may be weak, which allows us to develop a number of approximate models of flow [1–5]. At last, the studied motion admits a relatively easy experimental realization [2].

In spite of an apparent simplicity of formulation, there are no results concerning the global in time solvability of problem (1.1)–(1.7) even for small initial data. At the same time, the local existence theorem for this problem seems to be very plausible in the light of remarkable results of V. A. Solonnikov [6, 7].

As for the stationary analogue of the problem (1.1)–(1.7), the most successful situation takes place in a two-dimensional case where all unknown functions do not depend on z. In this case, the existence theorem is proved if parameters $R>0$ and $C\geq 0$ are arbitrary while the Galileo number G is sufficiently small [1]. We underline that this theorem is true even if $C=0$ (i.e., the surface tension is absent). We are not sure that a similar result remains valid in a three-dimensional case. However, if $C>0$ and G is small, it is not so difficult to prove the existence of a stationary solution using the results of paper [8].

2. Equation of the thin film

At the beginning, we define characteristic scales of length, time and velocity as a, $1/\omega$ and $a\omega$ respectively, so that $r' = r/a$, $z' = z/a$, $t' = \omega t$ are dimensionless variables. Let us suppose that the initial value of function h_0 has the form $\varepsilon a H_0$, where $\varepsilon > 0$ is a small parameter. The initial point of derivation of the thin film equation is the representation

$$h = \varepsilon a H, \quad r' = 1 + \varepsilon y, \quad u = \varepsilon a\omega U, \quad v = a\omega V, \quad w = a\omega W, \tag{2.1}$$

where H, U, V, and W are dimensionless unknown functions. We assume further that functions H, U, V, W and their derivatives with respect to θ, z', t' as well as the new independent variable r keep finite values as $\varepsilon \to 0$.

Now it is necessary to determine a characteristic value of pressure p. To this end, we represent p in the form $p = p_\Gamma + \rho\nu\omega p'$, where p_Γ is the value of pressure at free surface Γ, which could be determined basically by the balance of capillary and gravity forces while $p' \sim 1$ is the dynamical dimensionless component of pressure. Then we notice that

$$p_\Gamma = \frac{\sigma}{a} - \frac{\varepsilon\sigma}{a}\left(H_{\theta\theta} + H_{zz} + H\right) + \rho g a \sin\theta + \mathrm{O}\left(\varepsilon\right) \tag{2.2}$$

as $\varepsilon \to 0$ (here and in future we omit the symbol prime over z' and t').

Substitution of equalities (2.1), (2.2) into equations (1.1) brings to the relations

$$U_{yy} - p'_y = \mathrm{O}\left(\varepsilon\right) + \mathrm{O}\left(R\varepsilon^2\right),$$

$$V_{yy} + C\varepsilon^3\left(\Delta H + H\right)_\theta - G\varepsilon^2 \cos\theta = \mathrm{O}\left(\varepsilon\right) + \mathrm{O}\left(R\varepsilon^2\right),$$

$$W_{yy} + C\varepsilon^3\left(\Delta H + H\right)_z = \mathrm{O}\left(\varepsilon\right) + \mathrm{O}\left(R\varepsilon^2\right),$$

$$U_y + V_\theta + W_z = \mathrm{O}\left(\varepsilon\right)$$

(here Δ is Laplacian in θ, z). Now we postulate that

$$C\varepsilon^3 \to 3\alpha, \quad G\varepsilon^2 \to 3\gamma, \quad R\varepsilon^2 \to 0 \quad \text{if} \quad \varepsilon \to 0,$$

where α and γ are some nonnegative constants. Passing to the limit $\varepsilon = 0$ we obtain the following system of equations:

$$U_{yy} = p'_y, \tag{2.3}$$

$$V_{yy} = 3\gamma\cos\theta - 3\alpha\left(\Delta H + H\right)_\theta, \tag{2.4}$$

$$W_{yy} = -3\alpha\left(\Delta H + H\right)_z, \tag{2.5}$$

$$U_y + V_\theta + W_z = 0. \tag{2.6}$$

Relations (1.2), (1.4), (1.5), and (2.1) produce the boundary conditions for functions U, V, and W:

$$U = 0, \quad V = 1, \quad W = 0 \quad \text{if} \quad y = 0, \tag{2.7}$$

$$V_y = 0, \quad W_y = 0 \quad \text{if} \quad y = H. \tag{2.8}$$

Integration of equations (2.4)–(2.6) in y with conditions (2.7), (2.8) results in the expressions of functions U, V, W via H:

$$U = \frac{3y^2}{2}\left(H - \frac{y}{3}\right)\left[\gamma \sin\theta - \alpha\,\Delta\left(\Delta H + H\right)\right] +$$

$$+\frac{3y^2}{2}\left[\gamma H_\theta \cos\theta - \alpha \nabla H \cdot \nabla\left(\Delta H + H\right)\right],$$

$$V = 3\left(\frac{y^2}{2} - yH\right)\left[\gamma \cos\theta - \alpha\left(\Delta H + H\right)_\theta\right] + 1,$$

$$W = -3\alpha\left(\frac{y^2}{2} - yH\right)\left(\Delta H + H\right)_z,$$

where $\nabla = (\partial/\partial\theta,\ \partial/\partial z)$. Substituting these relations into equality (1.3) we come to the differential equation for function $H(\theta, z, t)$,

$$H_t + \left(H - \gamma H^3 \cos\theta\right)_\theta + \alpha \operatorname{div}\left[H^3 \nabla\left(\Delta H + H\right)\right] = 0. \tag{2.9}$$

Equation (2.9) should be completed by the initial condition,

$$H = H_0(\theta, z) \quad \text{if} \quad t = 0, \tag{2.10}$$

periodicity conditions

$$H(\theta + 2\pi, z, t) = H(\theta, z, t), \tag{2.11}$$

$$H(\theta, z + 2\pi l, t) = H(\theta, z, t), \tag{2.12}$$

and the evenness condition,

$$H(\theta, -z, t) = H(\theta, z, t). \tag{2.13}$$

At last, we obtain from the equation (2.3) and condition $p' = 0$ at $y = H$ that $p' = U_y(y, \theta, z, t) - U_y(0, \theta, z, t) + y - H$.

Before the investigation of problem (2.9)–(2.13), we should notice that we eliminate all evolution derivations from equations (2.3)–(2.5) due to the assumption $R\varepsilon^2 \to 0$ as $\varepsilon \to 0$. Consequently, we cannot satisfy initial conditions for the velocity components. However, if we assume that $R\varepsilon^2 \to \kappa$ as $\varepsilon \to 0$, where $\kappa > 0$ is a small parameter, we are able to fulfil initial conditions (1.7) by matching the asymptotic expansions for the solution of the problem (1.1)–(1.7) in the real time t and the short time $\tau = \kappa^{-1}t$. This procedure is described in [1] for the two-dimensional analogue of problem (1.1)–(1.7).

3. Space-periodic stationary solutions and their stability

The rest part of the paper is devoted to analysis of solutions of the thin film equation (2.9). Let us assume that the initial function H_0 in (2.10) is strictly positive and that it satisfies periodicity and evenness conditions. If $H_0 \in C^{4+\delta}\left(\bar{\Pi}\right)$, $0 < \delta < 1$ where $\Pi = \{\theta, z: \ \ 0 < \theta < 2\pi, \ \ 0 < z < 2\pi l\}$, there exists $T > 0$ such that the problem (2.9), (2.10) has a unique solution $H \in C^{4+\delta,\ 2+\delta/2}\left(\bar{\Pi} \times [0,T]\right)$, and $H > 0$ in this domain. This statement is proved by the method of a priori successive estimates of linear parabolic equations [9].

Since function H is proportional to the film thickness, only nonnegative solutions of the problem (2.9)–(2.10) have a physical sense. Positiveness of H means that the film covers the surface of a cylinder entirely. It would be interesting to prove the global in time existence theorem for the problem (2.9), (2.10) and also to clarify if the solution of this problem with $H_0 > 0$ could take a zero value at $t > 0$.

Further we concentrate our attention on stationary solutions of the equation (2.9) and their stability. The stationary problem consists of the equation

$$\alpha \mathrm{div}\left[H^3 \nabla\left(\Delta H + H\right)\right] + \left(H - \gamma H^3 \cos\theta\right)_\theta = 0 \tag{3.1}$$

and additional conditions

$$\begin{gathered} H\left(\theta + 2\pi, z\right) = H\left(\theta, z\right) = H\left(\theta, z + 2\pi l\right), \\ H\left(\theta, -z\right) = H\left(\theta, z\right). \end{gathered} \tag{3.2}$$

At first, we consider positive solutions of this problem at $\gamma = 0$. Multiplying equation (3.1) by $\Delta H + H$ and integrating the result over rectangular Π we conclude that

$$\int\limits_\Pi H^3 \left|\nabla\left(\Delta H + H\right)\right|^2 d\theta dz = 0.$$

Thanks to $H > 0$ we have $\Delta H + H = \text{const}$ and $H = C_0 + C_1 \cos z + C_2 \cos \theta + C_3 \sin \theta$ if $l = 1$, $H = C_0 + C_2 \cos \theta + C_3 \sin \theta$ if $l \neq 1$. But both constants C_2 and C_3 are equal to zero by virtue of equation (3.1). Therefore, all positive solutions of problem (3.1), (3.2) at $\gamma = 0$ are given by formulas

$$\begin{gathered} H_* = C_0 + C_1 \cos z \quad \text{if} \quad l = 1, \\ H_* = C_0 \quad \text{if} \quad l \neq 1 \end{gathered} \tag{3.3}$$

where C_0 and C_1 are constants, moreover, $C_0 > 0$, $|C_1| < C_0$.

Proposition 1. *Problem* (3.1) *and* (3.2) *has a two-parameter family of solutions $H_\gamma(\theta, z)$ in case $l = 1$ and a one-parameter family of solutions in the opposite case if $\gamma \in [0, \gamma_0]$ and γ_0 is sufficiently small. As $\gamma \to 0$, $H_\gamma \to H_*$ where H_* is defined by formulas* (3.3).

To prove Proposition 1, we use the modification of Newton–Kantorovich method [10]. Linearization of equation (3.1) on its exact solution (3.3) at $\gamma = 0$ leads to the equation

$$L\zeta \equiv \alpha \operatorname{div} \left[H_*^3 \nabla (\Delta \zeta + \zeta) \right] - \zeta_\theta = f.$$

Operator L acts from Sobolev space of periodic functions $H^2_{\text{per}}(\Pi)$ into the space $H^{-2}_{\text{per}}(\Pi)$ and has a two-dimensional kernel spanned by functions $\{1, \cos z\}$ if $l = 1$ and one-dimensional kernel $\{1\}$ if $l \neq 1$. The interactive procedure consists of the solution of equations

$$H_{(n+1)} = C_0 + C_1 \cos z + \gamma P L^{-1} \left(H^3_{(n)} \cos \theta \right)_\theta \qquad n = 0,\ 1,\ 2, \ldots$$

$$\text{and} \quad H_{(0)} = 0 \quad (C_1 = 0 \quad \text{if} \quad l \neq 1).$$

Here P is a projector in $L^2_{\text{per}}(\Pi)$ on the orthogonal supplement to the kernel of L. Since function $\left(H^3 \cos \theta\right)_\theta$ is orthogonal to l and $\cos z$ for any H, the expression $PL^{-1}\left(H^3_{(n-1)} \cos \theta\right)_\theta$ is determined uniquely and this function belongs to $H^2_{\text{per}}(\Pi)$. Convergence of the interactive procedure for small $\gamma > 0$ is established in a standard way. In fact, function $H_\gamma(\theta, z) = \lim_{n \to \infty} H_{(n)}$ has more smoothness than H^2; namely, $H_\gamma \in C^\infty\left(\bar{\Pi}\right)$.

Proposition 1 deserves a special comment. We suspect that problem (3.1), (3.2) has no solutions if γ is sufficiently large, though we are not able to prove it. However, as it will be shown below, this conjecture is true for a one-dimensional analogue of problem (3.1), (3.2).

Let us consider now evolution equation (2.9) with $\gamma = 0$ and linearize this equation on its exact solution $H = C_0 > 0$. The result of linearization is

$$\zeta_t + \alpha C_0^3 \Delta \left(\Delta \zeta + \zeta\right) = 0.$$

The last equation has a solution

$$\zeta = \exp\,(\lambda t) \cos\,(z/l)$$

where

$$\lambda = \alpha C_0^3 \left(l^2 - 1\right) l^{-4}$$

so that $\lambda > 0$ if $l > 1$. It means that the constant solution of Eq. (2.9) with $\gamma = 0$ is unstable with respect to perturbations in z direction, which length is more than 2π. This result is not surprising in the light of paper [11] where a long-wave instability of a uniformly rotating liquid with a cylindrical free surface was studied on the base of the full Navier–Stokes equations.

The statement concerning the long-wave instability of stationary solutions of Eq. (2.9) close to a constant one remains to be true if γ is sufficiently small. The case of near critical waves length is more delicate. As before, we start from the case $\gamma = 0$ and pass to new coordinate $\varphi = \theta - t$ instead of θ in Eq. (2.9). Function $F(\varphi, z, t) = H(\varphi + t, z, t)$ satisfies the equation

$$F_t - \gamma \left[F^3 \cos\,(\varphi + t)\right]_\varphi + \alpha \mathrm{di}\,v' \left[F^3 \nabla' \left(\Delta' F + F\right)\right] = 0, \tag{3.4}$$

where prime indicates differential operations in variables φ and z. We add to this equation the periodicity and evenness conditions,

$$F(\varphi + 2\pi, z, t) = F(\varphi, z, t)\,, \tag{3.5}$$

$$F(\varphi, z + 2\pi, t) = F(\varphi, z, t)\,, \tag{3.6}$$

$$F(\varphi, -z, t) = F(\varphi, z, t)\,. \tag{3.7}$$

Proposition 2. *All positive stationary solutions of the problem* (3.4)–(3.7) *have the form*

$$F_* = C_0 + C_1 \cos\, z + C_2 \cos\, \varphi + C_3 \sin \varphi, \tag{3.8}$$

where $C_0, \ldots, C_3$ *are constants and* $C_0 > 0$, $|C_1| + \left(C_2^2 + C_3^2\right)^{1/2} < 2C_0$.

Proof follows immediately from the identity

$$\int\limits_\Pi F^3 \left|\nabla' \left(\Delta' F + F\right)\right|^2 d\varphi dz = 0,$$

which is fulfilled for stationary solutions of the problem (3.4)–(3.7) at $\gamma = 0$.

Equality (3.8) describes all equilibrium shapes of rotating capillary film on the surface of a circular cylinder in weightlessness. Proposition 2 is in a good correlation with the result of paper [12], where the three dimensional equilibrium shapes of a rotating column of a capillary viscous liquid branching from the trivial one were found.

This result admits another interpretation. Returning to the original coordinate system, we conclude that equation (2.9) with $\gamma = 0$ has a family of solutions

$$H(\theta, z, t) = H_*(z) + C_2 \cos(\theta - t) + C_3 \sin(\theta - t), \qquad (3.9)$$

where H_* is defined by the first of the equalities (3.3). It means that the solution $H_*(z)$ is not more than neutrally stable with respect to perturbation in θ direction. One of the interesting open questions related to Eq. (2.9) is the following: do there exist its time periodic solutions, which converge to solutions (3.9) as $\gamma \to 0$?

4. Localized stationary solutions

Let us return to Eq. (3.1) with $\gamma = 0$. The function

$$H_{\text{loc}} = \begin{cases} C_0(1 + \cos z), & |z| < \pi, \\ 0, & |z| \geqslant \pi, \end{cases} \qquad (4.1)$$

where $C_0 > 0$ is a constant, satisfies this equation everywhere except for the lines $z = \pm\pi$. Moreover, as it will be shown further, H_{loc} is a generalized solution of the equation (3.9) in the sense of the definition given below. If γ is small, it is reasonable to construct a formal solution of this equation,

$$H \sim H_{\text{loc}} + \sum_{n=1}^{\infty} \gamma^n \xi^{(n)}.$$

Function $\xi^{(1)}$ is a solution of the equation

$$\alpha \operatorname{div}\left[m \nabla\left(\Delta\xi^{(1)} + \xi^{(1)}\right)\right] - \xi^{(1)}_\theta = -m \sin\theta, \qquad (4.2)$$

where $m(z) = H^3_{\text{loc}}$. Looking for a 2π-periodic in θ solution of the equation (4.2) in the form

$$\xi^{(1)} = A(z)\sin\theta + B(z)\cos\theta$$

we arrive to the system of equations

$$\begin{aligned} &(mA''')' - mA'' - B = -m, \\ &(mB''')' - mB'' + A = 0 \end{aligned} \tag{4.3}$$

(here prime denotes differentiation in z). According to (4.1), $m = \mathrm{O}\,(z \pm \pi)^6$ as $z \to \mp\pi$. It makes the boundary conditions

$$A \to 0, \quad B \to 0 \quad \text{as} \quad z \to \pm\pi \tag{4.4}$$

to be natural for the system (4.3). It is easy to prove that the uniform system (4.3) has only a trivial solution in the class of functions, which satisfy conditions (4.4) and have bounded derivatives up to the third order as $z \to \pi - 0$ and $z \to -\pi + 0$. Besides, the uniform system has four linearly independent solutions, which decay as $\exp\left[-k\,(z-\pi)^{-3/2}\right]$ ($k = \text{const} > 0$) when $z \to \pi - 0$ and four solutions, which decay as $\exp\left[-k\,(z+\pi)^{-3/2}\right]$ when $z \to -\pi + 0$. This allows us to prove the existence of solution of the singular boundary value problem (4.3), (4.4) and to establish the estimates of its solution,

$$A = \mathrm{O}\,(z \pm \pi)^8 \quad \text{if} \quad z \to \mp\pi,$$

$$B = \mathrm{O}\,(z \pm \pi)^6 \quad \text{if} \quad z \to \mp\pi.$$

We are able to call the solution (4.1) a localized solution of degenerate elliptic equation (3.1) at $\gamma = 0$. Comparing the main term H_{loc} with the second term $\gamma\xi^{(1)}$ of an asymptotic expansion of solution of this equation with respect to parameter γ, we see that function $\xi^{(1)}$ decreases in the end points of the interval $[-\pi, \pi]$ faster than H_{loc}. It turns out that next term of the expansion has a faster decay than $\xi^{(1)}$, $\xi^{(2)} = \mathrm{O}\,(z \pm \pi)^{10}$ when $z \to \mp\pi$. This tendency remains for a higher order terms of the asymptotic expansion. It means that perturbations of the solution (4.1) in parameter γ cannot change the boundary of its support, $z = \pm\pi$, $0 \leqslant \theta < 2\pi$.

To define the localized solution of the equation (3.1) in the square $\Pi = \{\theta, z: \ 0 < \theta < 2\pi, \ |z| < \pi\}$ for the case $\gamma > 0$, we introduce the functional space $\overset{\circ}{H}{}^2_{\mathrm{per}}(\Pi)$ as the closure of functions $\varphi\,(\theta, z) \in C^\infty\left(\bar{\Pi}\right)$, which are 2π-periodic in θ and vanish near the lines $z = \pm\pi$, in the norm of Sobolev space $H^2\,(\Pi)$. We say that function $H\,(\theta, z) \in \overset{\circ}{H}{}^2_{\mathrm{per}}\,(\Pi)$ is a localized solution of the equation (3.1) if the identity

$$\int\limits_{\Pi} \{\alpha[\nabla(H^3\nabla\Phi) : \nabla(\nabla H) - H^3\nabla\Phi \cdot \nabla H] +$$

$$+ \left(H - \gamma H^3 \cos\theta\right)_\theta \Phi\} \, d\theta dz = 0$$

is satisfied for any $\Phi \in \overset{\circ}{H}{}^2_{\text{per}}(\Pi)$. Apparently, function H_{loc} given by formulas (4.1) satisfies this identity if $\gamma = 0$. The open question is to prove the existence of localized solutions of the equation (3.1) for small $\gamma > 0$.

5. One-dimensional problem

If the film thickness does not depend on z, the governing equation (2.9) transforms into

$$H_t + \alpha\left[H^3\left(H_{\theta\theta\theta} + H_\theta\right)\right]_\theta + \left(H - \gamma H^3 \cos\theta\right)_\theta = 0. \tag{5.1}$$

The behaviour of solutions of this equation were well-studied in paper [5], and we cannot add something essential to its results. At the same time, some aspects of a one-dimensional stationary problem (nonexistence and nonuniqueness of its solutions, their asymptotic behaviour as $\alpha \to 0$ and $\alpha \to \infty$, etc.) were not completely investigated. Now we try to fill this gap. Setting $H_t = 0$ in (5.1) and integrating once with respect to θ yields

$$\alpha H^3\left(H''' + H'\right) = \gamma H^3 \cos\theta - H + Q, \tag{5.2}$$

where constant Q is the dimensionless flux of a liquid over a cross section of the film (here and further prime denotes differentiation in θ). To bring equation (5.2) to a more convenient form we introduce notations

$$\eta = Q^{-1}H, \quad \delta = \alpha Q^3, \quad \beta = \gamma Q^2.$$

New unknown function $\eta(\theta)$ satisfies the equation

$$\delta\left(\eta''' + \eta'\right) = \beta\cos\theta - \frac{1}{\eta^2} + \frac{1}{\eta^3} \tag{5.3}$$

and the periodicity condition

$$\eta(\theta + 2\pi) = \eta(\theta). \tag{5.4}$$

Besides, we are interested in positive solutions of the problem (5.3), (5.4) only.

Despite of the relative simplicity of the problem (5.3), (5.4), its study is nontrivial since the differential operator in (5.3) has a three-dimensional kernel spanned by functions $\{1, \cos\theta, \sin\theta\}$. This property implies the following relations:

$$\int \frac{d\theta}{\eta^2} = \int \frac{d\theta}{\eta^3}, \tag{5.5}$$

$$\pi\beta = \int \frac{\eta - 1}{\eta^3} \cos\theta \, d\theta, \tag{5.6}$$

$$0 = \int \frac{\eta - 1}{\eta^3} \sin\theta \, d\theta. \tag{5.7}$$

(if the limits of integration are not indicated, it means that they are 0 and 2π). Identities (5.5)–(5.7) are obtained by multiplication of (5.3) by 1, $\cos\theta$, $\sin\theta$ respectively, and integration of the results over $(0, 2\pi)$. Using positiveness of η and Hölder inequality, we obtain a priori estimates from (5.5):

$$\int \frac{d\theta}{\eta^n} \leqslant 2\pi, \quad n = 1, 2, 3. \tag{5.8}$$

Proposition 3. *If $\beta > 4$ then equation* (5.3) *has no positive 2π-periodic solutions.*

To prove Proposition 3, we estimate the modulus in right side of equality (5.6) and then apply inequalities (5.8):

$$\pi\beta \leqslant \int \frac{d\theta}{\eta^2} + \int \frac{d\theta}{\eta^3} \leqslant 4\pi.$$

Now the required statement is evident.

If $\delta = 0$ (that corresponds to the case of zero surface tension), equation (5.3) transforms into an algebraic one. It was shown in [2] that this cubic equation has a unique positive solution if $0 \leqslant \beta \leqslant 4/27$ and has no single-valued solutions if $\beta > 4/27$. The critical value $\beta = 4/27$ corresponds to appearance of a corner point in the graph of function $\eta(\theta)$ at $\theta = 0$. It is naturally to expect an extension of the existence domain in the plane β, δ for the problem (5.3), (5.4) if $\delta > 0$. We delay the discussion of this question up to next section and now we establish the local existence theorem for small values of β and any positive δ. Its proof is based on a reformulation of the problem (5.3), (5.4) in terms of the new unknown function $\xi = \eta - 1$. Function ξ satisfies the equation

$$\delta(\xi''' + \xi') + \xi = \beta\cos\theta + \frac{\xi^2\left(3 + 3\xi + \xi^2\right)}{(1 + \xi)^3}. \tag{5.9}$$

The operator

$$L=\delta\left(\frac{d^3}{d\theta^3}+\frac{d}{d\theta}\right)+1$$

is reversible in the space of 2π-periodic functions. Inverse operator L^{-1} is bounded as the operator acting from $L^2(0,2\pi)$ into $H^3(0,2\pi)$. However, it will be sufficient for our purpose to estimate the norm $L^{-1}: C[0,2\pi]\to H^1(0,2\pi)$. Let us consider the equation $Lw=f$, where $f\in L^2(0,2\pi)$ and $f\perp\{1,\ \cos\theta,\ \sin\theta\}$. Then

$$\|w\|_{C[0,2\pi]}\leqslant\frac{1}{2}\sqrt{\frac{\pi}{2}}\|w\|_{H^1(0,2\pi)}\leqslant\left[\frac{\pi}{2\left(1+36\delta^2\right)}\right]^{1/2}\|f\|_{L^2(0,2\pi)}\leqslant$$

$$\leqslant\frac{\pi}{\left(1+36\delta^2\right)^{1/2}}\|f\|_{C[0,2\pi]}\,.$$

On the other hand, $w=f$ if $f\in\{1,\ \cos\theta,\ \sin\theta\}$. Therefore, the following estimate takes place for the norm of the operator $L^{-1}: C[0,2\pi]\to C[0,2\pi]$:

$$\left\|L^{-1}\right\|_{C\to C}\leqslant\max\left[1,\ \frac{\pi}{\left(1+36\delta^2\right)^{1/2}}\right].\tag{5.10}$$

Converting the operator L in (5.9) we transform it into the equation

$$\xi=\beta\cos\theta+L^{-1}\left[\frac{\xi^2\left(3+3\xi+\xi^2\right)}{(1+\xi)^3}\right],\tag{5.11}$$

where operator in the right side is Lipschitz-continuous in $C[0,2\pi]$. Application of Caccipolli–Banach theorem to this equation yields

Proposition 4.. *Assume that numbers k, β and δ are chosen such that the following inequalities are fulfilled*

$$\left\|L^{-1}\right\|k\left(6+6k+4k^2+k^3\right)(1-k)^{-4}<1,$$

$$\left\|L^{-1}\right\|k^2\left(3+3k+k^2\right)(1-k)^{-3}+\beta<k,$$

where $\left\|L^{-1}\right\|$ satisfies the estimate (5.10). Equation (5.11) has a solution $\xi(\theta)\in C[0,2\pi]$, which is unique in the ball $\|\xi\|_C<k$.

We notice that $\left\|L^{-1}\right\|_{C\to C}=1$ if $\delta\geqslant 1/2$. Consequently, if we take $k=1/11$, assumptions of proposition 4 will be fulfilled in the range of governing parameters of the equation (5.11) $\delta\geqslant 1/2$, $\beta\leqslant 1/11-37/1000$.

Further, any continuous solution of (5.11) is an analytic function of θ. Moreover, this solution is analytic in parameter β in the point $\beta = 0$. Restricting ourselves to three terms of the Taylor expansion, we get:

$$\eta = 1+\beta \cos\,\theta+\beta^2 \left(\frac{3}{2}+\frac{3}{2}\cdot\frac{1}{1+36\delta^2}\cos\,2\theta-\frac{9\delta}{1+36\delta^2}\sin 2\theta\right)+\mathrm{O}\,\left(\beta^3\right)$$

when $\beta \to 0$. Fourth term of this expansion was found in [5] with the help of MAPLE; it is not presented here.

6. A priori estimates

On the base of previous analysis, we can present a solution of the problem (5.3), (5.4) as

$$\eta = s + q\cos\,\theta - \tau\sin\theta + \zeta, \tag{6.1}$$

where s, q, τ are unknown constants and function $\zeta\,(\theta)$ is orthogonal to 1, $\cos\,\theta$, $\sin\theta$ in $L^2\,(0, 2\pi)$. The problem (5.3), (5.4) is equivalent to the equation

$$\delta\,(\zeta''' + \zeta') = \beta\cos\,\theta - \frac{1}{\eta^2} + \frac{1}{\eta^3} \tag{6.2}$$

with the periodicity condition $\zeta\,(\theta + 2\pi) = \zeta\,(\theta)$, and additional conditions

$$\int \zeta d\theta = \int \zeta\cos\,\theta d\theta = \int \zeta\sin\theta d\theta = 0. \tag{6.3}$$

Last conditions are equivalent to equalities (5.5)–(5.7), which can be considered as the branching equations in the process of solution of the problem (5.3), (5.4) by Lyapunov–Schmidt method. (We recollect that the operator $\frac{d^3}{d\theta^3} + \frac{d}{d\theta}$ is invertible on the subspace S of 2π-periodic functions, which belong to $L^2\,(0, 2\pi)$ and orthogonal to 1, $\cos\,\theta$, $\sin\theta$.) Unfortunately, unknown parameters s, q, τ enter the branching equations very implicitly. However, the problem of their determination can be solved effectively if the proportional to the surface tension coefficient parameter δ is sufficiently large.

Below we obtain a priori estimates of parameters s, q, and τ and solution ζ of problem (6.1)–(6.3). Let us expand the right-hand side of Eq. (6.2) in Fourier series:

$$\beta\cos\,\theta - \frac{1}{\eta^2} + \frac{1}{\eta^3} \equiv f\,(\theta) = \sum_{n=2}^{\infty} (f_n \cos\,n\theta + g_n\sin n\theta).$$

(Here the absence of terms with $n = 0,\ 1$ is caused by equalities (5.5)–(5.7).) Thanks to (5.8),

$$|f_n| \leqslant 4, \quad |g_n| \leqslant 4 \quad \text{if} \quad n = 2, 3, \ldots .$$

Therefore, coefficients of the expansion

$$\zeta = \sum_{n=2}^{\infty} (a_n \cos\, n\theta + b_n \sin n\theta),$$

where $\zeta\,(\theta)$ is a 2π-periodic solution of the equation

$$\delta\,(\zeta''' + \zeta') = f\,(\theta)$$

admit the estimates

$$|a_n| \leqslant \frac{4}{\delta n\,(n^2-1)}, \quad |b_n| \leqslant \frac{4}{\delta n\,(n^2-1)} \quad \text{if} \quad n = 2, 3, \ldots$$

and

$$\|\zeta''\|_{L^2} < \frac{8\sqrt{\pi}}{\delta} \tag{6.4}$$

as far as

$$\sum_{n=2}^{\infty} \frac{n^2}{(n^2-1)^2} < 1.$$

Estimate (6.4) has as a consequence the following inequalities

$$0 \leqslant \beta\tau \leqslant \frac{64}{\delta}, \tag{6.5}$$

where τ is a coefficient before $\sin\theta$ in formula (6.1). Indeed, multiplying equation (5.3) by η' and integrating over $(0, 2\pi)$, taking into account (6.1) yield

$$\delta \int \left(\eta''^2 - \eta'^2\right)\, d\theta = \pi\beta\tau.$$

However, in view of (6.1),

$$\int \left(\eta''^2 - \eta'^2\right)\, d\theta = \int \left(\zeta''^2 - \zeta'^2\right)\, d\theta \geqslant 0$$

since $\zeta \perp \{1,\ \cos\,\theta,\ \sin\theta\}$. From here and (6.4) we derive (6.5).

It should be noted that value τ characterizes the degree of asymmetry of the (5.3), (5.4) problem's solution. It is evident that function $\eta\,(\theta)$ is even as $\delta = 0$. If $\delta \to \infty$, this function is close to an even one as follows from (6.4), (6.5). Thus, the most asymmetry takes place for a moderate δ.

Now we obtain the bilateral estimates for the coefficient in (6.1),

$$0 \leqslant \beta q \leqslant 2. \tag{6.6}$$

Multiplication of (5.3) by $\eta - 1$ and integration of the result over $(0, 2\pi)$ gives

$$\pi\beta q = \beta \int (\eta - 1) \cos\, \theta d\theta = \int \frac{(\eta-1)^2}{\eta^3} d\theta \geqslant 0,$$

i.e., the lower estimate (6.6). If we multiply equation (5.3) by and integrate the result again, we find

$$\pi\beta q + \int \frac{d\theta}{\eta^2} = \int \frac{d\theta}{\eta}.$$

Taken together, this equality and (5.8) imply the upper estimate (6.6).

At last, we derive the estimate from below for the mean value $\bar{\eta} = s$ of the function η on the interval $(0, 2\pi)$,

$$s \geqslant 1. \tag{6.7}$$

To this end, we remark that $\bar{\eta} = 1 + \bar{\xi}$ where ξ is a 2π-periodic solution of Eq. (5.9). But inequality $\bar{\xi} \geqslant 0$ is the direct corollary of (5.9) that proves (6.7).

Inequalities (6.4)–(6.7) and relation (6.1) give a useful information about the qualitative behaviour of solution of the problem (5.3), (5.4). We exploit this information for analysis of its solution as $\delta \to \infty$.

7. Amplitude equations

Let us introduce notations $\rho = \left(q^2 + \tau\right)^{1/2}$, $\chi = \mathrm{arctg}\, (\tau/q)$ and rewrite (6.1) in the form

$$\eta = s + \rho \cos\, (\theta + \chi) + \zeta. \tag{7.1}$$

Further, we multiply equation (5.3) by η^3, $\eta^3 \cos\, \theta$, $\eta^3 \sin\theta$ consecutively and integrate results within $(0, 2\pi)$. In accordance with (6.4), (6.5) we deduce the amplitude equations,

$$\begin{gathered} s = 1 + \frac{3}{2}\beta s^2 \rho + \frac{3}{8}\beta\rho^3 + \mathrm{O}\left(\frac{1}{\delta}\right), \\ \rho = \beta s^3 + \frac{9}{4}\beta s \rho^2 + \mathrm{O}\left(\frac{1}{\delta}\right), \quad \sin\chi = \mathrm{O}\left(\frac{1}{\delta}\right) \end{gathered} \tag{7.2}$$

as $\delta \to \infty$ uniformly in β, $0 \leqslant \beta \leqslant 4$. Parallel with (7.2) we consider the truncated system

$$s = 1 + \frac{3}{2}\beta s^2 \rho + \frac{3}{8}\beta \rho^3, \quad \rho = \beta s^3 + \frac{9}{4}\beta s \rho^2 \tag{7.3}$$

obtained from (7.2) in the limit $\delta \to \infty$. The condition of reality of roots

$$\rho_+ = \frac{2}{9\beta s}(1 + R), \quad \rho_- = \frac{2}{9\beta s}(1 - R)$$

($R = \left(1 - 9\beta^2 s^4\right)^{1/2}$) of the second equation (7.3) is $\beta \leqslant 1/3s^2$. But according to (6.7) $s \geqslant 1$ (moreover, this inequality is strict if $\beta > 0$). Hence, the shortened system (7.3) has no solutions if $\beta > 1/3$. This statement remains true for the initial system (7.2) if δ is sufficiently large.

Let now $\beta \leqslant 1/3$. Substitution of the value ρ_+ into the first equation (7.3) leads to the equation for determination of s:

$$\frac{1}{s} = \frac{7}{9} - \frac{4}{243\beta^2 s^4} - \left(\frac{8}{27} + \frac{4}{243\beta^2 s^4}\right)\left(1 - 9\beta^2 s^4\right)^{1/2}. \tag{7.4}$$

It is clear that the solution $s = s_+$ of Eq. (7.4) satisfies inequality $s_+ > 1$. However, this is not sufficient for the nonnegativity of approximate solution $\eta = s + \rho \cos\theta$ of the problem (5.3), (5.4) as $\delta \to \infty$. Requiring $\rho_+ \leqslant s_+$, we get $\beta s_+^2 \geqslant 4/13$. This demand restricts the range of admissible values of parameter β:

$$0.00423 \approx 11^2 \cdot 13^{-4} = \beta_1 \leqslant \beta \leqslant \beta_2 = 17^2 \cdot 3^{-7} \approx 0.132.$$

If these inequalities are fulfilled, equation (7.4) has a solution s_+ such that $27/17 \leqslant s_+ \leqslant 26/11$.

Taking the root ρ_- of the second equation (7.3), we come to the following relation for finding s:

$$\frac{1}{s} = \frac{7}{9} - \frac{4}{243\beta^2 s^4} + \left(\frac{8}{27} + \frac{4}{243\beta^2 s^4}\right)\left(1 - 9\beta^2 s^4\right)^{1/2}. \tag{7.5}$$

It turns out that equation (7.5) has a solution $s = s_- \geqslant 1$ for all β, $0 \leqslant \beta \leqslant \beta_2 = 17^2 \cdot 3^{-7}$, moreover, $s_- = 1$ as $\beta = 0$, $s_- = 27/17$ as $\beta = \beta_2$. In this case, the condition $\rho_- \leqslant s_-$ is fulfilled for all $\beta \in [0, \beta_2]$.

Returning to the original system of an amplitude equation (7.2), we can conclude that its solutions will be close to solutions of the truncated system (7.3) if parameter δ is sufficiently large. Summarizing above mentioned results we arrive at

Proposition 5. *If* $11^2 \cdot 13^{-4} < \beta_* \leqslant \beta \leqslant \beta_{**} < 17^2 \cdot 3^{-7}$ *and* δ *is sufficiently large, problem* (5.3), (5.4) *has at least two solutions. If* $\beta > 1/3$ *and* δ *is sufficiently large, problem* (5.3), (5.4) *has no solutions.*

Proposition 5 deserves some comments. The solution s_+, ρ_+ of the shortened system (7.3) exists even if $\beta \in (0, \beta_1)$, however $\rho_+ > s_+$ in this case, moreover, $s_+ \to 0$ as $\beta \to 0$. Hence, there are no solutions of the initial system (7.2) at large δ corresponding to $s = s_+$, $\rho = \rho_+$, $\chi = 0$, which have a physical sense. On the other hand, the upper bound $\beta = 17^2 \cdot 3^{-7}$ of the guaranteed existence of positive solutions of the problem (5.3), (5.4) at large δ is close to the one in the case $\delta = 0$, $\beta = 4/27$. The shape of the free boundary $r = \eta(\theta)$ is close to the curve $r = s + q\cos\theta$ if δ is sufficiently large.

8. Small surface tension

In this section, problem (5.3) and (5.4) will be considered in the opposite limiting case, $\delta \to 0$. A formal solution of this problem can be performed as the asymptotic series

$$\eta \sim \sum_{n=0}^{\infty} \delta^n \eta_n,$$

where η_0 is a solution of the cubic equation

$$\eta_0^3 \beta \cos\theta - \eta_0 + 1 = 0 \tag{8.1}$$

and η_k $(k = 1, 2, \ldots)$ are expressed by explicit recurrent formulas. In particular,

$$\eta_1 = -(3 - 2\eta_0)^{-1}\, \eta_0^3 \left(\eta_0''' + \eta_0'\right).$$

The multiplier $(3 - 2\eta_0)^{-1}$ appears in expressions of η_k via $\eta_{k-1}, \ldots, \eta_0$ $(k = 2, 3, \ldots)$ also. As shown in [2] function η_0 is analytic in θ and satisfies inequalities $0.894 < \eta_0 < 3/2$ if $0 \leqslant \beta < 4/27$. It implies the analyticity of η_n for any $n = 1, 2, \ldots$ Below we introduce the following notation

$$\eta^{(N)} \sim \sum_{n=0}^{N} \delta^n \eta_n, \qquad (N = 0, 1, 2, \ldots)$$

which denotes an approximate solution of the problem (5.3), (5.4) at small δ.

Proposition 6. *If $0 < \beta < 4/27$ there exists $\delta_0 = \delta_0(\beta)$ such that if $\delta \in (0, \delta_0]$ the problem (5.3), (5.4) has a positive solution $\eta(\theta)$ and the following estimate is true:*

$$\max_{[0,2\pi]} \left| \eta - \eta^{(N)} \right| \leqslant C(\beta, N)\, \delta^{N+1}, \qquad N = 0, 1, 2, \ldots$$

as $\delta \in (0, \delta_0]$ where $C > 0$ does not depend on δ.

The proof of Proposition 6 has a technical character; it is not performed here. The key point of this proof is obtaining an a priori estimate for 2π-periodic solution ξ of the linear equation

$$\delta(\xi''' + \xi') + \eta_0^{-4}(3 - 2\eta_0)\,\xi = \psi(\theta), \tag{8.2}$$

as $\delta \to 0$ (here ψ is a smooth 2π-periodic function of θ). The desired estimate is

$$\|\xi\|_{C^3[0,2\pi]} \leqslant K\delta^{-3} \max_{[0,2\pi]} |\psi|, \qquad 0 < \delta \leqslant \delta_0$$

where $K > 0$ does not depend on δ. If $0 \leqslant \beta < 4/27$, Eq. (8.2) has no turning points within the interval $[0, 2\pi]$. Thus, the well-known asymptotic methods are applicable to analysis of its solutions as $\delta \to 0$ (see, for example, [13]).

9. Removing singularity by capillary forces

As mentioned above, solution of the equation (8.1), which corresponds to the zero surface tension, does not exist if $\beta > 4/27$ because of a singularity formation at the point $\theta = 0$. Could the capillary force remove this singularity? There are reasons to give a positive answer to this question.

The expansion of function η_0 defined by Eq. (8.1) with $\beta = 4/27$ near the point $\theta = 0$ is [2]

$$\eta_0 = \frac{3}{2} - \frac{1}{2}\sqrt{\frac{3}{2}}\,|\theta| + \mathrm{O}\left(\theta^2\right). \tag{9.1}$$

A formal solution of the problem (5.3), (5.4) when δ is small and β is close to the critical value 4/27 could be found by the method of matching asymptotic expansions.

Let us introduce the new independent variable φ and gravity parameter λ by the relations

$$\theta = \delta^{1/4}\varphi, \qquad \beta = \frac{4}{27} + \lambda\delta^{1/2}$$

and look for the solution of the problem (5.3), (5.4) in the form

$$\eta = \frac{3}{2} + \delta^{1/4} v + \mathrm{O}\left(\delta^{1/2}\right), \qquad \delta \to 0. \tag{9.2}$$

Function $v(\varphi)$ satisfies the equation

$$\frac{d^3 v}{d\varphi^3} = \frac{16}{81} v^2 - \frac{2}{27}\varphi^2 + \lambda, \qquad -\infty < \varphi < \infty. \tag{9.3}$$

Boundary conditions for function v are obtained from the matching condition of representations (9.1) and (9.2) as $\theta \to 0$ and $|\varphi| \to \infty$, respectively:

$$v + \frac{1}{4}\sqrt{3\left(2\varphi^2 - 27\lambda\right)} \to 0 \quad \text{if} \quad \varphi \to \pm\infty. \tag{9.4}$$

There is the only result concerning solvability of the problem (9.3), (9.4), which is given here without a proof.

Proposition 7. *If $\lambda < 0$ and $|\lambda|$ is sufficiently large, problem* (9.3), (9.4) *has a solution.*

Problem (9.3) and (9.4) was studied numerically by S. J. Chapman [14] and P. Howell [15]. S. J. Chapman found its numerical solution for moderate negative values of λ and for positive λ, which do not exceed $\lambda_* \approx 0.2053$. P. Howell discovered that this problem has many solutions for small $|\lambda|$. He stated a very likely conjecture: problem (9.3) and (9.4) has the unique solution if $\lambda < \lambda_{**} < 0$, has a finite number of solutions if $\lambda_{**} \leqslant \lambda \leqslant \lambda_*$, $\lambda \neq 0$, has no solutions if $\lambda > \lambda_*$, and has a countable set of solutions if $\lambda = 0$.

Acknowledgment.
Results of this paper were presented on the seminars of Universities of Cambridge, Nottingham and Oxford. Author is very grateful to the participants of these seminars S. J. Chapman, E. J. Hinch, P. Howell, J. R. King, J. B. McLeod, H. K. Moffatt, and J. R. Ockendon for fruitful discussions.

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Поступило 1 октября 2003