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CONVERGENCE OF DISCRETIZED ATTRACTORS FOR PARABOLIC EQUATIONS ON THE LINE

Abstract. We show that, for a semilinear parabolic equation on the real line satisfying a dissipativity condition, global attractors of time-space discretizations converge (with respect to the Hausdorff semi-distance) to the attractor of the continuous system as the discretization steps tend to zero. The attractors considered correspond to pairs of function spaces (in the sense of Babin-Vishik) with weighted and locally uniform norms (taken from Mielke-Schneider) used for both the continuous and discrete systems.

**Dedicated to the memory of
Olga Aleksandrovna Ladyzhenskaya**

1. Introduction

Let Σ be the evolutionary system generated by a semilinear parabolic equation

$$u_t = \Delta u + f(u), \quad t \geq 0, \quad x \in \mathbb{R}, \quad \Delta u = \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

so that $\Sigma(t, u_0)$, $t \geq 0$, is the solution with initial value $u_0 = u_0(x)$.

Let $\mathcal{S} = \mathcal{S}_{h,d}$ be the dynamical system generated by the implicit space-time discretization

$$(u^{n+1} - u^n)/h = Au^{n+1} + \overline{f}(u^{n+1}) \tag{2}$$

of Eq. (1) with time step h and space step d, where A is the standard three point difference approximation of the operator Δ (see the exact definition below) and $(\overline{f}(u))_i = f(u_i)$.

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In this paper, we study the convergence of the global attractors $\mathcal{A}(h,d)$ of the systems $\mathcal{S}_{h,d}$ (for their existence see [3]) to the global attractor $\mathcal{A}$ of the system Σ as the stepsizes $h, d \to 0$.

For bounded domains, the study of global attractors for a wide class of discretizations of parabolic equations was begun by Ladyzhenskaya in [9]. In this case, the problem of convergence of "approximate" global attractors to the "exact" attractor has been studied extensively for various approximations in x and t (see, for example, [6, 12, 13]). The main new feature for an unbounded domain compared to the bounded domain is that the evolutionary system lacks compactness properties.

We adopt the approach of Babin and Vishik [1] and consider global attractors corresponding to pairs of function spaces. Our choice of weighted spaces follows Mielke and Schneider [11].

Let us first describe the spaces we work with. We fix $\varepsilon > 0$ (to be specified later) and a weight function

$$\rho(x) = (1+\varepsilon^2 x^2)^{-\gamma}, \quad \text{where} \quad \gamma \geq \frac{1}{2}.$$

For shifts and finite differences we use the following notation: if $u = \{u_k : k \in \mathbb{Z}\}$ is a sequence and $v(x), x \in \mathbb{R}$, is a function, then

$$(\partial_+ u)_k = (u_{k+1} - u_k)/d, \quad (\partial_- u)_k = (u_k - u_{k-1})/d,$$

$$(\partial_+ v)(x) = (v(x+d) - v(x))/d, \quad (\partial_- v)(x) = (v(x) - v(x-d))/d,$$

$$(T_y u)_k = u_{k+y}, \quad \text{and} \quad (T_y v)(x) = v(x+y),$$

where $y \in \mathbb{Z}$ in the first case and $y \in \mathbb{R}$ in the second case.

With this notation, the operator A in Eq. (2) is defined by the formula $Au = \partial_+ \partial_- u$.

We consider two Hilbert spaces of sequences $u = \{u_k : k \in \mathbb{Z}\}$: H_ρ with the norm defined by

$$\|u\|_{0,\rho}^2 = d \sum_{k \in \mathbb{Z}} \rho_k u_k^2,$$

where $\rho_k = \rho(kd)$, and Z_ρ with the norm defined by

$$\|u\|_{1,\rho}^2 = \|u\|_{0,\rho}^2 + \|\partial_- u\|_{0,\rho}^2,$$

and two Hilbert spaces of functions $u = u(x), x \in \mathbb{R}$: $\mathcal{H}_\rho$ with the norm defined by

$$\|u\|_{0,\rho}^2 = \int_{\mathbb{R}} \rho(x) u(x)^2 \, dx$$

and $\mathcal{Z}_\rho$ with the norm defined by

$$\|u\|_{1,\rho}^2 = \|u\|_{0,\rho}^2 + \|\nabla u\|_{0,\rho}^2.$$

Note that, for any of the spaces above, the norms corresponding to two different choices of ε are equivalent.

Finally, we introduce the space Z_u of sequences $u = \{u_k : k \in \mathbb{Z}\}$ with the norm defined by

$$\|u\|_{1,u} = \sup_{y \in \mathbb{Z}} \|T_y u\|_{1,\rho}$$

and the space Z_u of functions $u = u(x), x \in \mathbb{R}$, with the norm defined by

$$\|u\|_{1,u} = \sup_{y \in \mathbb{R}} \|T_y u\|_{1,\rho}$$

We assume that the nonlinearity f in Eq. (1) satisfies the following main conditions:

(AI) f is in $C^1(\mathbb{R})$ with globally bounded derivative;

(AII) for some $a, b > 0$, the function f satisfies the dissipativity condition

$$uf(u) \leq -au^2 + b, \quad u \in \mathbb{R}. \tag{3}$$

Condition (3) implies that $f(c) = 0$ for some $c \in \mathbb{R}$ and therefore, by the change of variables $u := u + c$, we may assume that $f(0) = 0$.

Let Φ be a general evolutionary system on a Banach space $\mathcal{R}$ and let $\mathcal{R}$ contain a Banach space $\mathcal{R}'$ with continuous embedding. Following [1], we say that $I \subset \mathcal{R}$ is the global $(\mathcal{R}', \mathcal{R})$-attractor of Φ if

(i) I is a compact set in $\mathcal{R}$;

(ii) I is positively invariant with respect to Φ, i.e., $\Phi(t, I) = I$ for $t \geq 0$;

(iii) I attracts bounded subsets of $\mathcal{R}'$ with respect to the topology of $\mathcal{R}$.

It is proved in [7] that, under conditions (AI),(AII), the system Σ has the global $(Z_u, \mathcal{Z}_\rho)$ attractor $\mathcal{A}$. Note that, for a different pair of function spaces, the existence of a global attractor of the system Σ has been established by Babin and Vishik in a pioneering paper [2]. The proof in [7] uses the choice of spaces and an abstract result from [11].

The main result of [3] shows that if conditions (AI) and (AII) are satisfied and h and d are small enough, then the system $\mathcal{S}$ has the global (Z_u, Z_ρ)-attractor $\mathcal{A}(h, d)$ and this attractor has a bound in Z_u that is uniform in h and d. In addition, the attractor $\mathcal{A}(h,d)$ is invariant under $\mathcal{S}$ in the sense

$$\mathcal{S}^t(\mathcal{A}(h,d)) = \mathcal{A}(h,d) \quad \text{for} \quad t \in \mathbb{Z}. \tag{4}$$

Notice that this property can be shown for all $t \in \mathbb{Z}$ due to invertibility of the system $\mathcal{S}$. For noninvertible systems such as Σ this property is replaced by the fact that the global attractor consists of complete orbits [12].

We embed the space H_ρ into $\mathcal{H}_\rho$ as follows. Define a partition of unity $\{\omega_k\}_{k\in\mathbb{Z}}$, where the hat functions $\omega_k(x)$, $x \in \mathbb{R}$, are given by

$$\omega_k(x) = \begin{cases} (x-(k-1)d)/d, & x \in [(k-1)d, kd], \\ ((k+1)d-x)/d, & x \in [kd, (k+1)d], \\ 0 & \text{otherwise}. \end{cases}$$

Then we define the interpolation operator $\mathcal{T} : H_\rho \to \mathcal{H}_\rho$ by

$$\mathcal{T}\{u_k\} = \sum_{k\in\mathbb{Z}} \omega_k(x) u_k. \tag{5}$$

For two sets $B_1, B_2 \subset \mathcal{Z}_\rho$, we introduce the Hausdorff semi-distance, which we call the deviation for short, by

$$\operatorname{dev}(B_1, B_2) = \sup_{u\in B_1} \operatorname{dist}(u, B_2).$$

Here "dist" is generated by the norm of the space $\mathcal{Z}_\rho$.

The main result of this paper is the following statement.

Theorem 1. *Under the assumptions* (AI) and (AII), *the attractors converge in the following sense:*

$$\operatorname{dev}(\mathcal{T}\mathcal{A}(h,d), \mathcal{A}) \to 0 \quad \text{as } h, d \to 0. \tag{6}$$

We have stated this theorem for scalar parabolic equations. It is, however, quite straightforward to extend the result to systems along the lines of [3]. A preliminary version of this paper is contained in the authors' preprint [4].

The structure of the paper is as follows. In Sec. 2, we summarize regularity estimates from [7] for solutions of Eq. (1) on finite time intervals and state error estimates for projectors and interpolation operators in weighted norms that have been derived in [8]. Moreover, we set up the basic technical results that are used in Sec. 3 to prove the main theorem.

Then sections 4 and 5 are devoted to the proof of the technical results from Sec. 2 – a finite time error estimate between solutions of Eq. (1) and the corresponding finite difference solution, a regularity estimate for the discrete solution, and an asymptotic compactness result. In all cases, we use the weighted and uniform norms defined above.

2. Preliminary Estimates

In our reasoning below, we apply the following regularity result for solutions of Eq. (1) on finite time intervals (see [7]). Note that all derivatives of a solution mentioned in Proposition 2 exist for almost all (t, x).

Proposition 2. *Assume that the nonlinearity f in Eq. (1) is Lipschitz continuous. Let $u(t, x)$ be a solution of Eq. (1) such that $u_0 \in \mathcal{Z}_\rho$. For any $T > 0$, there exists a constant $C(T) > 0$ such that the following estimates hold:*

$$||\Delta u(t)||_{0,\rho}^2 + ||u_t(t)||_{0,\rho}^2 \leq C(T)t^{-1}||u_0||_{1,\rho}^2, \quad 0 < t \leq T; \tag{7}$$

$$||\nabla u_t(t)||_{0,\rho}^2 \leq C(T)t^{-2}||u_0||_{1,\rho}^2, \quad 0 < t \leq T; \tag{8}$$

$$\int_0^T ||u_t(t)||_{p,\rho}^2 dt \leq C(T)||u_0||_{1,\rho}^2; \tag{9}$$

$$\int_0^T t^2||\Delta u_t(t)||_{0,\rho}^2 dt \leq C(T)||u_0||_{1,\rho}^2; \tag{10}$$

$$\int_0^T t^2||u_{tt}(t)||_{0,\rho}^2 dt \leq C(T)||u_0||_{1,\rho}^2. \tag{11}$$

For the discrete system in Eq. (2), we need a certain analog that yields estimates up to second order.

Proposition 3. *For h and d sufficiently small, Eq. (2) defines a solution operator $\mathcal{S}$ on H_ρ and on Z_ρ with Lipschitz constant $1 + Ch$ for both norms.*

For any fixed $T > 0$, there exists a constant $C = C(T) > 0$ such that solutions $u^{(n+1)} = \mathcal{S}(u^{(n)})$ of the discrete system (2) satisfy for

$0 < nh \leq T$ *the following estimates:*

$$\|u^{(n)}\|^2_{o,\rho} + h\sum_{k=1}^{n} \|\partial_+ u^{(k)}\|^2_{0,\rho} \leq C\|u^{(0)}\|^2_{0,\rho}, \tag{12}$$

$$\|\partial_+\partial_- u^{(n)}\|^2_{0,\rho} \leq \frac{C}{(nh)^2}\|u^{(0)}\|^2_{0,\rho}. \tag{13}$$

The first part essentially follows from [3], while a detailed proof of (13) will be given in Section 5.

In the following, it will be convenient to use second order spaces and norms

$$\mathcal{Y}_\rho = \{u \in \mathcal{Z}_\rho : \|u\|^2_{2,\rho} = \|u\|^2_{0,\rho} + \|\nabla u\|^2_{1,\rho} < \infty\},$$

$$Y_\rho = \{u \in Z_\rho : \|u\|^2_{2,\rho} = \|u\|^2_{0,\rho} + \|\partial_- u\|^2_{1,\rho} < \infty\},$$

with their uniform counterparts denoted by $(Y_u, \|\cdot\|_{2,u})$ and $(\mathcal{Y}_u, \|\cdot\|_{2,u})$.

Now consider the interpolation operator $\mathcal{T}$ defined by (5). The following Lemmas 4–7 are proved in [8].

Lemma 4. *There exists a constant $C > 0$ such that*

$$C^{-1}\|u\|_{0,\rho} \leq \|\mathcal{T}u\|_{0,\rho} \leq C\|u\|_{0,\rho} \quad \text{and} \quad C^{-1}\|u\|_{1,\rho} \leq \|\mathcal{T}u\|_{1,\rho} \leq C\|u\|_{1,\rho}$$

for any $u \in H_\rho$ and $u \in Z_\rho$, respectively.

Comment. Here and below, we denote by C various constants that are independent of h and d but may depend on the parameter ε in the weight function ρ.

Lemma 5. *The operator $\mathcal{T}$ is uniformly (in d) bounded from Z_u into Z_u.*

Introduce the subspace $\mathcal{V}_d = \mathcal{T}(H_\rho) \subset \mathcal{H}_\rho$ of piecewise linear functions. Lemma 4 implies that $\mathcal{V}_d$ is a closed subspace of $\mathcal{H}_\rho$ and that $\mathcal{T}$ is a homeomorphism between H_ρ and $\mathcal{V}_d$.

Let $\mathcal{P}_d$ be the orthogonal projector onto $\mathcal{V}_d$ in the space $\mathcal{H}_\rho$.

Lemma 6. *If $u \in \mathcal{Z}_\rho$, then*

$$\|\mathcal{P}_d u\|_{1,\rho} \leq C\|u\|_{1,\rho} \tag{14}$$

and

$$\|(I - \mathcal{P}_d)u\|_{0,\rho} \leq Cd\|u\|_{1,\rho}.$$

Note that the error estimates above as well as (15) below are classical in finite element analysis for the case of bounded domains and without weights (see e.g. [5]).

Lemma 7. *If $u \in \mathcal{Y}_\rho$, then*

$$\|(I - \mathcal{P}_d)u\|_{1,\rho} \leq Cd\|u\|_{2,\rho}, \tag{15}$$

and for any $K > 0$ there exists a constant $C(K) > 0$ such that

$$\|\mathcal{P}_d T_{kd}(I - \mathcal{P}_d)u\|_{0,\rho} \leq C(K)d^3\|u\|_{2,\rho}$$

for all integers k such that $|k| \leq K$.

If, in addition, $u \in \mathcal{Y}_u$, then for all $|y| \leq 1$,

$$\|T_y \mathcal{T} u - \mathcal{T} u\|_{1,u} \leqslant C\sqrt{|y|}\; \|u\|_{2,u}.$$

An easy consequence of these lemmas is the following statement (see Section 5).

Lemma 8. *The attractor $\mathcal{A}$ is contained in $\mathcal{Y}_\rho$, and*

$$\operatorname{dev}(\mathcal{P}_d\mathcal{A}, \mathcal{A}) \leqslant Cd.$$

For the following finite time estimate, we remind the reader that the operators $\mathcal{T}$ and $\mathcal{S}$ depend on d and on d, h respectively.

Proposition 9. *If we fix $v \in \mathcal{Z}_\rho$ and a number $T > 0$, then*

$$\sup_{0<nh\leq T} \|\mathcal{T}\,\mathcal{S}^n(\mathcal{T}^{-1}\mathcal{P}_d v) - \mathcal{P}_d\Sigma(nh, v)\|_{1,\rho} \to 0 \quad \text{as} \quad h, d \to 0.$$

Let us note that, for the case of a parabolic equation on a bounded (in x) domain, explicit (in terms of the steps) estimates of finite-time discretization errors were obtained, for example, in [10]. Proposition 9 will be proved in Section 4.

Finally, consider a sequence (h_m, d_m) of discretization steps such that $h_m, d_m \to 0$ as $m \to \infty$, and let $\mathcal{T}_m$ denote the interpolation operator corresponding to $d = d_m$.

Proposition 10. *If $u_m \in \mathcal{A}(h_m, d_m)$, then the sequence $v_m = \mathcal{T}_m u_m$ is precompact in $\mathcal{Z}_\rho$.*

An essential tool in the proof of Proposition 10 (cf. Section 5) is the following compactness result from [7].

Proposition 11. *Any bounded set $B \subset Z_u$ that satisfies*

$$\sup_{u\in B} \|T_y u - u\|_{1,u} \to 0 \quad \text{as} \quad y \to 0$$

is precompact in $\mathcal{Z}_\rho$.

3. Proof of the main theorem

To prove the main theorem, let us assume that relation (6) does not hold. In this case, there exists a positive number c and a sequence $(h_m, d_m) \to (0,0)$ such that

$$\operatorname{dev}(\mathcal{T}_m \mathcal{A}(h_m, d_m), \mathcal{A}) \geq 2c.$$

Find points $u'_m \in \mathcal{A}(h_m, d_m)$ such that

$$\operatorname{dist}(\mathcal{T}_m u'_m, \mathcal{A}) \geq c. \tag{16}$$

Since $\mathcal{T}_m$ is uniformly (in d) bounded from Z_u into Z_u (see Lemma 4) and the Z_u-size of the attractors $\mathcal{A}(h_m, d_m)$ is uniformly bounded for large m [3], there exists a closed bounded ball B of the space Z_u such that

$$\mathcal{T}_m \mathcal{A}(h_m, d_m) \subset B.$$

Find a number $T > 1$ such that

$$\operatorname{dev}(\Sigma(t, B), \mathcal{A}) < c/C \quad \text{for} \quad t \geq T-1,$$

where C is from (14) in Lemma 6.

If $h_m < 1$, we can find integers $\tau(m)$ such that $T - 1 \leq \tau(m) h_m \leq T$. Let

$$u_m := \mathcal{S}_m^{-\tau(m)}(u'_m) \in \mathcal{A}(h_m, d_m) \quad \text{and} \quad v_m := \mathcal{T}_m u_m,$$

where $\mathcal{S}_m$ is the solution operator for $d = d_m, h = h_m$.

Since $u_m \in \mathcal{A}(h_m, d_m)$, it follows from Proposition 10 that the sequence v_m contains a subsequence convergent in $\mathcal{Z}_\rho$; we assume that $v_m \to v$ as $m \to \infty$. It is easy to show that $v \in B \subset Z_u$.

Thus, there exist points $w_m \in \mathcal{A}$ such that

$$\|\Sigma(\tau(m) h_m, v) - w_m\|_{1,\rho} < c/C. \tag{17}$$

Note that $\mathcal{T}_m \mathcal{S}_m^{\tau(m)}(u_m) = \mathcal{T}_m u'_m$.

Let us estimate

$$\begin{aligned}\operatorname{dist}(\mathcal{T}_m u'_m, \mathcal{A}) \leq{}& \|\mathcal{T}_m \mathcal{S}_m^{\tau(m)}(u_m) - \mathcal{T}_m \mathcal{S}_m^{\tau(m)}(\mathcal{T}_m^{-1} \mathcal{P}_{d_m} v)\|_{1,\rho} + \\ &+ \|\mathcal{T}_m \mathcal{S}_m^{\tau(m)}(\mathcal{T}_m^{-1} \mathcal{P}_{d_m} v) - \mathcal{P}_{d_m} \Sigma(\tau(m) h_m, v)_{1,\rho} + \\ &+ \|\mathcal{P}_{d_m} \Sigma(\tau(m) h_m, v) - \mathcal{P}_{d_m} w_m\|_{1,\rho} + \operatorname{dist}(\mathcal{P}_{d_m} w_m, \mathcal{A}).\end{aligned}$$

By Proposition 3, the mapping $\mathcal{S}_m$ has a Lipschitz constant of the form $1 + C h_m$ with C independent of m. Hence, the mappings $\mathcal{S}_m^{\tau(m)}$ have uniform Lipschitz constants for small h_m and d_m. Since $u_m = \mathcal{T}_m^{-1}\mathcal{P}_{d_m} v_m$ and the operators $\mathcal{T}_m$, $\mathcal{T}_m^{-1}$ and $\mathcal{P}_{d_m}$ are uniformly bounded, the first term on the right in the above inequality tends to 0 as $m \to \infty$.

By Proposition 9, the second term on the right tends to 0 as $m \to \infty$. It follows from inequality (17) that $\|\mathcal{P}_{d_m}\Sigma(\tau(m)h_m, v) - \mathcal{P}_{d_m} w_m\|_{1,\rho} < c$.

By Lemma 8, we have $\operatorname{dist}(\mathcal{P}_{d_m} w_m, \mathcal{A}) \to 0$ as $m \to \infty$. Thus,

$$\operatorname{dist}(\mathcal{T}_m u'_m, \mathcal{A}) < c$$

for large m, and we obtain a contradiction with inequalities (16). This completes the proof.

4. An error estimate with weighted norms

This section is devoted to the proof of Proposition 9. Let $\mathcal{L}$ be a Lipschitz constant of f. Denote

$$A_d = \mathcal{T}(\partial_+\partial_-)\mathcal{T}^{-1}$$

and note that $A_d = \partial_+\partial_-$ on $\mathcal{V}_d$. We further define

$$f_d(u) = \mathcal{T}\{f((\mathcal{T}^{-1}u)_k)\} \quad \text{for} \quad u \in \mathcal{V}_d.$$

and use the notation $f(u)(x) = f(u(x))$ for $u \in \mathcal{H}_\rho$.

Fix a function $u_0 \in \mathcal{Z}_\rho$ (this function will play the role of v). Take $u^0 = \mathcal{T}^{-1}\mathcal{P}_d u_0$ and consider the corresponding trajectory $\{u^n : n \geq 0\}$ of the discretized equation (2). Denote $v^n = \mathcal{T}u^n \in \mathcal{V}_d \subset \mathcal{H}_\rho$. Applying $\mathcal{T}$ to (2), we see that the functions v^n satisfy the following equation:

$$(v^{n+1} - v^n)/h = \mathcal{T}(\partial_+\partial_-)u^{n+1} + \mathcal{T}\overline{f}(u^{n+1}) = A_d v^{n+1} + f_d(v^{n+1}).$$

Let us write this equation as follows:

$$(v^{n+1} - v^n)/h = A_d v^{n+1} + \mathcal{P}_d f(v^{n+1}) + \sigma_1^{n+1}, \tag{18}$$

where

$$\sigma_1^{n+1} = f_d(v^{n+1}) - \mathcal{P}_d f(v^{n+1}).$$

Let $u(t,x)$ be the solution of Eq. (1) with initial value $u_0(x)$ at $t = 0$. Denote $u^{(n)}(x) = \mathcal{P}_d u(nh, x)$. Applying $\mathcal{P}_d$ to (1) at $t = (n+1)h$, we see that

$$\begin{gathered}(u^{(n+1)}(x) - u^{(n)}(x))/h = \\ = A_d u^{(n+1)}(x) + \mathcal{P}_d f(u((n+1)h, x)) - \sigma_2^{n+1}(x) - \sigma_3^{n+1}(x), \quad (19)\end{gathered}$$

where

$$\sigma_2^{n+1}(x) = A_d u^{(n+1)}(x) - \mathcal{P}_d \Delta u((n+1)h, x)$$

and

$$\sigma_3^{n+1}(x) = \mathcal{P}_d u_t((n+1)h, x) - (u^{(n+1)}(x) - u^{(n)}(x))/h.$$

Let $\Theta^{(n)} = v^n - u^{(n)}$. Subtracting (19) from (18), we see that

$$(\Theta^{(n+1)} - \Theta^{(n)})/h =$$

$$= A_d\Theta^{(n+1)} + \mathcal{P}_d(f(v^{(n+1)}) - f(u((n+1)h, x))) + \sigma_1^{n+1} + \sigma_2^{n+1} + \sigma_3^{n+1} \quad (20)$$

and $\Theta^{(0)} = v^0 - \mathcal{P}_d u_0$. Below we take into account that $\Theta^{(0)} = 0$ due to our choice of u^0.

Now we fix $T > 0$ and estimate $\|\Theta^{(n)}\|_{1,\rho}$ for $0 < nh \le T$. Let us begin with preliminary estimates.

Estimation of σ_1^{n+1}.

Fix $x = (k+\theta)d$, where $\theta \in [0,1]$. Since $\mathcal{P}_d f_d = f_d$,

$$|\sigma_1^{n+1}(x)| = |\mathcal{P}_d(f_d(v^{n+1}) - f(v^{n+1}))(x)|.$$

Let us estimate

$$|f(u_{k+1}^{n+1})\theta + f(u_k^{n+1})(1-\theta) - f(u_{k+1}^{n+1}\theta + u_k^{n+1}(1-\theta))| \le$$

$$\le |f(u_{k+1}^{n+1}\theta + u_k^{n+1}(1-\theta)) - f(u_{k+1}^{n+1})|\theta +$$

$$+ |f(u_{k+1}^{n+1}\theta + u_k^{n+1}(1-\theta)) - f(u_k^{n+1})|(1-\theta) \le$$

$$\le 2\mathcal{L}\theta(1-\theta)|u_{k+1}^{n+1} - u_k^{n+1}| \le \mathcal{L}|u_{k+1}^{n+1} - u_k^{n+1}|/2 = \mathcal{L}d|(\partial_+ u^{n+1})_k|/2.$$

Since $\mathcal{P}_d$ is an orthogonal projector, $||\mathcal{P}_d|| = 1$, and we get the following estimate:

$$\|\sigma_1^{n+1}\|_{0,p}^2 \le \int_{\mathbb{R}} \rho(x)|(f_d(v^{n+1}) - f(v^{n+1}))(x)|^2 dx \le$$

(recall that $\rho(x+\theta d) \le C\rho(x)$ for $|\theta| \le 1$, [3])

$$\le C\mathcal{L}^2 d^3 \sum_{k\in\mathbb{Z}} \rho_k |(\partial_+ u^{n+1})_k|^2 \le Cd^2 \|\partial_+ u^{n+1}\|_{0,\rho}^2.$$

Finally, we arrive at the estimate

$$\|\sigma_1^{n+1}\|_{0,\rho}^2 \le Cd^2 \|\partial_+ u^{n+1}\|_{0,\rho}^2. \quad (21)$$

Estimation of σ_2^{n+1}.

Let us transform

$$\sigma_2^{n+1}(x) = A_d u^{(n+1)} - \mathcal{P}_d \Delta u((n+1)h, x) =$$
$$= \partial_+ \partial_- \mathcal{P}_d u((n+1)h, x) - \mathcal{P}_d \Delta u((n+1)h, x) =$$
$$= \partial_+ \partial_- \mathcal{P}_d u((n+1)h, x) - \mathcal{P}_d \partial_+ \partial_- u((n+1)h, x) +$$
$$+ \mathcal{P}_d(\partial_+ \partial_- u((n+1)h, x) - \Delta u((n+1)h, x)).$$

We denote

$$\sigma_{2,1}^{n+1} := \partial_+ \partial_- \mathcal{P}_d u((n+1)h, x) - \mathcal{P}_d \partial_+ \partial_- u((n+1)h, x),$$
$$\sigma_{2,2}^{n+1} := \mathcal{P}_d(\partial_+ \partial_- u((n+1)h, x) - \Delta u((n+1)h, x)). \tag{22}$$

Let us estimate the term $\sigma_{2,1}^{n+1}$. The following equalities hold:

$$\sigma_{2,1}^{n+1} = \mathcal{P}_d \partial_+ \partial_- (I - \mathcal{P}_d) u((n+1)h, x) =$$
$$= \frac{1}{d^2} \mathcal{P}_d (T_d - 2I + T_{-d})(I - \mathcal{P}_d) u((n+1)h, x) =$$
$$= \frac{1}{d^2} \mathcal{P}_d (T_d + T_{-d})(I - \mathcal{P}_d) u((n+1)h, x) - \frac{2}{d^2} \mathcal{P}_d (I - \mathcal{P}_d) u((n+1)h, x) =$$
$$= \frac{1}{d^2} \mathcal{P}_d (T_d + T_{-d})(I - \mathcal{P}_d) u((n+1)h, x).$$

The second estimate in Lemma 7 implies that if $u(x) = u((n+1)h, x)$ satisfies the inequality

$$\|u\|_{2,\rho}^2 = \|u\|_{0,\rho}^2 + \|\nabla u\|_{1,\rho}^2 < \infty,$$

then

$$\|\sigma_{2,1}^{n+1}\|_{0,\rho} \le \frac{1}{d^2} \Big(\|\mathcal{P}_d T_d (I - \mathcal{P}_d) u\|_{0,\rho} +$$
$$+ \|\mathcal{P}_d T_{-d} (I - \mathcal{P}_d) u\|_{0,\rho} \Big) \le 2Cd \|u\|_{2,\rho}. \tag{23}$$

Now let us estimate the term $\sigma_{2,2}^{n+1}$. Since

$$|\partial_+ \partial_- u((n+1)h, x)| = \Big| \int_0^1 \partial_- \nabla u((n+1)h, x + \theta d) d\theta \Big| =$$

$$= |\int_0^1 \int_0^1 \Delta u((n+1)h, x + (\theta + \theta_1)d - d) d\theta d\theta_1| =$$

(introduce $\theta_2 = \theta + \theta_1 - 1$)

$$= |\int_0^1 \int_{\theta-1}^{\theta} \Delta u((n+1)h, x + \theta_2 d) d\theta_2 d\theta|, \tag{24}$$

we obtain the following equalities:

$$|\partial_+ \partial_- u((n+1)h, x) - \Delta u((n+1)h, x)| =$$

$$= |\int_0^1 \int_{\theta-1}^{\theta} (\Delta u((n+1)h, x + \theta_2 d) - \Delta u((n+1)h, x)) d\theta_2 d\theta| =$$

$$= |\int_{-1}^0 \int_0^{1+\theta_2} \dots d\theta d\theta_2 + \int_0^1 \int_{\theta_2}^1 \dots d\theta d\theta_2| =$$

$$= d|\int_{-1}^1 \int_0^{\theta_2} u'''((n+1)h, x + \theta_3 d)(1 - |\theta_2|) d\theta_3 d\theta_2| =$$

$$= d| - \int_{-1}^0 \int_{\theta_2}^0 \dots + \int_0^1 \int_0^{\theta_2} \dots | =$$

$$= d|\int_0^1 \int_{\theta_3}^1 u'''((n+1)h, x + \theta_3 d)(1 - |\theta_2|) d\theta_2 d\theta_3 + \int_{-1}^0 \int_{-1}^{\theta_3} \dots | =$$

$$= d|\int_0^1 u'''(\dots)(1 - |\theta_3|)^2/2 d\theta_3 - \int_{-1}^0 u'''(\dots)(1 - |\theta_3|)^2/2 d\theta_3|.$$

It follows that

$$\|\sigma_{2,2}^{n+1}\|_{0,\rho}^2 = \|\partial_+ \partial_- u((n+1)h, x) - \Delta u((n+1)h, x)\|_{0,\rho}^2 = \int_{\mathbb{R}} \rho | \dots |^2 dx \leq$$

$$\leq d \int_{\mathbb{R}} \int_{-d}^{d} \rho |u'''((n+1)h, x+\theta_4)|^2 d\theta_4 dx \leq$$

(we differentiate Eq. (1))

$$\leq 2d \int_{\mathbb{R}} \int_{-d}^{d} \rho |\nabla u_t((n+1)h, x+\theta_4)|^2 d\theta_4 dx +$$

$$+2d \int_{\mathbb{R}} \int_{-d}^{d} \rho |\nabla f(u((n+1)h, x+\theta_4))|^2 d\theta_4 dx \leq$$

(apply Proposition 2)

$$\leq Cd^2((n+1)h)^{-2} \|u_0\|^2_{1,\rho} + 2\mathcal{L}^2 d \int_{\mathbb{R}} \int_{-d}^{d} \rho |\nabla u((n+1)h, x+\theta_4)|^2 d\theta_4 dx.$$

Finally, applying Proposition 2 once more, we have

$$\|\sigma^{n+1}_{2,2}\|^2_{0,\rho} \leq Cd^2(1 + ((n+1)h)^{-2}) \|u_0\|^2_{1,\rho}. \tag{25}$$

Estimation of σ_3^{n+1}.

The following estimates hold:

$$\|\sigma_3^{n+1}\|^2_{0,\rho} = \int_{\mathbb{R}} \rho |(u^{(n+1)} - u^{(n)})/h - \mathcal{P}_d u_t((n+1)h, x)|^2 dx =$$

$$= \int_{\mathbb{R}} \rho |\mathcal{P}_d \int_0^1 (u_t((n+\theta)h, x) - u_t((n+1)h, x) d\theta|^2 dx \leq$$

$$\leq h^2 \int_{\mathbb{R}} \rho | \int_0^1 \int_{\theta}^1 u_{tt}((n+\theta_1)h, x) d\theta d\theta_1|^2 dx =$$

$$= h^2 \int_{\mathbb{R}} \rho | \int_0^1 u_{tt}((n+\theta_1)h, x) \theta_1 d\theta_1|^2 dx \leq$$

$$\leq h^2 \int_0^1 \int_{\mathbb{R}} \rho |u_{tt}((n+\theta_1)h,x)|^2 \theta_1^2 dx d\theta_1.$$

Hence,

$$\|\sigma_3^{n+1}\|^2_{0,\rho} \leq h^2 \int_0^1 \int_{\mathbb{R}} \rho |u_{tt}((n+\theta_1)h,x)|^2 dx d\theta_1. \tag{26}$$

Estimation of $\Theta^{(n)}$.

Multiplying Eq. (20) by $\Theta^{(n+1)}$, we get the following estimate:

$$(|\Theta^{(n+1)}|^2 - |\Theta^{(n)}|^2)/(2h) \leq (|\Theta^{(n+1)}|^2 - \Theta^{(n+1)}\Theta^{(n)})/h =$$

$$= A_d \Theta^{(n+1)}\Theta^{(n+1)} - \mathcal{P}_d(f(v^{(n+1)}) - f(u((n+1)h,x)))\Theta^{(n+1)} +$$

$$+ (\sigma_1^{(n+1)} + \sigma_2^{(n+1)} + \sigma_3^{(n+1)})\Theta^{(n+1)}. \tag{27}$$

Now we multiply (27) by ρ and integrate over $\mathbb{R}$:

$$(\|\Theta^{(n+1)}\|^2_{0,\rho} - \|\Theta^{(n)}\|^2_{0,\rho})/(2h) \leq \int_{\mathbb{R}} \rho A_d \Theta^{(n+1)}\Theta^{(n+1)} dx -$$

$$- \int_{\mathbb{R}} \rho \mathcal{P}_d(f(v^{(n+1)}) - f(u((n+1)h,x))\Theta^{(n+1)} dx +$$

$$+ \int_{\mathbb{R}} \rho(\sigma_1^{(n+1)} + \sigma_2^{(n+1)} + \sigma_3^{(n+1)})\Theta^{(n+1)} dx \leq$$

$$\leq \int_{\mathbb{R}} \rho \partial_+ \partial_- \Theta^{(n+1)}\Theta^{(n+1)} dx -$$

$$- \int_{\mathbb{R}} \rho \mathcal{P}_d(\int_0^1 f'(u((n+1)h,x)(1-\chi) + v^{(n+1)}\chi) \times$$

$$\times (v^{(n+1)} - u((n+1)h,x)) d\chi)\Theta^{(n+1)} dx +$$

$$+ \|\Theta^{(n+1)}\|^2_{0,\rho}/2 + (\|\sigma_1^{(n+1)}\|^2_{0,\rho} + \|\sigma_2^{(n+1)} + \sigma_3^{(n+1)}\|^2_{0,\rho})/2. \tag{28}$$

Let us estimate the terms separately:

$$\int_{\mathbb{R}} \rho\partial_+\partial_-\Theta^{(n+1)}\Theta^{(n+1)}dx = -\int_{\mathbb{R}} \partial_-(\rho\Theta^{(n+1)})\partial_-\Theta^{(n+1)}dx =$$

$$= -\int_{\mathbb{R}} \rho|\partial_-\Theta^{(n+1)}|^2dx - \int_{\mathbb{R}}(\partial_-\rho)(T_{-d}\Theta^{(n+1)})\partial_-\Theta^{(n+1)}dx \le$$

$$\le -\int_{\mathbb{R}} \rho|\partial_-\Theta^{(n+1)}|^2dx + C\varepsilon^{-1}\int_{\mathbb{R}} \rho|\Theta^{(n+1)}|^2dx + C\varepsilon\int_{\mathbb{R}} \rho|\partial_-\Theta^{(n+1)}|^2dx \le$$

$$\le -\|\partial_-\Theta^{(n+1)}\|^2_{0,\rho}/2 + C\varepsilon^{-1}\|\Theta^{(n+1)}\|^2_{0,\rho} \tag{29}$$

by a proper choice of ε.

Further,

$$|\int_{\mathbb{R}} \rho\mathcal{P}_d(\int_0^1 f'(u((n+1)h,x)(1-\chi) + v^{(n+1)}\chi)\times$$

$$\times(v^{(n+1)} - u((n+1)h,x))d\chi)\Theta^{(n+1)}dx| \le$$

$$\le \int_{\mathbb{R}} \rho|\mathcal{P}_d\int_0^1 f'(u((n+1)h,x)(1-\chi) + v^{(n+1)}\chi)(v^{(n+1)}-$$

$$-u((n+1)h,x))d\chi|^2dx + \|\Theta^{(n+1)}\|^2_{0,\rho} \le$$

$$\le \mathcal{L}\int_{\mathbb{R}} \rho|v^{(n+1)} - \mathcal{P}_d u((n+1)h,x)|^2dx+$$

(recall that $v^{(n+1)} - \mathcal{P}_d u((n+1)h,x) = \Theta^{(n+1)}$)

$$+\mathcal{L}\int_{\mathbb{R}} \rho|\mathcal{P}_d u((n+1)h,x) - u((n+1)h,x)|^2dx + \|\Theta^{(n+1)}\|^2_{0,\rho} \le$$

$$\le Cd^2\|u_0\|^2_{1,\rho} + C\|\Theta^{(n+1)}\|^2_{0,\rho}. \tag{30}$$

In the last step we applied Lemma 6 to the second term and used that

$$\|u(kh,\cdot)\|^2_{1,\rho} \le C\|u_0\|^2_{1,\rho}$$

holds for $0 \le kh \le T$ with a constant C depending on T.

It follows from inequalities (28)–(30) and equality (22) that

$$(\|\Theta^{(n+1)}\|^2_{0,\rho} - \|\Theta^{(n)}\|^2_{0,\rho})/h + \|\partial_- \Theta^{(n+1)}\|^2_{0,\rho} \le$$

$$\le C\|\Theta^{(n+1)}\|^2_{0,\rho} + Cd^2\|u_0\|^2_{1,\rho} + 2\|\sigma_1^{n+1}\|^2_{0,\rho} + 2\|\sigma_{2,1}^{n+1}\|^2_{0,\rho} + 2\sigma^{n+1},$$

where

$$\sigma^{n+1} = \|\sigma_{2,2}^{n+1} + \sigma_3^{n+1}\|^2_{0,\rho}.$$

Summing the latter inequalities, we see that

$$\|\Theta^{(n+1)}\|^2_{0,\rho} \le \|\Theta^{(0)}\|^2_{0,\rho} + Ch\sum_{k=1}^{n+1}\|\Theta^{(k)}\|^2_{0,\rho} + CTd^2\|u_0\|^2_{1,\rho} +$$

$$+2h\sum_{k=1}^{n+1}(\sigma^{(k)} + \|\sigma_1^k\|^2_{0,\rho} + \|\sigma_{2,1}^k\|^2_{0,\rho}) \le$$

(we apply estimates (21) and (23))

$$\le \|\Theta^{(0)}\|^2_{0,\rho} + Ch\sum_{k=1}^{n+1}\|\Theta^{(k)}\|^2_{0,\rho} + CTd^2\|u_0\|^2_{1,\rho} +$$

$$+Chd^2\sum_{k=1}^{n+1}\|u(kh,x)\|^2_{2,\rho} + Chd^2\sum_{k=1}^{n+1}\|\partial_+ u^{(k)}\|^2_{0,\rho} + 2h\sum_{k=1}^{n+1}\sigma^{(k)}. \quad (31)$$

Note that Proposition 3 gives the estimate

$$h\sum_{k=1}^{n+1}\|\partial_+ u^k\|^2_{0,\rho} \le C\|u^0\|^2_{0,\rho}. \quad (32)$$

Next we estimate the term $hd^2\sum_{k=1}^{n+1}\|u(kh,x)\|^2_{2,\rho}$:

$$hd^2\sum_{k=1}^{n+1}\|u(kh,x)\|^2_{2,\rho} = hd^2\sum_{k=1}^{n+1}\int_{\mathbb{R}}\rho|\Delta u(kh,x)|^2 + hd^2\sum_{k=1}^{n+1}\|u(kh,x)\|^2_{1,\rho} \le$$

$$\le hd^2\sum_{k=1}^{n+1}\Big(\int_{\mathbb{R}}\rho|\Delta u(kh,x)|^2dx - \int_0^1\int_{\mathbb{R}}\rho|\Delta u((k+\theta)h,x)|^2dxd\theta\Big) +$$

$$+d^2\int_0^{T+h}\int_{\mathbb{R}}\rho|\Delta u(t,x)|^2dxdt+CTd^2\|u_0\|^2_{1,\rho}=$$

$$=hd^2\sum_{k=1}^{n+1}\int_{\mathbb{R}}\rho\int_0^1(|\Delta u(kh,x)|^2-|\Delta u((k+\theta)h,x)|^2)d\theta dx+$$

$$+d^2\int_0^{T+h}\int_{\mathbb{R}}\rho|\Delta u(t,x)|^2dxdt+CTd^2\|u_0\|^2_{1\rho}=$$

$$=-hd^2\sum_{k=1}^{n+1}\int_{\mathbb{R}}\rho\int_0^1\int_0^\theta 2h\Delta u_t((k+\theta_1)h,x)\Delta u((k+\theta_1)h,x)d\theta_1d\theta dx+$$

$$+d^2\int_0^{T+h}\int_{\mathbb{R}}\rho|\Delta u(t,x)|^2dxdt+CTd^2\|u_0\|^2_{1,\rho}\le$$

$$\le hd^2\sum_{k=1}^{n+1}\int_{\mathbb{R}}\rho\int_0^1\int_0^\theta(h^2|\Delta u_t((k+\theta_1)h,x)|^2+|\Delta u((k+\theta_1)h,x)|^2)d\theta_1d\theta dx+$$

$$+d^2\int_0^{T+h}\int_{\mathbb{R}}\rho|\Delta u(t,x)|^2dxdt+CTd^2\|u_0\|^2_{1,\rho}=$$

$$=hd^2\sum_{k=1}^{n+1}\int_{\mathbb{R}}\rho\int_0^1(h^2|\Delta u_t((k+\theta_1)h,x)|^2+|\Delta u((k+\theta_1)h,x)|^2)(1-\theta_1)d\theta_1dx+$$

$$+d^2\int_0^{T+h}\int_{\mathbb{R}}\rho|\Delta u(t,x)|^2dxdt+CTd^2\|u_0\|^2_{1,\rho}\le$$

$$\le d^2\int_h^{T+h}\int_{\mathbb{R}}\rho(h^2|\Delta u_t(t,x)|^2+|\Delta u(t,x)|^2)dxdt+$$

$$+d^2\int_0^{T+h}\int_{\mathbb{R}}\rho|\Delta u(t,x)|^2dxdt+CTd^2\|u_0\|^2_{1,\rho}\le$$

$$\leq d^2 \int\limits_0^{T+h} \int\limits_{\mathbb{R}} \rho t^2 |\Delta u_t(t,x)|^2 dx dt +$$

$$+2d^2 \int\limits_0^{T+h} \int\limits_{\mathbb{R}} \rho |\Delta u(t,x)|^2 dx dt + CTd^2 \|u_0\|^2_{1,\rho} \leq Cd^2 \|u_0\|^2_{1,\rho},$$

where Proposition 2 was employed in the last step.

Now we fix an arbitrary $\alpha > 0$ and denote

$$\Sigma' = \sum_{h \leq kh \leq \alpha} \quad \text{and} \quad \Sigma'' = \sum_{\alpha < kh \leq (n+1)h} .$$

Applying estimates (25) and (26) (and taking into account the expressions for $\sigma_{2,2}$ and σ_3 considering Σ'), we see that

$$h \sum_{k=1}^{n+1} \sigma^{(k)} = h(\Sigma' \sigma^{(k)} + \Sigma'' \sigma^{(k)}) \leq$$

$$= h\Sigma' \|\mathcal{P}_d \Delta u(kh,x) - \mathcal{P}_d u_t(kh,x) + (u^{(k)} - u^{(k-1)})/h - \mathcal{P}_d \partial_+ \partial_- u(kh,x)\|^2_{0,\rho} +$$

$$+ hd^2 \Sigma'' (1 + (kh)^{-2}) \|u_0\|^2_{1,\rho} + h^3 \Sigma'' \int\limits_0^1 \int\limits_{\mathbb{R}} \rho |u_{tt}((k+\theta_1-1)h,x)|^2 d\theta_1 dx) \leq$$

$$\leq h\Sigma' \|\mathcal{P}_d f(u(kh,x))\|^2_{0,\rho} + h\Sigma' \| \int\limits_0^1 u_t((k-1+\theta)h,x) d\theta \|^2_{0,\rho} +$$

$$+ h\Sigma' \|\mathcal{P}_d \partial_+ \partial_- u(kh,x)\|^2_{0,\rho} + CTd^2 (1+\alpha^{-2}) \|u_0\|^2_{1,\rho} +$$

$$+ h^2 \int\limits_{\alpha-h}^{T} \int\limits_{\mathbb{R}} \rho |u_{tt}(t,x)|^2 dt dx. \qquad (33)$$

We estimate the terms separately.

From the Lipschitz continuity of f we obtain:

$$h\Sigma' \|\mathcal{P}_d f(u(kh,x))\|^2_{0,\rho} \leq h\Sigma' \|f(u(kh,x))\|^2_{0,\rho} \leq C\alpha \|u_0\|^2_{0,\rho}. \qquad (34)$$

Further,

$$h\Sigma' \| \int\limits_0^1 u_t((k-1+\theta)h,x) d\theta \|^2_{0,\rho} =$$

$$= h\Sigma' \int_{\mathbb{R}} \rho | \int_0^1 u_t((k-1+\theta)h, x) d\theta|^2 dx \le$$

$$\le h\Sigma' \int_0^1 \int_{\mathbb{R}} \rho |u_t((k-1+\theta)h, x)|^2 d\theta dx \le$$

$$\le \int_0^{\alpha} \|u_t(t, \cdot)\|_{0,\rho}^2 dt. \tag{35}$$

Further,

$$h\Sigma' \|\mathcal{P}_d \partial_+ \partial_- u(kh, x)\|_{0,\rho}^2 \le h\Sigma' \int_{\mathbb{R}} \rho |\partial_+ \partial_- u(kh, x)|^2 \le$$

(according to equality (24))

$$\le h\Sigma' \int_{\mathbb{R}} \rho | \int_{-1}^1 \Delta u(kh, x+\theta d)(1-|\theta|) d\theta|^2 dx \le$$

$$\le 2h\Sigma' \int_{\mathbb{R}} \rho \int_{-1}^1 |\Delta u(kh, x+\theta d)|^2 (1-|\theta|)^2 d\theta dx \le$$

$$\le 2h\Sigma' \int_{-1}^1 \int_{\mathbb{R}} \rho | \int_0^1 \Delta u((k+\theta_1)h, x+\theta d) d\theta_1 +$$

$$+ \int_0^1 (\Delta u(kh, x+\theta d) - \Delta u((k+\theta_1)h, x+\theta d)) d\theta_1|^2 dx d\theta \le$$

$$\le 4h\Sigma' \int_{-1}^1 \int_{\mathbb{R}} \rho \int_0^1 |\Delta u((k+\theta_1)h, x+\theta d)|^2 dx d\theta_1 d\theta +$$

$$+4h\Sigma' \int_{-1}^1 \int_{\mathbb{R}} \rho | h \int_0^1 \int_0^{\theta_1} \Delta u_t((k+\theta_2)h, x+\theta d) d\theta_2 d\theta_1|^2 dx d\theta \le$$

$$\leq 4h\Sigma' \int_{-1}^{1}\int_{0}^{1}\int_{\mathbb{R}} \rho|\Delta u((k+\theta_1)h, x+\theta d)|^2 dx d\theta_1 d\theta +$$

$$+4h\Sigma' \int_{-1}^{1}\int_{0}^{1}\int_{\mathbb{R}} \rho(1-\theta_2)^2 h^2 |\Delta u_t((k+\theta_2)h, x+\theta d)|^2 dx d\theta_2 d\theta \leq$$

(we introduce $x_1 = x + \theta d$)

$$\leq 4Ch\Sigma' \int_{-1}^{1}\int_{0}^{1}\int_{\mathbb{R}} \rho(x_1)|\Delta u((k+\theta_1)h, x_1)|^2 dx_1 d\theta_1 d\theta +$$

$$+4Ch\Sigma' \int_{-1}^{1}\int_{0}^{1}\int_{\mathbb{R}} \rho(x_1)(1-\theta_2)^2 h^2 |\Delta u_t((k+\theta_2)h, x_1)|^2 dx_1 d\theta_2 d\theta \leq$$

$$\leq 4C \int_{-1}^{1}\int_{h}^{\alpha+h}\int_{\mathbb{R}} \rho|\Delta u(t, x_1)|^2 dx_1 dt d\theta +$$

$$+4C \int_{-1}^{1}\int_{h}^{\alpha+h}\int_{\mathbb{R}} \rho h^2 |\Delta u_t(t, x_1)|^2 dx_1 dt d\theta \leq$$

$$\leq 8C \int_{0}^{\alpha+h} (\|\Delta u(t,\cdot)\|_{0,\rho}^2 + t^2 \|\Delta u_t(t,\cdot)\|_{0,\rho}^2) dt. \tag{36}$$

We estimate the remaining term in (33) as follows:

$$h^2 \int_{\alpha-h}^{T}\int_{\mathbb{R}} \rho|u_{tt}(t,x)|^2 dx dt \leq \frac{h^2}{(\alpha-h)^2} \int_{\alpha-h}^{T}\int_{\mathbb{R}} t^2 \rho |u_{tt}(t,x)|^2 dx dt \leq$$

$$\leq \frac{Ch^2}{(\alpha-h)^2}\|u_0\|_{1,\rho}^2. \tag{37}$$

Now we fix a bounded ball in $\mathcal{Z}_\rho$ and take the initial function u_0 for Eq. (1) from this ball. Below, the constants C depend on T and the size of this ball, i.e., they "accumulate" the terms $\|u_0\|_{0,\rho}^2$ and $\|u_0\|_{1,\rho}^2$.

It follows from (33) and our estimates that

$$h\sum_{k=1}^{n+1}\sigma^{(k)} \le C\alpha + \int_0^{\alpha} \|u_t(t,\cdot)\|_{0,\rho}^2 dt +$$

$$+\int_0^{\alpha+h} (\|\Delta u(t,\cdot)\|_{0,\rho}^2 + t^2\|\Delta u_t(t,\cdot)\|_{0,\rho}^2)dt +$$

$$+Cd^2 + \frac{Cd^2}{\alpha^2} + \frac{Ch^2}{(\alpha-h)^2}, \tag{38}$$

where the constant C does not depend on α, d, and h. Taking, for example, $\alpha = (d^2+h^2)^{1/4}$, we see that the value

$$h\sum_{k=1}^{n+1}\sigma^{(k)}$$

tends to 0 as $h, d \to 0$ (note that the integrands in (38) are summable due to Proposition 2).

Thus, it follows from (31) and the Gronwall lemma that if we take $u^0 = \mathcal{T}^{-1}(\mathcal{P}_d u_0)$ (so that $\Theta^{(0)} = 0$), then

$$\|\Theta^{(n)}\|_{0,\rho}^2 \le (\|\Theta^{(0)}\|_{0,\rho}^2 + Cd^2 + 2h\sum_{k=1}^{n+1}\sigma^{(k)})\exp(CT) \to 0 \tag{39}$$

as $h, d \to 0$.

Estimation of $\|\Theta^{(n)}\|_{1,\rho}$.

For this term, we use arguments similar to those for $\|\Theta^{(n)}\|_{0,\rho}$.

Multiply equality (20) by $\rho A_d \Theta^{(n+1)}$ and integrate over $\mathbb{R}$:

$$(\int_{\mathbb{R}} \rho(\Theta^{(n+1)} - \Theta^{(n)})\partial_+\partial_-\Theta^{(n+1)}dx)/h = \int_{\mathbb{R}} \rho|A_d\Theta^{(n+1)}|^2 dx +$$

$$+\int_{\mathbb{R}} \rho(\sigma_1^{n+1} + \sigma_2^{n+1} + \sigma_3^{n+1} + \mathcal{P}_d(f(v^{(n+1)}) - f(u((n+1)h,x)))A_d\Theta^{(n+1)}dx. \tag{40}$$

Let us "integrate by parts" on the left in (40):

$$(\int_{\mathbb{R}} \rho(\Theta^{(n+1)} - \Theta^{(n)})\partial_+\partial_-\Theta^{(n+1)}dx)/h =$$

$$= -(\int_{\mathbb{R}} \partial_-(\rho(\Theta^{(n+1)} - \Theta^{(n)}))\partial_-\Theta^{(n+1)}dx)/h =$$

$$= -(\int_{\mathbb{R}} \rho(\partial_-\Theta^{(n+1)} - \partial_-\Theta^{(n)})\partial_-\Theta^{(n+1)}dx)/h-$$

$$-(\int_{\mathbb{R}} (\partial_-\rho)(T_{-d}(\Theta^{(n+1)} - \Theta^{(n)}))\partial_-\Theta^{(n+1)}dx)/h \leq$$

(we change variables in the second integral and take (20) into account)

$$\leq -(\|\partial_-\Theta^{(n+1)}\|^2_{0,\rho} - \|\partial_-\Theta^{(n)}\|^2_{0,\rho})/(2h)-$$

$$-\int_{\mathbb{R}} (\partial_+\rho(A_d\Theta^{(n+1)} + \sigma_1^{n+1} + \sigma_2^{n+1} + \sigma_3^{n+1}+$$

$$+\mathcal{P}_d(f(v^{(n+1)}) - f(u((n+1)h, x))))\partial_+\Theta^{(n+1)}dx. \tag{41}$$

It follows from (40) and (41) that

$$(\|\partial_-\Theta^{(n+1)}\|^2_{0,\rho} - \|\partial_-\Theta^{(n)}\|^2_{0,\rho})/(2h) + \|A_d\Theta^{(n+1)}\|^2_{0\rho} \leq$$

$$\leq -\int_{\mathbb{R}} (\partial_+\rho)A_d\Theta^{(n+1)}\partial_+\Theta^{(n+1)}dx+$$

$$+\int_{\mathbb{R}} (\sigma_1^{n+1} + \sigma_2^{n+1} + \sigma_3^{n+1})(\rho A_d\Theta^{(n+1)} - (\partial_+\rho)\partial_+\Theta^{(n+1)})dx-$$

$$-\int_{\mathbb{R}} \mathcal{P}_d(f(v^{(n+1)}) - f(u((n+1)h, x)))((\partial_+\rho)\partial_+\Theta^{(n+1)} - \rho A_d\Theta^{(n+1)})dx \leq$$

(we take $\beta = 1/(2Cd)$ and apply the usual $2ab \leq \beta a^2 + b^2/\beta$ trick)

$$\leq C\beta d\|A_d\Theta^{(n+1)}\|^2_{0\rho} + Cd(1 + 1/\beta)\|\partial_+\Theta^{(n+1)}\|^2_{0,\rho}+$$

$$+C(1 + 1/\beta)\|\sigma_1^{n+1} + \sigma_2^{n+1} + \sigma_3^{n+1}\|^2_{0,\rho}+$$

$$+C(1+1/\beta)\|\mathcal{P}_d(f(v^{(n+1)})-f(u(n+1)h,x)))\|^2_{0,\rho}.$$

Estimating the squared norm in the latter term by

$$\mathcal{L}^2\|\Theta^{(n+1)}\|^2_{0,\rho}+Cd^2\|u_0\|^2_{1,\rho},$$

we arrive at the following estimate:

$$(\|\partial_-\Theta^{(n+1)}\|^2_{0,\rho}-\|\partial_-\Theta^{(n)}\|^2_{0,\rho})/(2h)+\|A_d\Theta^{(n+1)}\|^2_{0,\rho}/2\le$$

$$\le Cd\|\partial_-\Theta^{(n+1)}\|^2_{0,\rho}+C\|\sigma_1^{n+1}+\sigma_2^{n+1}+\sigma_3^{n+1}\|^2_{0,\rho}+C\|\Theta^{(n+1)}\|^2_{0,\rho}+Cd^2.$$

By the Gronwall lemma,

$$\|\partial_-\Theta^{(n)}\|^2_{0,\rho}\le C(\|\partial_-\Theta^{(0)}\|^2_{0,\rho}+$$

$$+\sum_{k=1}^{n}\|\sigma_1^k+\sigma_2^k+\sigma_3^k\|^2_{0,\rho}+h\sum_{k=1}^{n}\|\Theta^{(k)}\|^2_{0,\rho}+d^2)\exp(CT).$$

The first term in parentheses on the right vanishes, while the remaining terms tend to 0 as $h,d\to 0$, see (38) and (39).

Finally, we apply Lemma 4 to show that

$$\|\nabla\Theta^{(n)}\|^2_{0,\rho}=$$

$$=\sum_{k\in\mathbb{Z}}\int\limits_{kd}^{(k+1)d}\rho|((\mathcal{T}^{-1}\Theta^{(n)})_{k+1}-(\mathcal{T}^{-1}\Theta^{(n)})_k)/d|^2dx\le$$

$$\le C\sum_{k\in\mathbb{Z}}\rho(kd)|((\mathcal{T}^{-1}\Theta^{(n)})_{k+1}-(\mathcal{T}^{-1}\Theta^{(n)})_k)/d|^2=$$

$$=C\|\mathcal{T}\partial_-\mathcal{T}^{-1}\Theta^{(n)}\|^2_{0,\rho}=C\|\partial_-\Theta^{(n)}\|^2_{0,\rho}\to 0$$

as $h,d\to 0$.

To complete the proof of Proposition 9, it remains to note that if $T>0$, then the estimates obtained above hold for any $nh\in(0,T]$.

5. Regularity estimates and compactness

Let us start with the

Proof of Proposition 3.

The proof of Lemma 2.3 in [3], with inequality (2.15) replaced by the inequality $\langle B^*Bv,v\rangle_\rho\ge(1-Ch)\langle v,v\rangle_\rho$, shows that the operator $\mathcal{S}$ has a Lipschitz constant $1+Ch$. For the a priori estimate (12), we use the energy estimate from Lemma 2.1 in [3]

$$\langle Av, \rho v\rangle \leq -C\|\partial_- v\|^2_{0,\rho} + C\varepsilon\|v\|^2_{0,\rho}, \quad v \in H_\rho. \tag{42}$$

Here we used the inner product $\langle u, v\rangle = \sum_{k=0}^{\infty} u_k v_k$. Multiply (2) by ρu^{n+1} and use (AI) and (42) to obtain

$$\begin{aligned}
&\frac{1}{2h}(\|u^{(n+1)}\|^2_{0,\rho} - \|u^{(n)}\|^2_{0,\rho}) \leqslant \\
&\quad\leqslant \frac{1}{2h}\langle u^{(n+1)} - u^{(n)}, \rho(u^{(n+1)} - u^{(n)} + u^{(n+1)} + u^{(n)})\rangle = \\
&\quad= \langle \rho u^{n+1}, \overline{f}(u^{n+1})\rangle + \langle \rho u^{n+1}, Au^{n+1}\rangle \leq \\
&\quad\leq C\|u^{(n+1)}\|^2_{0,\rho} - C\|\partial_- u^{(n+1)}\|^2_{0,\rho}.
\end{aligned}$$

Summing up leads to

$$\|u^{(n+1)}\|^2_{0,\rho} + Ch\sum_{j=1}^{n}\|\partial_- u^{(j)}\|^2_{0,\rho} \leq \|u^{(0)}\|^2_{0,\rho} + Ch\sum_{j=1}^{n}\|u^{(j)}\|^2_{0,\rho},$$

from which (12) follows by a discrete Gronwall estimate.

For the proof of (13), consider $w_k^{(n)} := \partial_+ u_k^{(n)}$ which satisfies the following equation:

$$\begin{aligned}
\frac{w_k^{(n+1)} - w_k^{(n)}}{h} &= \partial_+\partial_- w_k^{(n+1)} - \frac{f(u_{k+1}^{(n+1)}) - f(u_k^{(n+1)})}{h} = \\
&= \partial_+\partial_- w_k^{(n+1)} - \int_0^1 f'(u_{k+1}^{(n+1)}\theta + u_k^{(n+1)}(1-\theta))d\theta \cdot w_k^{(n+1)}.
\end{aligned}$$

Therefore, the sequence $v_k^{(n)} := hn\partial_+ u_k^{(n)}$, $n \geq 0$, $k \in \mathbb{Z}$, is a solution of the equation

$$\frac{v_k^{(n+1)} - v_k^{(n)}}{h} = \partial_+\partial_- v_k^{(n+1)} - \int_0^1 f'(u_{k+1}^{(n+1)}\theta + u_k^{(n+1)}(1-\theta))v_k^{(n+1)}d\theta + \partial_+ u_k^{(n)},$$

$$v^{(0)} = 0.$$

As usual, we multiply the last equation by $\rho_k\partial_+\partial_- v_k^{(n+1)}$, sum the expressions obtained for all $k \in \mathbb{Z}$, and obtain the following inequalities:

$$\frac{\|u\|^2_{0,\rho}\partial_+ v^{(n+1)} - \|u\|^2_{0,\rho}\partial_+ v^{(n)}}{2h} = d\sum_{k\in\mathbb{Z}}\rho_k\frac{|\partial_+ v_k^{(n+1)}|^2 - |\partial_+ v_k^{(n)}|^2}{2h} \leq$$

$$\leq d\sum_{k\in\mathbb{Z}}\rho_k\frac{\partial_+ v_k^{(n+1)}(\partial_+ v_k^{(n+1)}-\partial_+ v_k^{(n)})}{h}=$$

$$=-d\sum_{k\in\mathbb{Z}}\partial_-(\rho_k\partial_+ v_k^{(n+1)})\frac{v_k^{(n+1)}-v_k^{(n)}}{h}=$$

$$=-d\sum_{k\in\mathbb{Z}}\left(\partial_-\rho_k\partial_- v_k^{(n+1)}+\rho_k\partial_+\partial_- v_k^{(n+1)}\right)\frac{v_k^{(n+1)}-v_k^{(n)}}{h}=$$

$$=-d\sum_{k\in\mathbb{Z}}\left(\partial_-\rho_k\partial_- v_k^{(n+1)}+\rho_k\partial_+\partial_- v_k^{(n+1)}\right)\Big(\partial_+\partial_- v_k^{(n+1)}+$$

$$+\int_0^1 f'(u_{k+1}^{(n+1)}\theta+u_k^{(n+1)}(1-\theta))d\theta\cdot v_k^{(n+1)}+\partial_+ u_k^{(n)}\Big)=$$

$$=-d\sum_{k\in\mathbb{Z}}\rho_k|\partial_+\partial_- v_k^{(n+1)}|^2-\sum_{k\in\mathbb{Z}}\partial_+\partial_- v_k^{(n+1)}\Big(\partial_-\rho_k\partial_- v_k^{(n+1)}+$$

$$+\rho_k\int_0^1 f'(u_{k+1}^{(n+1)}\theta+u_k^{(n+1)}(1-\theta))d\theta\cdot v_k^{(n+1)}+\rho_k\partial_+ u_k^{(n)}\Big)-$$

$$-d\sum_{k\in\mathbb{Z}}\partial_-\rho_k\partial_- v_k^{(n+1)}\Big(\int_0^1 f'(u_{k+1}^{(n+1)}\theta+u_k^{(n+1)}(1-\theta))d\theta\cdot v_k^{(n+1)}+\partial_+ u_k^{(n)}\Big)\leq$$

$$\leq -d\sum_{k\in\mathbb{Z}}\rho_k|\partial_+\partial_- v_k^{(n+1)}|^2+C\varepsilon d\sum_{k\in\mathbb{Z}}\rho_k|\partial_+\partial_- v_k^{(n+1)}|^2+$$

$$+C\varepsilon^{-1}d\sum_{k\in\mathbb{Z}}\rho_k\left(|\partial_+ v_k^{(n+1)}|^2+|v_k^{(n+1)}|^2+|\partial_+ u_k^{(n)}|^2\right)+$$

$$+Cd\sum_{k\in\mathbb{Z}}\rho_k\left(|\partial_+ v_k^{(n+1)}|^2+|v_k^{(n+1)}|^2+|\partial_+ u_k^{(n)}|^2\right)\leq$$

(we take $\varepsilon < C^{-1}$ and continue for $(n+1)h \leq T$):

$$\leq Cd\sum_{k\in\mathbb{Z}}\rho_k\left(|\partial_+ v_k^{(n+1)}|^2+|v_k^{(n+1)}|^2+|\partial_+ u_k^{(n)}|^2\right)=$$

$$= Cd \sum_{k\in\mathbb{Z}} \rho_k \left(|\partial_+ v_k^{(n+1)}|^2 + |h(n+1)\partial_+ u_k^{(n+1)}|^2 + |\partial_+ u_k^{(n)}|^2 \right) \leq$$

$$\leq Cd \sum_{k\in\mathbb{Z}} \rho_k \left(|\partial_+ v_k^{(n+1)}|^2 + |\partial_+ u_k^{(n+1)}|^2 + |\partial_+ u_k^{(n)}|^2 \right).$$

Summarizing, we have shown

$$\frac{\|\partial_+ v^{(n+1)}\|_{0,\rho}^2 - \|\partial_+ v^{(n)}\|_{0,\rho}^2}{2h} \leq$$
$$\leq C\big(\|\partial_+ v^{(n+1)}\|_{0,\rho}^2 + \|\partial_+ u^{(n+1)}\|_{0,\rho}^2 + \|\partial_+ u^{(n)}\|_{0,\rho}^2\big). \tag{43}$$

Finally we multiply inequality (43) by $2h$, take the sum over all $n = 0, \ldots, N-1$, $Nh \leq T$, and with $v^{(0)} = 0$ obtain the following inequality:

$$\|\partial_+ v^{(N)}\|_{0,\rho}^2 \leq \|\partial_+ v^{(0)}\|_{0,\rho}^2 + Ch \sum_{n=1}^{N} \|\partial_+ v^{(n)}\|_{0,\rho}^2 + Ch \sum_{n=0}^{N} \|\partial_+ u^{(n)}\|_{0,\rho}^2 \leq$$

$$\leq Ch \sum_{n=1}^{N} \|\partial_+ v^{(n)}\|_{0,\rho}^2 + C\|u^{(0)}\|_{0,\rho}^2.$$

Applying the discrete Gronwall inequality, we get the estimate:

$$\|\partial_+ v^{(N)}\|_{0,\rho}^2 \leq C\|u^{(0)}\|_{0,\rho}^2$$

for all numbers N such that $0 < Nh \leq T$. This means that

$$\|\partial_+ \partial_- u^{(N)}\|_{0,\gamma}^2 \leq C(Nh)^{-2}\|u^{(0)}\|_{0,\gamma}^2,$$

and Proposition 3 is proved. □

Proof of Proposition 10. We apply Proposition 11 to the sequence $v_m = \mathcal{T}_m u_m$. Since the attractors $\mathcal{A}(h_m, d_m)$ are uniformly bounded in the space Z_u, Lemma 5 shows that $\|\mathcal{T}_m u_m\|_{1,u}$ is also bounded. Moreover, using the invariance of the attractor $\mathcal{A}(h_m, d_m)$ under translation and iteration, Proposition 3 implies that $\|u_m\|_{2,u}$ is uniformly bounded as well (use (13) with $nh = 1$). Therefore, we can apply Lemma 7 and obtain, uniformly in m,

$$\|T_y v_m - v_m\|_{1,\rho} \leq \sqrt{|y|}\, \|u_m\|_{2,u} \to 0 \quad \text{as} \quad y \to 0.$$

Then Proposition 11 yields the assertion. □

Proof of Lemma 8. We know that the attractor $\mathcal{A}$ is bounded with respect to $\|u\|_{1,u}$ and therefore (similarly to the discrete case) the regularity estimate (7) in Proposition 2 together with the translation invariance of the attractor shows that $\mathcal{A}$ is bounded with respect to $\|\cdot\|_{2,u}$. Lemma 7 then implies

$$\sup_{u\in\mathcal{A}} \|T_y u\|_{1,\rho}\mathcal{P}_d u - u \leq Cd. \qquad \square$$

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Сходимость дискретизированных аттракторов для параболических уравнений на оси. Бейн В. Ю., Колежук В. С., Пилюгин С. Ю.

Показано, что если полулинейное параболическое уравнение на оси удовлетворяет условию диссипативности, то глобальные аттракторы его дискретизаций по времени и пространству сходятся (в смысле отклонения по Хаусдорфу) к глобальному аттрактору точного уравнения, если шаги дискретизации стремятся к нулю. Рассматриваемые аттракторы соответствуют парам функциональных пространств (в смысле Бабина–Вишика) с весовыми локально равномерными нормами (введенными Мильке–Шнейдером), используемыми как для точного, так и для дискретизированного уравнений. Библ. 13 – назв.

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