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D. Leites

ON COMPUTER-AIDED SOLVING DIFFERENTIAL EQUATIONS AND STABILITY STUDY OF MARKETS

ABSTRACT. For any nonholonomic manifold, i.e., a manifold with nonintegrable distribution, I define an analog of the Riemann curvature tensor and refer to Grozman's package **SuperLie** with the help of which the tensor had been computed in several cases. Being an analog of the usual curvature tensor this invariant characterizes (in)stability of any nonholonomic dynamical system, in particular, of markets. Similar invariants give criteria for formal integrability of differential equations whose symmetries are induced by contact transformations similar to Goldshmidt's criteria for formal integrability of differential equations whose symmetries are induced by point transformations. As a byproduct, one gets an approximate solution of the equation whose integrability is under study.

INTRODUCTION. MAIN IDEAS: AN OVERVIEW

This contribution to L. V. Kantorovich memorial conference touches two topics that were of acute interest to L. V. Kantorovich (see [16, 4][1]) and are of interest to some of the organizers (see [36, 35, 32]), namely, new methods of numerical solutions (here: of differential equations) and conditions for stability in market economy, a model of which is a particular example of a dynamical system with constraints on generalized "velocities."

Let us start with solving differential equations. Given an ordinary or partial differential equation, one usually can not solve it analytically and the engineer or economist uses computers to solve it numerically, replacing all functions involved by polynomials.

I am thankful to MPIM-Bonn for hospitality which enabled to finish the first draft of documentation of **SuperLie** ([10, 11]) – the package capable to compute not only the invariants discussed here but a lot more. I am thankful to I. Shchepochkina and P. Grozman for generous help.

[1]Kantorovich's enthusiasm was very inspiring. Due to L. V. Kantorovich, one of my brightest diploma students (E. Korkina) changed her field of research and got her Ph.D. in his laboratory.

As the degree of the polynomials-solutions grows, we get an approximation which ultimately should become a formal series but in practice remains a polynomial or a collections of polynomials.

To approximate a given function, we can use, for example, Taylor series expansion, or Fourier series, or wavelets, etc. Each method has its own pros and cons, depending on the context. I could never understand why engineers (or any other customer who needs to solve differential equations in earnest, in real life) never use criteria for formal integrability of differential equations Goldschmidt suggested in late 1960's (for an exposition, see [3]). As a byproduct, these criteria yield an approximate solution. It is just inconceivable that these criteria are only good for nothing but theoretical discussions and no type of differential equations is "convenient" enough for these criteria to be implemented in real computations.

Perhaps, the following facts constituted the stumbling block: (a) psychological threshold: these criteria are formulated in terms of Spencer *cohomology* – a new and scary term, not clear if worth learning; (b) even now no efficient code for computing cohomology exists (although the problem obviously involves lots of repeated computations and sparse matrices, none of these features was ever exploited); at the time Goldschmidt suggested his criteria the situation was much worse; (c) the criteria embraced only "one half" of all equations.

To explain (c), recall that a theorem of S. Lie states that all differential equations are of the two types: for one, all its symmetries are induced by point transformations, for the other one, all its symmetries are induced by contact transformations. (For details, see [17].) Goldschmidt's criteria fit only differential equations whose symmetries are induced by point transformations, they did not embrace differential equations whose symmetries are induced by contact transformations.

Now, recall that the contact structure is the simplest of *nonholonomic* structures. Recall also that it was Hertz [15] who, in his attempts to exorcise the dubious notion of "force," divided the dynamical systems into *holonomic* (no constraints on velocities or only integrable constraints) and *nonholonomic* (with nonintegrable constraints on velocities).

The first nonholonomic examples were from mechanics (see, e.g., [4]): a body rolling with friction over other body. Among various images that spring to mind, a simplest is that of a bike, or just a ball, rolling on an asphalt road. At the point of tangency of the wheel or the ball with asphalt the velocity is zero. This is a *linear* constraint.

More generally, a manifold (no dynamics) is said to be *nonholonomic* if endowed with a nonintegrable distribution (a subbundle of the tangent bundle). A famous theorem of Frobenius gives criteria of local integrability: the sections of the subbundle should span a Lie algebra.

One often encounters *nonlinear* constraints: switching the cruise control of your car ON you single out in the phase space of your car a distribution of spheres over the configuration space. Though it is difficult even to imagine what does it mean that such a distribution is integrable or not, at the end of this paper we give the corresponding definition.

For a historical review, see [37] and more recent [2] and especially [18]. See also a brilliant paper by Vershik [35] with first rigorous mathematical formulations of nonholonomic geometry and indications to various similar structures in several unexpected, at the time [35] was written, areas (like optimal control or macro-economics, where nonlinear constraints are also natural).

In [35], Vershik summarizes about 100 years of studies of nonholonomic geometry (Hertz, Carathéodory, Vrănceanu, Wagner, Schouten, Faddeev, Griffiths, and Godbillon; to his list we should add that Internet returns about 16,400 entries for "nonholonomic," and more for its synonyms: 2,500 more for "sub-Riemannian," 10,000 for "Finsler," several hundred for "cat's problem" and "autoparallel"; there seems to be more, actually, nonholonomic dynamical systems than holonomic ones; finally "supergravity" is also a nonholonomic structure, albeit on *super*manifolds – ca 122,000 more entries).

The term "nonholonomic" bridges the problem of formal integrability of differential equations with (stability problem of) market economy. Indeed, in his book [32], Sergeev demonstrated that macro-economy can be considered as an ideal gas and it was Carathérodory who long ago demonstrated that statistical thermodynamics is a nonholonomic system.

Prehistory: supergravity and rolling balls

To me, this story started with two seemimngly unrelated topics: Ogievetsky's lectures on supergravity and Vershik's dazzling lectures in Voronezh winter school on dynamics with constraints, see [35] (also a topic of interest to L. V. Kantorovich, cf. [4]).

Recall that Wess and Zumino (see [39]) were the first to realize and explain to others that *we live on a supermanifold* and thus made one of the most fundamental steps in science, not only in physics. Physicists realized that there is not one, but a whole hierarchy of *standard* Minkowski *super*spaces, plus a number of exotic models. The standard (at the moment) models are labeled by a parameter N running (conventionally) 1 to 8. The most interesting model is the largest one, for $N = 8$: it should embrace all.

Ensuing enthusiasm was overwhelming.

The mass media (from Pravda to New York Times) heralded that soon Einstein's dream – the Grand Unified Theory (or theories, now officially called SUSY GUTs for their SUperSYmmetry) — will come true. But eventually fanfare subsided. The reason: SUGRA equations for $N = 1$ were established during 1974–76. After much work in various research centers, it was Ogievetsky's group in JINR, Dubna, who, guided by their GUT feeling only for lack of ready mathematical recipes, realized by 1984 that to fulfill natural requirements for $N = 2$ (and to an extent $N = 3$) one has to *enlarge* the Minkowski space underlying Minkowski superspace by adding the what was called "harmonic" coordinate. What to do for $N > 3$ was unclear, see [7].

To apply for grants subsidizing studies of nonexisting (so far) equations became more difficult with time and most of the researchers left the field. What was the problem? (Not with grants, all is clear here; what is the *mathematical* difficulty?)

One of the best experts, Wess, honestly lectured [39]: "*we do not know how to express the Riemannian tensor (the left hand side of the equations to be written) in the general case.*"

In [1], Witten wrote: "*Direct experimental confirmation of supersymmetry is one of the prime missions of the proposed Superconducting supercollider.*" And then added: "*More fundamentally, I believe that the main obstacle is that the core geometrical ideas – which must underline string theory the way Riemannian geometry underlines general relativity – have not yet been unearthed.*"

I believe that I was lucky to have discovered some of these ideas [19]. They are related to (a) *nonholonomic* mechanics, (b) the presentation of the curvature tensor in terms of Lie algebra cohomology.

Relation with nonholonomic mechanics is occasioned by the fact that Minkowski *super*spaces (whatever they are) are **nonholonomic**, unlike the Minkowski *space* which is holonomic.

Analytical study of holonomic systems has progressed much further than that of nonholonomic. Moreover, only *linear* constraints are usually considered. Though Internet lists thousands of entries for nonholonomic and its synonyms, though nonholonomic systems are important in various branches of applied physics and engineering, they have not been sufficiently studied in mathematics, even the best text books on analytical mechanics just mention several examples and pass to holonomic systems.

One of the reasons: the qualitative study, such as stability problems, require the notion of curvature tensor which was not defined for the nonholonomic systems in the literature, cf. [33], the standard textbook for this purpose, see also [3]. Moreover, the attempts to define it lasted through the XX century and resulted in total frustration of the researchers: in the best (to my knowledge) mathematical description of nonholonomic systems [35] (this seems to be one of the latest work discussing the analog of the Riemann tensor in nonholonomic case), Vershik conjectured that, though known in some cases, "*it is probably impossible to construct an analog of the Riemann tensor for the general nonholonomic manifold.*"

However, in 1989, during my stay at IAS, I gave such a general definition (published much later in [19]) and, with co-authors, applied it to SUGRA (see [21, 11]); the result matched the one obtained earlier by *ad hoc* methods. Recently I found out that one of the numerous attempts to give a general definition of a nonholonomic version of the curvature tensor, the one due to Wagner (see [5]), was successful (or almost: it almost coincides with definitions below) but to decipher Wagner's results, expressed in terms of tensors with lots of indices, is very difficult and the answer is hardly usable.

Much later than Wagner, Tanaka [34] tackled the same, actually, problem for totally different reasons (even the term "nonholonomic" was never used) and Tanaka's results (especially their lucid exposition in [40]) are easier to understand because he used the hieroglyphics of Lie algebra cohomology which are much more graphic than tensor notations. Still, Tanaka only considered flags of depth 2 and of very particular form at that.

As a byproduct of her classification of simple infinite dimensional Lie superalgebras of vector fields ([25, 24]), Shchepochkina introduced several generalizations of Cartan prolongation ([29, 30]). These generalizations are precisely what is needed to define the nonholonomic analog of the Riemann tensor in the general case.

Observe that our nonholonomic invariants, though natural analogs of the curvature and torsion tensors, do not coincide on nonholonomic manifolds with the classical ones and bearing the same name. Indeed, by definition, on any nonholonomic manifold, there is a nonzero classical torsion while every contact manifold is flat in *our* sense. To avoid confusion, we should always add adjective "nonholonomic" for the invariants introduced below. Since this is too long, we will briefly say *N-torsion*, *N-curvature*, etc.

So, now we are able to compute the N-curvature for SUGRA: it is as difficult as to compute same for the phase space of a rolling ball.

Nonholonomic structures "forbid" time machines. When Gödel demonstrated that Einstein's theory does not preclude time machines, this was considered as a disaster: everyday experience hints that it is hardly possible to enter the same river twice. Probably this artefact of a mathematical model would be totally hushed up were Gödel less famous.

Casting Prigozin's and the like ideas in precise terms, A. Nordmark [26] demonstrated that nonholonomic systems have a preferred direction of time. Since "we live on a *nonholonomic* supermanifold," this probably explains why time machines are impossible to construct.

To markets and their stability

Sergeev [32] demonstrated that any market is a nonholonomic system (is singled out by (a system of) Pfaff equations), thus making it possible to use our techniques to investigate its stability.

Here we immediately encounter a technical problem. Though to compute the (nonholonomic) curvature boils down to a simple problem of linear algebra, this simple problem is repeated so many times that even computers give up. Besides, this "simple linear algebra" requires, actually, a considerable portion of representation theory. The code (**SuperLie**) capable to perform the task in some simplest cases was designed by Grozman first for the purposes of supergravity. Now, with more down-to-earth applications in view than a romantic supergravity, we refer the reader to [10] for the documentation of **SuperLie** and the package itself.

A general picture of some markets

"I called it borrowing because it was what Pap called it,
but Tom said it warn't borrowing, it was stealing."
Mark Twain, *Adventures of Huckleberry Finn.*

Lately, we observe an alarming destabilization of scientific milieu. The scientific market is corroded, notably, by the following factors: (1) some scientists live up to the motto "publish or perish," often on the verge of piracy or beyond it, (2) the general degeneration of the level of teaching of mathematics through the whole ladder from elementary schools to Ph.D. awarding universities and (3) aggressive proliferation of aggressive mediocrity.

Since all these phenomena threaten to destroy the creative atmosphere of the scientific market, it is more than interesting to find out the means for stabilization of the situation. The task is especially difficult since the oppressive "law enforcing" measures, though indispensable, are clearly insufficient.

Elimination of governmental support for math research might pass unobserved by general public, not only by shortsighted politicians (Dagens Nyheter – the Swedish daily with the largest circulation – refused to accept an article of a (Swedish) colleague of mine on the catastrophe ongoing in Sweden, where I live and work, as a "too narrow topic"). On the contrary, the fact that increased volatility of the usual, nonscientific, market threatened to shatter the foundation of our lives is understood by virtually everybody.

Before we begin studying stability of general markets, let me consider the scientific market in mathematics. The situation in other sciences is, it seems, even more disquieting.

Recently I received a letter from an eminent mathematician who requested to shed some light on the priority of a result. This is not the first such letter that my co-authors and I receive from time to time. Fortunately, each time the priority in question is easy to establish from dates of submission of published or preprinted papers (if one has time to read them).

The parenthesized remark is crucial. Modern scientific society exhorts almost unbearable pressure on the researcher forcing one to publish more often than one usually can. To ease one's burden to read and quote, which was one's duty in olden times, the mechanisms of public reproach or even ostracism for even unintentional failure to make a reference are nowadays slackened or lifted. As a result, abundant is faked or feigned ignorance of earlier rival results, the results that sometimes indicated direction or constitute the most part of the published result obtained later. Far more often we encounter reproduction of easier to understand parts of rival results (even the great Grothendieck complained). In *The Intelligencer*, V. Arnold gives examples with the time lag up to 10 years, his list can be easily continued and the lag of 18 years, or even 30 years is not rare.

To fight against individuals who "borrows" other's results without due credit does not appeal to the scientist. Besides, such individuals in their strife for higher salary and new awards often acquire what Freud, with his specific metaphors for gold in dreams, would, no doubt, have dubbed as syndrome of "inverted Midas touch," and are very dexterous in disputes and corrupting co-authors by threat and bribery. Even lost cases do not diminish their ability to obtain seats at various boards enabling them to determine invited speakers to conferences, steer publishers and tip hiring committees.

Instead, in this longish digression I would like to draw attention of not only "working mathematicians" but also of Administrators and Teachers to the alarming situation that budded always but currently flourished so ripe that AMS established a new body, Ethics committee.

The problem I address is rooted deep in the nature of the human ape, and therefore is difficult to solve.

Nevertheless, the very fact that there were periods of time in the past when society more actively disapproved patterns of behavior akin to that of yahoos, and the very fact that market economy is run, mainly, not by crooks and thieves, leaves some hope for success of the efforts of AMS's committees in future.

The absolute necessity of rigid moral rules for smooth functioning of markets was demonstrated by those whom we are used to least suspect as being ethics carriers: during the transition period from the state run economy to a market one the bandits popularly demonstrated, repeatedly, that markets agents must behave ([31]). The objects of their lessons are not with us, after soldering iron and other graphic means was applied to them, but the majority of those still alive took the hint.

This might be an unjustified optimism (in view of various and multiplying examples of huge corporations and even leading countries run by very dubious characters) could I not substantiate my optimism with mathematical results that might lead, among other things, to qualitative and even quantitative study of stability of markets. This mathematical description of markets shows that under certain conditions it is more profitable to behave honestly.

Our task is to help to create such circumstances that severe punishments or ethics enforcing committees become rudimentary.

Observe that under certain circumstances even brainless molecules of water range in order. They do it not because they heard a Call or read moral philosophers cum economists like A. Smith but because it is energetically more profitable for them to form ice.

Repressive measures alone do not, actually, ensure decent behavior of masses or individuals, and probably never will completely. (Even in countries with strict Sharia law some people steal at the price of an arm chopped off. But not on the alarming mass scale, though.) To achieve this goal, we must resolve an Herculean task and change the current instruments of and grounds for hiring, promotion and awarding.

Let me start with several examples. These examples might be, perhaps, of more acute interest if supplied with more names, especially, of our contemporaries. The names, however, would divert attention of the reader from the essence of the story whereas every expert will easily recognize the characters involved.

For further, more optimistic, reading on history with illustration of the fact that science consists of (at least) two parts: selected problems and theories, and how this division influenced the rewarding bodies, and how the idea of what is an important part of the scientific work changed with time I recommend inspiring books [8, 6], see also [20].

Examples from ethology. It is well known now, thanks to TV, even to non-ethologists, that every group of chimpanzees is partly ordered. It is not such a common knowledge that if one of the group happens to make a discovery or invention, the outcome for the society of chimps depends on the place of the discoverer in the hierarchy.

If, for example, it is an ω-chimp who has learned how to put the sticks together to reach bananas, the mates will expropriate these bananas, break the sticks, bite the discoverer and throw faeces at him or her, they might even harass the discoverer sexually.

If the discoverer is an α-chimp, the selected mates would be magnanimously awarded by the discoverer with bits of banana procured and the whole congregation will assiduously study the trick with the sticks and learn it in no time.

Most of the researchers are aware of this phenomenon, and therefore are eager to climb the hierarchical ladder or, at least, acquire an already esteemed co-author. There are many ways to do so, let me name just a few of them.

Publicity and marketing. A weak but industrious chimp discovered an empty tin. Chimps do not like noise (at least, not produced by themselves) and quickly became ready to do anything to please the one who started drumming the tin whenever he wanted to subject his mates to obedience. (The oftener one speaks about a result, the higher the level of conferences one delivers talks, the sooner the audience will identify the speaker as the discoverer of the result.)

PR: grooming and lobbying. On a more advanced level, a word in a right (or even left) ear is no less efficient in promotion than a louse taken out of the boss's fur. And it is clear, even to an idealist, that to join a pack (mafia, a political or religious group) considerably simplifies promotion and facilitates gaining resources.[2]

The above might sound funny were not the modern scientific community, partly manned by werewolves, no less ruthless sometimes than most notorious carnivorous species: B. Russell recollects:

"In the year 1922, Mussolini, who had **not** yet accomplished his coup d'état, sent word to the organizers of the [Mathematical] Congress that, while no harm should be done to me[3], any Italian who spoke to me should be assassinated."

Since 1922 there was some progress: now one never writes openly, like Mr. Mussolini. One does what one wants on the quiet.

Bose condensation of mediocrities. It is interesting to understand the nature of this phenomenon. I believe its nature lies in thermodynamics, rather than in biology or psychology.

Observe that the usually encountered most regrettable Fermi repulsion of geniuses is not universal. To find out "What are the *mathematical* conditions for its occurence?" is an intriguing open problem.

Is computing cohomology a road to salvation ?

The examples described above, everyday experience (Enron-like affairs, search for non-existing weapons as a pretext for an army excercise in the spirit of Orwell's "1984," though such excercises provoke advent of new terrible wars, etc.), may make one think that the situation is hopeless.

A pessimistic, but active approach is to consider the problem as impossible to be resolved without strict law-enforcing measures, or without attempts to spread some moral teaching. Any (unless overzealous) activity in this direction helps, I believe, to improve the situation but it is difficult to work if deep inside you suspect that the task is hopeless.

It seems to me that now we know enough mathematics to find out the causes for (in)stability of markets, and therefore can transplant the problem from a realm of guessing to the realm where an answer, however

[2] Göthe expressed this poetically: "Mit kleinen tut man kleine taten, mit grossen wird der kleine gross..."

[3] B. Russell, *The autobiography of Bertrand Russell*, 1872–1914, G. Allen & Unwin, London (1967), p. 135.

grim, might be obtained. This is going to be not an easy job, but the perspective of a definite answer (WHAT are the parameters that govern stability or markets) makes it very tempting to try.

§1. Structure functions of G-structures

In mid-1970s, Gindikin formulated a problem of local characterization of compact Hermitean symmetric domains $X = S/P$. Goncharov solved this problem [9] having considered the fields of certain quadratic cones and having computed the structure functions (obstructions to flatness) of the corresponding G-structures, where G is the Levi (reductive) subgroup of the simple Lie group S.

For $S = O(n+2)$ and $G = CO(n) = O(n) \times \mathbb{K}^\times$, these structure functions were known; they constitute the Weyl tensor – the conformally invariant part of the Riemann curvature tensor.

For $S = SL(n+m)$ and $G = S(GL(n) \times GL(m))$, these structure functions are obstructions to integrability of multidimensional analogs of Penrose's α- and β-planes on the Grassmannian Gr_n^{n+m}.

Not any S and P can form a classical domain: S is (almost any of) the simple complex Lie group(s) and $P = P_i$ is the maximal parabolic subgroup generated by all but one (say, negative, ith) Chevalley generators of S. In Goncharov's studies, the group P can not be arbitrary, either. To describe the admissible P's, let us label vertices of the Dynkin graph of S with the coefficients of the maximal root expressed in terms of the simple roots. The omitted generator should correspond to the vertex of the Dynkin graph with label 1.

It is natural to ask:

for any S and any its parabolic subgroup $P \subset S$,
what is the analog of the conformal
structure Goncharov introduced, (1)
which of these structures are flat,
and what are the obstructions to their flatness?

Let M^n be a manifold over a field $\mathbb{K}$. Let $F(M)$ be the frame bundle over M, i.e., the principal $GL(n;\mathbb{K})$-bundle. Let $G \subset GL(n;\mathbb{K})$ be a Lie group. A *G-structure* on M is a reduction of the principal $GL(n;\mathbb{K})$-bundle to the principal G-bundle. Another formulation is more appealing: if one can select transition functions from one coordinate patch to another so that they belong to G for every intersecting pair of patches, then we have a G-structure on M.

The simplest G-structure is the *flat* G-structure defined as follows. Let F be a fixed frame in $V=\mathbb{K}^n$. Having identified V with T_vV via parallel translation along v, we get a fixed frame in every T_vV. In other words, we consider V as a commutative Lie group with the Lie algebra T_0V, and either left or right translations identify T_0V with T_vV. Then we fix a frame in V, identify it with a frame in T_0V, and thus distribute over every T_vV. The *flat G-structure* is the bundle over V whose fiber over $v\in V$ consists of all frames obtained from the fixed one under the G-action.

In textbooks on differential geometry (e.g., in [33]), the obstructions to identification of the kth infinitesimal neighborhood of a point $m\in M$ on a manifold M with G-structure with that of a point $v\in V$ with the above describe flat G-structure are called *structure functions* of order k. It is also explained there that such an identification is possible provided all structure functions of lesser orders vanish.

To precisely describe the structure functions, we need some definitions. Let S^i denote the operator of the ith symmetric power. Set

$$\mathfrak{g}_{-1}=T_mM,\quad \mathfrak{g}_0=\mathfrak{g}=\mathrm{Lie}(G);$$

for any $i>0$ and $k=1,\dots,i$, set:

$$\begin{aligned}\mathfrak{g}_i=\{X\in\mathrm{Hom}(\mathfrak{g}_{-1},\mathfrak{g}_{i-1})\mid X(v_1)(v_2,v_3,\dots,v_i)=\\ =X(v_2)(v_1,v_3,\dots,v_i)\ \text{ for any }\ v_k\in\mathfrak{g}_{-1}\}.\end{aligned}$$

Now, set $(\mathfrak{g}_{-1},\mathfrak{g}_0)_*=\mathop{\oplus}\limits_{i\geq -1}\mathfrak{g}_i$.

Suppose that the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is faithful. Then, clearly,

$$(\mathfrak{g}_{-1},\mathfrak{g}_0)_*\subset\mathfrak{vect}(n)=\mathfrak{der}\,\mathbb{K}[[x_1,\dots,x_n]],\quad\text{where } n=\dim\ \mathfrak{g}_{-1}.$$

It is subject to an easy verification that the Lie algebra structure on $\mathfrak{vect}(n)$ induces same on $(\mathfrak{g}_{-1},\mathfrak{g}_0)_*$. The Lie algebra $(\mathfrak{g}_{-1},\mathfrak{g}_0)_*$ will be called the *Cartan's prolong* (the result of Cartan's *prolongation*) of the pair $(\mathfrak{g}_{-1},\mathfrak{g}_0)$. The Cartan prolong is the Lie algebra of symmetries the G-structure induces at T_mM.

Let Λ^i be the operator of the ith exterior power; set $C^{k,s}_{(\mathfrak{g}_{-1},\mathfrak{g}_0)_*}=\mathfrak{g}_{k-s}\otimes\Lambda^s(\mathfrak{g}^*_{-1})$. Define the differential $\delta_s: C^{k,s}_{(\mathfrak{g}_{-1},\mathfrak{g}_0)_*}\longrightarrow C^{k,s+1}_{(\mathfrak{g}_{-1},\mathfrak{g}_0)_*}$ setting for any $v_1,\dots,v_{s+1}\in V$ (as usual, the slot with the hatted variable is ignored):

$$(\delta_sf)(v_1,\dots,v_{s+1})=\sum_i(-1)^i[f(v_1,\dots,\widehat{v_{s+1-i}},\dots,v_{s+1}),v_{s+1-i}].$$

As expected, $\delta_s\delta_{s+1} = 0$. The homology of this bicomplex is called *Spencer cohomology* of the pair $(\mathfrak{g}_{-1}, \mathfrak{g}_0)$ and denoted by $H^{k,s}_{(\mathfrak{g}_{-1},\mathfrak{g}_0)_*}$.

1.1. Proposition ([33]). *The order k structure functions of the G-structure at every point $m \in M$ span* $H^{k,2}_{(\mathfrak{g}_{-1},\mathfrak{g}_0)_*}$.

Example. The values of the Riemann tensor on any n-dimensional ($n > 3$) Riemannian manifold M belong at every point $m \in M$ to $H^{2,2}_{(V,\mathfrak{o}(V))_*}$, where $V = T_mM$. The fact that $H^{1,2}_{(V,\mathfrak{o}(V))_*} = 0$ (no torsion) is called the Levi-Cevita theorem.

Remark (cf. [9]). The $\mathbb{Z}$-grading of $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$ induces a $\mathbb{Z}$-grading of $H^s(\mathfrak{g}_{-1}; (\mathfrak{g}_{-1}, \mathfrak{g}_0)_*)$. Let H^s_k be the degree k component of $H^s(\mathfrak{g}_{-1}; (\mathfrak{g}_{-1}, \mathfrak{g}_0)_*)$. Clearly, $H^{k,s}_{(\mathfrak{g}_{-1},\mathfrak{g}_0)_*} = H^s_k$, so

$$\overset{k}{\oplus} H^{k,s}_{(\mathfrak{g}_{-1},\mathfrak{g}_0)_*} = H^s(\mathfrak{g}_{-1}; (\mathfrak{g}_{-1}, \mathfrak{g}_0)_*).$$

This remark considerably simplifies calculations, in particular, if the Lie algebra $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$ is simple and finite dimensional: we can apply the Borel–Weil–Bott (BWB) theorem, retold in [13]. In the nonholonomic case considered in what follows the remark provides with a compact definition[4] of structure functions. We can recover the bigrading (explicitly preserved in [33]) at any moment but to work with just one grading is much simpler, as with generating functions of series instead of individual terms. Moreover, there are a few general theorems that help to calculate Lie algebra cohomology and no theorems on Spencer cohomology.

§2. Structure functions of nonholonomic structures

To embrace contact-like – nonholonomic – structures, and generalize Goldshmidt's criteria to all differential equations, we have to generalize the notion of Cartan prolongation: the tangent space to a point of a nonholonomic manifold is naturally rigged with a nilpotent Lie algebra structure (e.g., for any odd dimensional contact manifolds this is the Heisenberg algebra), cf. [38].

[4] Cf. with the problems encountered in the pioneer papers [34], where only $d = 2$ is considered. Wagner's tensors (for any d) look even more horrible.

2.1. Nonholonomic manifolds ([35, 37]). Let M^n be an n-dimensional manifold with a nonintegrable distribution $\mathcal{D}$. Let

$$\mathcal{D} = \mathcal{D}_1 \subset \mathcal{D}_2 \subset \mathcal{D}_3 \dots \subset \mathcal{D}_d$$

be the sequence of strict inclusions, where the fiber of $\mathcal{D}_i$ at a point $x \in M$ is

$$\mathcal{D}_{i-1}|_x + [\mathcal{D}_1, \mathcal{D}_{i-1}]|_x$$

(here $[\mathcal{D}_1, \mathcal{D}_{i-1}] = \{[X, Y] \mid X \in \Gamma(\mathcal{D}_1), Y \in \Gamma(\mathcal{D}_{i-1})\}$) and d is the least number such that

$$\mathcal{D}_d|_x + [\mathcal{D}_1, \mathcal{D}_d]|_x = \mathcal{D}_d|_x.$$

In case $\mathcal{D}_d = TM$ the distribution is called *completely nonholonomic*. The number $d = d(M)$ is called the *nonholonomicity degree*. A manifold M with a distribution D on it will be referred to as *nonholonomic* one if $d(M) \neq 1$.

For our purposes, it suffices to consider only completely nonholonomic manifolds. This does not diminish generality, actually, since if the distribution in question is not completely nonholonomic, we can take the integral manifold corresponding to it; the restriction of the same distribution onto this integral manifold is completely nonholonomic. If, nevertheless, we *have to*, for some reason, consider the general case it is easy *mutatis mutandis*; it corresponds to *non-transitive* generalized prolongs, cf. below, and the general machinery is easily adjusted to them.

Let $n_i(x) = \dim \mathcal{D}_i|_x$. The distribution $\mathcal{D}$ is said to be *regular* if all the dimensions n_i are constants on M. In what follows we will only consider regular distributions. This is also not a restriction if everything is smooth.

The tangent bundle over a nonholonomic manifold $(M, \mathcal{D})$ can be naturally endowed with a bundle of nilpotent Lie algebras as follows. Fix a point $x \in M$. The ususal adic filtration by powers of the maximal ideal $\mathfrak{m} := \mathfrak{m}_x$ consisting of functions that vanish at x should be modified because distinct coordinates may have distinct "weights." The distribution $\mathcal{D}$ induces the following filtration in $\mathfrak{m}$:

$$\mathfrak{m}_i = \{f \in \mathfrak{m} \mid f = X_1^{a_1} \dots X_d^{a_d}(f) = 0 \\ \text{for any } \ X_j \in \Gamma(\mathcal{D}_j) \ \text{ and } \ \sum a_j \leq i\}, \quad (2)$$

where $\Gamma(\mathcal{D}_i)$ is the space of germs at x of sections of the bundle D_i.

Let us show that every nonholonomic structure on a manifold M determines a nilpotent Lie algebra structure in every tangent space to M.

Given two vectors $v_1, v_2 \in T_mM$, take two vector fields $X_1, X_2 \in \Gamma(TM)$ such that

$$X_i|_m = v_i \text{ for } i = 1, 2 \tag{3}$$

and set

$$[v_1, v_2] = [X_1, X_2]|_m. \tag{4}$$

Exercise. The bracket (4) does not depend on the choice of the fields satisfying (3).

Digression: Natural bases in T_mM: the D_i's and the Q_i's. A natural basis in every tangent space T_mM to any manifold M is given by partial derivatives. If M is endowed with a nonholonomic structure, then there are **two** types of natural bases in T_mM. Moreover, the tangent space to every point of any nonholonomic manifold is endowed with a nilpotent Lie algebra structure. The elements of these above-mentioned bases that generate the Lie algebra T_mM are denoted in physical literature on supersymmetry and supergravity the D_i's and the Q_i's.

Let us consider the simplest example. Let $\dim M = 2n+1$ and let the nonholonomic structure on M be given by the contact form $\alpha = dt - \sum(p_i dq_i + q_i dp_i)$. The vector fields that belong to the distribution $\mathcal{D}$ are the fields

$$X = f\partial_t + \sum(g_i\partial_{q_i} + h_i\partial_{p_i})$$
$$\text{such that } \alpha(X) = f - \sum(p_i g_i + q_i h_i) = 0. \tag{5}$$

In particular, we see that neither ∂_{q_i} nor ∂_{p_i} belongs to $\mathcal{D}$, but rather

$$D_{p_i} = \partial_{q_i} + p_i\partial_t \text{ and } D_{q_i} = \partial_{p_i} - q_i\partial_t.$$

These D_{p_i} and D_{q_i} are examples of the D-type basis vectors.

Now, the Lie algebra that preserves $\mathcal{D}$ consists of vector fields X such that

$$L_X(\alpha) = 0. \tag{6}$$

The corresponding vector fields in our particular case of the contact distribution are contact vector fields and the basis of the tangent space is spanned by

$$K_{p_i} = \partial_{q_i} - p_i\partial_t \text{ and } K_{q_i} = \partial_{p_i} + q_i\partial_t$$

and their brackets. These K_{p_i} and K_{q_i} are examples of the Q-type basis vectors.

How to interprete the D-type and the Q-type vectors? Let

$$\mathfrak{n} = \bigoplus_{-d\leq i\leq -1} \mathfrak{n}_i$$

be a nilpotent Lie algebra generated by $\mathfrak{n}_{-1}$. Let $B = \{b_1, \dots, b_n\}$ be a graded basis of $\mathfrak{n}$ (the basis is said to be *graded* if its first $n_1 := \dim \mathfrak{n}_{-1}$ elements span $\mathfrak{g}_{-1}$, the next $n_2 := \dim \mathfrak{n}_{-2}$ elements span $\mathfrak{n}_{-2}$, and so on. Let N be the connected and simply connected Lie group with the Lie algebra $\mathfrak{n}$. On N, consider the two systems of vector fields: the left-invariant fields D_i and the right-invariant fields Q_i such that (e is the unit of N)

$$D_i(e) = Q_i(e) = b_i \text{ for all } i = 1, \dots, n.$$

NB: Here we deviate from the conventions in physical papers in which the symbols D_i and Q_i are only applied to the generators of $\mathfrak{n}$, i.e., to the first n_1 elements.

Let $\mathfrak{g}_-$ be a realization of $\mathfrak{n}$ by **left-invariant** vector fields, so the vectors $D_i(e)$ span $\mathfrak{g}_-$. Let θ^i be right-invariant 1-forms on N such that

$$\theta^i(Q_j) = \delta^i_j.$$

Now, any vector field X on N is of the form

$$X = \sum_{i=1}^{n} \theta^i(X) Q_i.$$

Since each D_i commutes with each Q_j (and they also **commute**, not supercommute when $\mathfrak{n}$ is a Lie superalgebra), it follows that

$$\theta^i([D_j, X]) = D_j(\theta^i(X)).$$

Now, let us determine a right-invariant distribution $\mathcal{D}$ on N such that $\mathcal{D}|_e = \mathfrak{n}_{-1}$. Clearly, $\mathcal{D}$ is singled out in TN by eqs.

$$\theta^{n_1+1} = \dots = \theta^n = 0.$$

Since each D_i commutes with each Q_j, the algebra $\mathfrak{g}_-$ preserves $\mathcal{D}$. The coordinates (2) on N described above determine two embeddings of $\mathfrak{n}$ into $\mathfrak{vect}(n)$: one is spanned by the D_i and the other one by the Q_i.

The grading of the coordinates (2) determines a nonstandard grading of $\mathfrak{vect}(n)$:

$$\begin{gathered}\deg x_1 = \ldots = \deg x_{n_1} = 1,\\ \deg x_{n_1+1} = \ldots = \deg x_{n_2} = 2,\\ \dots\dots\dots\dots\dots\dots\\ \deg x_{n-n_d+1} = \ldots = \deg x_n = d.\end{gathered} \tag{7}$$

Denote by $\mathfrak{g} = \underset{i\geq -d}{\oplus} \mathfrak{g}_i$ the algebra $\mathfrak{vect}(n)$ with the grading (7). Then $\mathfrak{g}_- = \underset{i<0}{\oplus} \mathfrak{g}_i$ preserves $\mathcal{D}$.

For nonholonomic manifolds, an analog of G from the term "G-structure," or rather of $\mathfrak{g} = \mathrm{Lie}(G)$, is the pair $(\mathfrak{g}_-, \mathfrak{g}_0)$, where $\mathfrak{g}_0$ is a subalgebra of the Lie algebra which preserves the $\mathbb{Z}$-grading of $\mathfrak{g}_-$, i.e., $\mathfrak{g}_0 \subset (\mathfrak{der}\ \mathfrak{g}_-)_0$. If $\mathfrak{g}_0$ is not explicitly indicated we assume that $\mathfrak{g}_0 = (\mathfrak{der}\ \mathfrak{g}_-)_0$, i.e., is the largest one.

Thus we see that, with every nonholonomic manifold $(M, \mathcal{D})$, a natural G-structure is associated, its Lie algebra is $\mathrm{Lie}(G) = (\mathfrak{der}\ \mathfrak{g}_-)_0$. But the structure functions of this G-structure do not reflect the nonholonomic nature of M.

Indeed, recall an example from [33]. Let $W_1 \subset W$ be a subspace of dimension k and $G \subset GL(W)$ the parabolic subgroup that preserves the subspace. Then any G-structure on M, where $\dim M = \dim W$, is the same as a differential k-system or a k-dimensional distribution. A fixed frame f in T_mM determines an isomorphism $f : W \longrightarrow T_mM$. Given a G-structure on M, we set $\mathcal{D}(m) = f(W_1)$. Since G preserves W_1, the subspace $\mathcal{D}(m)$ indeed depends only on m, not on f.

The other way round, given a distribution $\mathcal{D}$, consider the frames f such that $f^{-1}(\mathcal{D}(m)) = W_1$. They form a G-structure. The flat G-structures correspond to integrable distributions.

To take the nonholonomic nature of M into account, we need an analog of the above proposition for the case where the natural basis of the tangent space consists not of partial derivatives but rather of covariant derivatives corresponding to the connection determined by the same Pfaff equations that determine the distribution, and therefore instead of $T_mM = \mathfrak{g}_{-1}$ we have $T_mM = \mathfrak{g}_-$.

So we have to define

(1) the simplest nonholonomic structure – the "flat" one,

(2) the analog of Cartan prolong when $\mathfrak{g}_{-1}$ is replaced by $\mathfrak{g}_-$,

(3) what is the analog of $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$,

(4) what is the analog of $H^{k,2}_{(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*}$.

Here are the answers:

1) Let $\mathcal{D}$ be a nonholonomic distribution in M, let F be the flag $\mathcal{D}$ determines at a point $m \in M$. Let $N =: \mathbb{C}^n$ with a fixed flag F and a fixed frame f. Having identified T_nN with N by means of the translation by n considered as an element of the *nilpotent* Lie group N whose Lie algebra is $\mathfrak{g}_-$ (since the group N is not commutative now, we select, say,

left translations) we fix a frame and a flag — the images of f and F — in each T_nN. A *flat nonholonomic structure* on N is the *pair* of bundles (the frame bundle, the distribution $\mathcal{D}$); the fibers of both bundles over n are obtained from the fixed frame and flag, respectively, by means of the G-action, where G is the (connected and simply connected) Lie group whose Lie algebra $\mathfrak{g}_0$ is defined in step 2).

2) If only a distribution $\mathcal{D}$ is given, we set $\mathfrak{g}_0 := \mathfrak{der}_0\mathfrak{g}_-$; it is often interesting to consider an additional structure on the distribution, say Riemannian, cf. [37], as in the case of Carnot-Carathéodory metric. This means we take for $\mathfrak{g}_0$ a subalgebra of $= \mathfrak{der}_0\mathfrak{g}_-$.

3) Given a pair $(\mathfrak{g}_-, \mathfrak{g}_0)$ as above, define its kth *Shchepochkina prolong* (first published by I. Shchepochkina [29] for Lie superalgebras; see also more accessible [30] and [25]) for $k > 0$ to be:

$$\mathfrak{g}_k = (i(S^*(\mathfrak{g}_-)^* \otimes \mathfrak{g}_0) \cap j(S^*(\mathfrak{g}_-)^* \otimes \mathfrak{g}_-))_k,$$

where the subscript singles out the component of degree k and where

$$i : S^{k+1}(\mathfrak{g}_{-1})^* \otimes \mathfrak{g}_{-1} \longrightarrow S^k(\mathfrak{g}_{-1})^* \otimes \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}$$

and

$$j : S^k(\mathfrak{g}_{-1})^* \otimes \mathfrak{g}_0 \longrightarrow S^k(\mathfrak{g}_{-1})^* \otimes \mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}$$

are natural embeddings.

Similarly to the case where $\mathfrak{g}_-$ is commutative, define $(\mathfrak{g}_-, \mathfrak{g}_0)_*$ to be $\underset{k\geq -d}{\oplus} \mathfrak{g}_k$; then, as is easy to verify, $(\mathfrak{g}_-, \mathfrak{g}_0)_*$ is a Lie algebra.

4) By the same reasons as for the manifolds with G-structure ([33]), $H^2(\mathfrak{g}_-; (\mathfrak{g}_-, \mathfrak{g}_0)_*)$ is the space of values of the structure functions – obstructions to the identification of the infinitesimal neighborhood of a point m of the manifold M with a nonholonomic structure (given by $\mathfrak{g}_-$ and $\mathfrak{g}_0$) with the infinitesimal neighborhood of a point of a flat nonholonomic manifold with the same $\mathfrak{g}_-$ and $\mathfrak{g}_0$.

The space $H^2(\mathfrak{g}_-; (\mathfrak{g}_-, \mathfrak{g}_0)_*)$ naturally splits into homogeneous components whose degrees will be called the *orders* of the structure functions; the order ranges $2 - d$ to $N + 2d$ or ∞, where $(\mathfrak{g}_-, \mathfrak{g}_0)_* = \overset{N}{\underset{k\geq -d}{\oplus}} \mathfrak{g}_k$. As in the case of a commutative $\mathfrak{g}_- = \mathfrak{g}_{-1}$, the structure functions of order k can be interpreted as obstructions to flatness of the nonholonomic manifold with the $(\mathfrak{g}_-, \mathfrak{g}_0)$-structure provided the obstructions of lesser orders vanish. Observe that, for nonholonomic manifolds, the order of structure functions is no more in direct relation with the numbers of the infinitesimal neighborhoods of the points we wish to identify: distinct partial derivatives bear different "weights."

Different filtered algebras L with the same graded $\mathfrak{g}_-$ are governed precisely by the coboundaries responsible for filtered deformations, and all of them vanish in cohomology, so the above nonholonomic structure functions are well-defined.

§3. The Riemann and Weyl tensors. Projective structures

The conformal case. For the classical domains $X = S/P$ Goncharov considered, the structure functions are generalizations of the Weyl tensor – the conformally invariant part of the Riemann tensor (the case $S = O(n+2)$ and $G = CO(n)$). In most of these cases

$$(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \mathfrak{s}\ (:= \mathrm{Lie}(S)) \tag{$*$}$$

and the description of the structure functions is a particular case of the BWB theorem.

In particular, if $(*)$ holds, the space $H^2(\mathfrak{g}_{-1}; (\mathfrak{g}_{-1}, \mathfrak{g}_0)_*)$, considered as a $\mathfrak{g}_0$-module, has the same number of irreducible components and the same dimension as $\Lambda^2(\mathfrak{g}_{-1})$; only weights differ.

The generalized Riemannian case. When we reduce $\mathfrak{g}_0$, by taking its semisimple part $\hat{\mathfrak{g}_0}$ and deleting the center, we can not directly apply the BWB theorem, because $(\mathfrak{g}_{-1}, \hat{\mathfrak{g}_0})_* = \mathfrak{g}_{-1} \oplus \hat{\mathfrak{g}_0}$ is not simple. But we can use the formula

$$H^2(\mathfrak{g}_{-1}; (\mathfrak{g}_{-1}, \hat{\mathfrak{g}_0})_*) = H^2(\mathfrak{g}_{-1}; \mathfrak{s}) \oplus S^2(\mathfrak{g}_{-1}^*).$$

For the nonholonomic case, the general answer is given by Premet's theorem cited in [13]. (This answer is implicit; the first explicit calculations for flag varieties whose group of motions is simple will soon be available.[5])

The projective case. When $(*)$ fails, $\mathfrak{s}$ is a subalgebra of $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$. This happens if $S = SL(n+1)$ and $X = \mathbb{C}P^n$; in this case $\mathfrak{g}_0 = \mathfrak{gl}(n)$ and $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \mathfrak{vect}(n)$, the Lie algebra of vector fields in n indeterminates.

We have $H^2(\mathfrak{g}_{-1}; \mathfrak{vect}(n)) = 0$ whereas $H^2(\mathfrak{g}_{-1}; \mathfrak{s}) \neq 0$, these structure functions correspond to obstructions to the *projective structure*.

[5] **Added in proof**: See P. Grozman, D. Leites, *The nonholonomic Riemann and Weyl tensors for flag manifolds*. In: J. Bernstein, V. Hinich, A. Melnikov (eds.), A. Joseph Festschrift, Birkhäuser (2005).

The Riemannian version of this case, corresponding to $\hat{\mathfrak{g}_0} = \mathfrak{sl}(n)$ gives $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \mathfrak{svect}(n)$, the Lie algebra of divergence free vector fields. In this case $H^2(\mathfrak{g}_{-1}; \mathfrak{svect}(n)) = 0$.

In what follows, for manifolds $X = S/P$ with nonholonomic structure, we say "N-Weyl" or "N-conformal," for cohomology of $\mathfrak{g}_-$ with coefficients in $(\mathfrak{g}_-, \mathfrak{g}_0)_*$, "N-Riemannian" for nonholonomic structure functions $(\mathfrak{g}_-, \hat{\mathfrak{g}_0})_*$, where $\hat{\mathfrak{g}_0}$ is the semisimple part of $\mathfrak{g}_0$, and "N-projective" for the coefficients in $\mathfrak{s} = \mathrm{Lie}(S)$ or any partial Cartan prolong (for the definition, see [25]) of $(\mathfrak{g}_-, \mathfrak{g}_0)$.

The simplest examples. Let $\mathfrak{g}_- = \mathfrak{hei}(2n)$, the Heisenberg Lie algebra. Then $\mathfrak{g}_0 = \mathfrak{csp}(2n) = \mathfrak{sp}(2n) \oplus \mathbb{C}z$ and $(\mathfrak{g}_-, \mathfrak{g}_0)_*$ is the Lie algebra $\mathfrak{k}(2n+1)$ of contact vector fields.

So far, there is no analog of Serre's theorem on involutivity for simple $\mathbb{Z}$-graded Lie algebras of depth > 1, cf. [22], and examples from [14] show that if exists, it will be more involved.

Nevertheless, it is easy to show that $H^2(\mathfrak{hei}(2n); \mathfrak{k}(2n+1)) = 0$. In our terms this is an explanation why the Pfaff equation $\alpha = 0$, where α is the contact form, can be locally reduced to a canonical form.

For the nonholonomic analog of Riemann tensor in this case, we have: $\hat{\mathfrak{g}_0} = \mathfrak{sp}(2n)$ and $(\mathfrak{g}_-, \hat{\mathfrak{g}_0})_*$ is the Poisson Lie algebra $\mathfrak{po}(2n)$. Since the contact form can be locally reduced to a canonical form (Darboux's theorem, cf. [41]), $H^2(\mathfrak{hei}(2n); \mathfrak{po}(2n))$ must vanish. (It does.)

§4. Integrability of the fields of spheres or cones

Given a k-dimensional distribution D (a rank k subbundle of the tangent bundle TM^n), we say that D is *integrable at point* $m \in M$ if there is a dimension k submanifold $N \subset M$ such that $T_m N = D|_m$.

If S is a distribution of spheres or cones or similar bodies (at every point m we have an embedding $S_m \subset T_m M$), there seems to be no equally graphic picture illustrating what should we consider as integrability of such a distribution. However, since the problem is local, it suffices to consider ∞-jets of the embedding $S_m \subset T_m M$ – a linearization of this embedding. So we have reduced the problem to a linear distribution, but on the ∞-dimensional manifold.

How to solve the arising problem is technically unclear, but all the paraphernalia is ready: in the above definitions, we "just" have to consider ∞-dimensional $\mathfrak{g}_-$ and $\mathfrak{g}_0$.

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Sophus-Lie-Professor, MPIMiS,
Leipzig, Germany
(on leave from
University of Stockholm,
Sweden)

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E-mail: mleites@math.su.se