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ON GLOBAL BEHAVIOR OF SOLUTIONS TO AN INVERSE PROBLEM FOR SEMI-LINEAR HYPERBOLIC EQUATIONS

ABSTRACT. This paper is concerned with global in time behavior of solutions for a semi-linear hyperbolic inverse source problem. We prove two types of results, the first is a global nonexistence result for smooth solutions when the data is chosen appropriately. The second type of result is on asymptotic stability of solutions when the integral constraint vanishes as t goes to infinity.

**Dedicated to the memory of
Olga Aleksandrovna Ladyzhenskaya**

1. INTRODUCTION

In this paper we study the global in time behavior of solutions for a semi-linear hyperbolic inverse source problem. We prove two types of results, the first is a global nonexistence result for smooth solutions when the data is chosen appropriately. The second type of result is on asymptotic stability of solutions when the integral constraint vanishes as t goes to infinity. In contrast with the extensive literature on the stability and global nonexistence results for direct problems, we do not know any result on the global nonexistence for solutions of hyperbolic inverse problems.

The inverse problem we consider consist of finding a pair of functions $\{u(x,t), F(t)\}$ satisfying

$$u_{tt} - \Delta u - |u|^p u + b(x,t,u,\nabla u) = F(t)w(x), \quad x \in \Omega, \quad t > 0, \tag{1}$$

$$u(x,t) = 0, \quad x \in \partial\Omega, \quad t > 0, \tag{2}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{3}$$

$$\int_\Omega u(x,t)w(x)dx = 1, \quad t > 0, \tag{4}$$

where $\Omega \subseteq R^n$ is a bounded (or unbounded) domain of R^n with smooth boundary $\partial\Omega$, $p > 0$, $w(x)$, and $b(x, t, u, \mathbf{p})$ are given functions satisfying

$$w \in H^2(\Omega) \cap H_0^1(\Omega) \cap L^{p+2}(\Omega), \quad \int_\Omega w^2(x)dx = 1, \qquad (A1)$$

and for some M_1 and $M_2 > 0$

$$|b(x, t, u, \mathbf{p})| \leq M_1|\mathbf{p}| + M_2|u|, \ \forall x \in \Omega, \ t \in R^+, \ u \in R, \ \mathbf{p} \in R^n. \ (A2)$$

The initial functions satisfy the conditions

$$u_0 \in H_0^1 \cap L^{p+2}(\Omega), \quad u_1 \in L^2(\Omega), \quad \text{and} \quad \int_\Omega u_0(x)w(x)dx = 1. \quad (A3)$$

In [18], Prilepko et al. the local existence of solutions for similar inverse problems is established. The key observation is that the given problem (1)–(4) is equivalent to (1)–(3) where the unknown function $F(t)$ in (1) is replaced by

$$F(t) = -\langle u, \Delta w\rangle - \int_\Omega |u|^p uw dx + \langle b(x, t, u, \nabla u), w\rangle. \qquad (5)$$

Once the unknown function $F(t)$ is eliminated, the standard theory of nonlinear hyperbolic equations also becomes applicable to deduce local in time existence of solutions.

Stability results for hyperbolic inverse problems are few (see [2, 7]), among them the closest to our setting are linear problems considered by Guvenilir and Kalantarov [7].

In Belov and Shipina [2], the asymptotic behavior of solution $\{u^{(1)}(x, t), u^{(2)}(x, t), g(t)\}$ to the following inverse source problem is studied:

$$u_t^{(1)}(x, t) + \sum_{k=1}^{2} a_{1k}(t)u_x^{(k)} = \nu(t)u_{xx}^{(1)} + g(t)f(x, t), x \in R, t \in (0, T),$$

$$u_t^{(2)}(x, t) + \sum_{k=1}^{2} a_{2k}(t)u_x^{(k)} + \sum_{k=1}^{2} b_{2k}(t)u^{(k)} = 0, x \in R, t \in (0, T),$$

$$u^{(k)}(x, 0) = u_0^{(k)}(x), x \in R, k = 1, 2; \quad u^{(1)}(0, t) = \beta(t), t \in (0, T).$$

In Guvenilir and Kalantarov [7], global asymptotic stability of the zero solution to inverse source problem for linear parabolic and hyperbolic type equations is established in H^1-norm.

The global nonexistence result we obtained relies on generalized concavity argument developed in [11]. The classical concavity argument leading to the nonexistence of solutions in finite time starts with a positive functional $\Psi(t)$ of a solution that exists locally in time and proceeds by establishing the differential inequality $\Phi''(t) \leq 0$, where $\Phi(t) = \Psi(t)^{-\alpha}$. This inequality allows one to conclude that $\Psi(t) \to \infty$ as $t \to t_1^-$ hence the global nonexistence. The concavity method was introduced by Levine in [14, 15] to establish the global nonexistence both for parabolic and hyperbolic equations. The essence of the method lies in the following lemma.

Lemma 1 (Levine [14, 15]). *Let $\Psi(t)$ be a positive, twice differentiable function,which satisfies, for $t > 0$, the inequality*

$$\Psi''(t)\Psi(t) - (1+\alpha)\left[\Psi'(t)\right]^2 \geq 0$$

with some $\alpha > 0$. If $\Psi(0) > 0$ and $\Psi'(0) > 0$, then there exists a time $t_1 \leq \frac{\Psi(0)}{\alpha\Psi'(0)}$ such that $\Psi(t) \to +\infty$ as $t \to t_1^-$.

Here we will only cite some of the works on hyperbolic equations. In [14], Levine using the concavity method obtained a global nonexistence result for the Cauchy problem for the second order nonlinear differential-operator equation of the form

$$Pu_{tt} + Au = F(u),$$

where A is assumed to be symmetric and nonnegative. The concavity method was applied by Levine in [15] to an equation of the form

$$Pu_{tt} + Cu_t + Au = F(u),$$

where C is a positive symmetric operator. In [20], Straughan obtained global nonexistence result for the abstract wave equation of the form $u_{tt} + k(t)u_t + A(t)u = F(u)$. In [17], Levine and Payne applied the concavity method to the study of the initial-boundary value problems for the equation

$$u_{tt} = \sum_{i=1}^{n} \left(|u_{x_i}|^{p-2}u_{x_i}\right)_{x_i} + u^{q+1},$$

where $p > 2$, $q > 0$, and $q + 2 > p$.

Recently Yang in [22] by using the concavity method obtained global nonexistence result for a class of the nonlinear wave equations with a nonlinear damping term. By the different method first global nonexistence result for the nonlinear wave equation with the nonlinear damping term is obtained in[6].

Hu and Yin [8], on the other hand, by using the concavity method studied the problem of global nonexistence for parabolic equation with Neumann boundary condition instead of (2) and when $w(x) \equiv 1$.

A critical observation made in [11] results in the following modification of the Lemma 1.

Lemma 2 (Kalantarov and Ladyzhenskaya [11]). *Let $\alpha > 0$, $C_1, C_2 \geq 0$, and $C_1 + C_2 > 0$. Suppose that $\Psi(t)$ is a twice differentiable positive function satisfying*

$$\Psi^{''}\Psi - (1+\alpha)[\Psi^{'}]^2 \geq -2C_1\Psi\Psi^{'} - C_2[\Psi]^2 \tag{6}$$

for all $t \geq 0$. If

$$\Psi(0) > 0 \quad and \quad \Psi^{'}(0) + \gamma_2\alpha^{-1}\Psi(0) > 0, \tag{7}$$

then as

$$t \to t_1 \leq t_2 = \frac{1}{2\sqrt{C_1^2 + \alpha C_2}} \ln \frac{\gamma_1\Psi(0) + \alpha\Psi^{'}(0)}{\gamma_2\Psi(0) + \alpha\Psi'(0)}, \quad \Psi(t) \to +\infty. \tag{8}$$

Here

$$\gamma_1 = -C_1 + \sqrt{C_1^2 + \alpha C_2}, \quad \gamma_2 = -C_1 - \sqrt{C_1^2 + \alpha C_2}.$$

However, here the function $\Phi(t) = \Psi(t)^{-\alpha}$ satisfies a second order constant coefficient differential inequality of the form

$$\Phi^{''}(t) + 2C_1\Phi^{'}(t) - C_2\alpha\Phi(t) \leq 0.$$

The modified concavity method developed in [11] already in [11] was applied to many problems of the form

$$Pu_{tt} + Au + B(u) + aPu_t = F(t,u).$$

Global nonexistence theorem for solutions of the problem

$$u_{tt} - \Delta u + a(x,t,u,\nabla u) = 0, \quad x \in \Omega, \quad t > 0,$$

$$\frac{\partial u}{\partial n} = f(t,u), \quad x \in \partial\Omega, \quad t > 0$$

is studied in [9].

In [10], global nonexistence result is obtained for the Cauchy problem for differential-operator equation of the form

$$Pu_{tt} + Qu_t + Au = B(u, u_t) + F(t, u).$$

This set up is applicable to some classes of coupled hyperbolic equations and pseudo-hyperbolic equations.

In [5], Erdem and Kalantarov has applied the the modified concavity method to the following equation

$$u_{tt} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\left(d_0 + |\nabla u|^{m-2} \right) \frac{\partial u}{\partial x_i} \right) + h(u, \nabla u) = f(u),$$

where

$$f(u)u \geq 2(\alpha + 1) \int_0^u f(s)ds, \quad \forall u \in R^1, \quad \alpha > (m-2)/2, \quad m \geq 2$$

and

$$|h(u, \mathbf{p})| \leq C(|u|^{m/2} + |\mathbf{p}|^{m/2}).$$

In [1] and [3], the concavity method and modified concavity method was used to show global nonexistence of solutions to nonlinear wave equations with the damping on the boundary.

By using the Lemma 2, Qin and Rivera [19] studied the blow up of solutions to the Cauchy problem for the following one-dimensional thermoelasticity system with a nonautonomous forcing and a thermal memory:

$$u_{tt} = au_{xx} + b\theta_x + +du_x - mu_t + f(t, u),$$

$$c\theta_t = k\theta_{xx} + g * \theta_{xx} + bu_{xt} + pu_x + q\theta_x.$$

A nonexistence result to a Cauchy problem for the thermoelasticity system without memory that is the system above with $g = 0$ is obtained in [12].

Finally, in [4], Eden and Kalantarov has applied the modified concavity method to the parabolic analog of the inverse source problem. Our paper is organized as follows: in the second section, we prove the main theorem on global nonexistence of solutions. In the last section, we state and prove a stability result for the solutions when the integral constraint vanishes at infinity. Our main result is the following theorem:

Theorem 1. *Let the conditions* (A1)–(A3) *be satisfied and assume that*

$$\|u_0\| > 0, \quad E(0) \geq D_1 + \frac{2B_1}{p+4}, \tag{A4}$$

where

$$E(0) = \left[-\frac{1}{2}\|u_1\|^2 - \frac{1}{2}\|\nabla u_0\|^2 - \frac{\lambda^2}{2}\|u_0\|^2 + \frac{1}{p+2}\|u_0\|_{p+2}^{p+2}\right],$$

$$D_1 = \frac{\|\Delta w\|^2}{M_2(M_1+M_2+2\lambda)} + C(p)\frac{\|w\|_{p+2}^{p+2}}{M_1+M_2+2\lambda p} + \frac{M_1\lambda^2+M_2}{2M_1+2M_2+8\lambda}\|w\|^2,$$

$$B_1 = \frac{2\lambda}{p}\|\Delta w\|^2 + \frac{1}{(p+2)\left[\frac{p}{2p+2}\right]^{p+1}}\|w\|_p^p + \left(\frac{2}{p}M_1^2 + \frac{2M_2^2}{\lambda p}\right)\|w\|^2,$$

and

$$\lambda = \max\left\{\frac{p+2}{p}(M_1+M_2+1); \left(\frac{M_2}{2}\right)^{\frac{1}{3}}\right\},$$

$$\gamma_{1,2} = -C_1 \pm \sqrt{C_1^2 + \alpha C_2}, C_1 = M_1 + \frac{2}{p}M_2^2,$$

$$C_2 = (p+4)\left[\frac{3}{2}\frac{M_2^2}{4\epsilon_1} + 1 + \left(3\frac{M_1^2}{4\epsilon_1} + 1/2\right)\left(M_1^2 + M_2^2\right)\right].$$

Then there exists a finite t_1 *such that*

$$\|u(t)\| \to +\infty \quad as \quad t \to t_1^-.$$

Here and for the rest of the paper we will adhere to the following notational conventions:

$$\langle f, g\rangle = \int_\Omega f(x)g(x)dx, \|f\| = \left(\int_\Omega f^2(x)dx\right)^{\frac{1}{2}} \quad \text{for} \quad f,\, g \in L^2(\Omega),$$

$$\langle F, G\rangle = \int_\Omega F(x)\cdot G(x)dx, \|F\| = \left(\int_\Omega F(x)\cdot F(x)dx\right)^{\frac{1}{2}}$$

$$\text{for} \quad F,\, G \in L^2(\Omega)^n.$$

Also for convenience, we will suppress the dependence of the functions on x and t as we have already done in [5].

2. A global nonexistence result

In order to study the global nonexistence we consider the following problem that is obtained from (1)–(4) by substituting $v(x,t) = u(x,t)e^{-\lambda t}$:

$$v_{tt} - \Delta v + \lambda^2 v + 2\lambda v_t - e^{\lambda pt}|v|^p v + b(x,t,e^{\lambda t}v, e^{\lambda t}\nabla v)e^{-\lambda t} =$$

$$= Fwe^{-\lambda t}, \quad x \in \Omega, \quad t > 0, \tag{9}$$

$$v(x,t) = 0, \quad x \in \partial\Omega, \quad t > 0, \tag{10}$$

$$v(x,0) = u_0(x), \quad v_t(x,0) = u_1(x) - \lambda u_0(x), \quad x \in \Omega, \tag{11}$$

$$\langle v, w\rangle = e^{-\lambda t}, \quad t > 0. \tag{12}$$

The value of the parameter λ will be prescribed later. As mentioned in the introduction, multiplying (9) by w and using (12) one can obtain that

$$F(t) = -e^{\lambda t}\langle v, \Delta w\rangle - e^{\lambda(p+1)t}\langle |v|^p v, w\rangle + \langle b(x,t,e^{\lambda t}v, e^{\lambda t}\nabla v), w\rangle. \tag{13}$$

Substituting (13) into (9) one obtains a problem that is shown to be equivalent to (9)–(12). (See for example [18].)

The choice for the functional is standard (see [14, 11]) we define

$$\Psi(t) = \|v(t)\|^2. \tag{14}$$

It is clear that

$$\Psi'(t) = \frac{d}{dt}\|v(t)\|^2 = 2\langle v_t(t), v(t)\rangle, \tag{15}$$

$$\Psi''(t) = 2\frac{d}{dt}\langle v_t(t), v(t)\rangle = 2\langle v_{tt}(t), v(t)\rangle + 2\|v_t(t)\|^2. \tag{16}$$

In order to obtain a differential inequality of the form (6) we need to bound from below (16) for solutions of (9)–(12).

The following notations are introduced to simplify the estimates; let

$$\widehat{b}(t,v) \equiv b(x,t,e^{\lambda t}v, e^{\lambda t}\nabla v),$$

$$G(t,v) \equiv e^{\lambda pt}\int_\Omega |v|^{p+2}dx, \tag{17}$$

$$L(v(t)) \equiv \|v_t\|^2 + \|\nabla v(t)\|^2 + \lambda^2\|v(t)\|^2, \tag{18}$$

$$E(t) \equiv -\frac{1}{2}L(v) + \frac{1}{p+2}G(t,v). \tag{19}$$

First we estimate $\langle v_{tt}, v\rangle$ from below. Using in addition to (12) the fact that

$$\langle w, v_t\rangle = \frac{d}{dt}\langle w, v\rangle = (e^{-\lambda t})' = -\lambda e^{-\lambda t}, \quad \langle w, v_{tt}\rangle = \lambda^2 e^{-\lambda t} \tag{20}$$

it is easy to get

$$\langle v_{tt}, v\rangle = -\|\nabla v\|^2 - \lambda^2\|v\|^2 - 2\lambda\langle v_t, v\rangle + e^{\lambda p t}\int_\Omega |v|^{p+2}dx + e^{-\lambda t}\langle \widehat{b}(t,v), v\rangle -$$

$$e^{-\lambda t}\langle v, \Delta w\rangle - e^{\lambda(p-1)t}\langle |v|^p v, w\rangle + e^{-2\lambda t}\langle \widehat{b}(t,v), w\rangle =$$

$$\Big(\frac{p}{2}+2\Big)E(t) + \frac{p}{4}\|\nabla v\|^2 + \frac{\lambda^2 p}{4}\|v\|^2 + \Big(\frac{p}{4}+1\Big)\|v_t\|^2 - 2\lambda\langle v_t, v\rangle +$$

$$+\frac{p}{2p+4}G(t,v) + e^{-\lambda t}\langle \widehat{b}(t,v), v\rangle - e^{-\lambda t}\langle v, \Delta w\rangle -$$

$$-e^{\lambda(p-1)t}\langle |v|^p v, w\rangle + e^{-2\lambda t}\langle \widehat{b}(t,v), w\rangle. \tag{21}$$

Multiplying (9) by v_t and integrating over Ω, we get

$$-\frac{d}{dt}E(t) + \frac{\lambda p}{p+2}G(t,v) + 2\lambda\|v_t\|^2 + \langle \widehat{b}(t,v), v_t\rangle e^{-\lambda t} = -\lambda F(t)e^{-2\lambda t}. \tag{22}$$

Again substituting (13) into,

$$-\frac{d}{dt}E(t) + \frac{\lambda p}{p+2}G(t,v) + 2\lambda\|v_t\|^2 + \langle \widehat{b}(t,v), v_t\rangle e^{-\lambda t} =$$

$$\lambda[e^{-\lambda t}\langle v, \Delta w\rangle + e^{\lambda(p-1)t}\langle |v|^p v, w\rangle - \langle \widehat{b}(t,v), w\rangle e^{-2\lambda t}]. \tag{23}$$

Now we are going to estimate various terms on the right hand side of (21) and (23) by the appropriate combinations of $G(t,v)$ and $j(t)$. Let us recall that Young's inequality implies that for $a, b \geq 0$

$$ab \leq \beta a^q + C(\beta, q)b^{q'}$$

for $\frac{1}{q} + \frac{1}{q'} = 1$ and where $C(\beta, q) = \frac{1}{q'(\beta q)^{q'/q}}$.

Taking $a = \|v\|, b = \|\Delta w\|e^{-\lambda t}, q = q' = 2$, and $\beta = \frac{p\lambda^2}{8}$, we get

$$|\langle v, \Delta w\rangle e^{-\lambda t}| \leq \frac{\lambda^2 p}{8}\|v\|^2 + \frac{2}{p\lambda^2}\|\Delta w\|^2 e^{-2\lambda t} \tag{24}$$

and taking

$$q = \frac{p+2}{p+1}, \quad q^{'} = p+2, \quad a = e^{(\frac{p^2+p}{p+2})t} \|v\|_{p+2}^{p+1}, \quad b = \lambda \|w\|_{p+2} e^{-\frac{2\lambda t}{p+2}},$$

and $\beta = \frac{p}{2(p+2)}$, we get

$$e^{\lambda(p-1)t} \int_\Omega |v|^p v w dx \leq$$

$$\leq \frac{p}{2(p+2)} G(t,v) + \frac{e^{-2\lambda p t}}{(p+2)[\frac{p}{2p+2}]^{p+1}} \int_\Omega |w|^{p+2} dx. \quad (25)$$

By using the condition (A2) similarly, we obtain

$$e^{-2\lambda t} |\langle \widehat{b}(t,v), w\rangle| \leq$$

$$\leq \frac{p}{8}\|\nabla v\|^2 + \frac{2}{p} M_1^2 \|w\|^2 e^{-2\lambda t} + \frac{\lambda p}{8}\|v\|^2 + \frac{2M_2^2}{\lambda^2 p}\|w\|^2 e^{-2\lambda t} \quad (26)$$

and

$$e^{-\lambda t} |\langle \widehat{b}(t,v), v\rangle| \leq M_1 \|\nabla v\| \|v\| + M_2 \|v\|^2 \leq$$

$$\leq \Big(M_2 + \frac{2M_1^2}{p}\Big)\|v\|^2 + \frac{p}{8}\|\nabla v\|^2. \quad (27)$$

Rewriting (21) and using (24), (25), (26), and (27)

$$\langle v_{tt}, v\rangle \geq \Big(\frac{p}{2} + 2\Big) E(t) - \Big(M_1 + \frac{2}{p} M_2^2\Big)\|v\|^2 +$$

$$+ \Big(1 + \frac{p}{4}\Big)\|v_t\|^2 - 2\lambda \langle v_t, v\rangle - B_1, \quad (28)$$

where

$$B_1 = \frac{2\lambda}{p}\|\Delta w\|^2 + \frac{1}{(p+2)\left[\frac{p}{2p+2}\right]^{p+1}} \|w\|_p^p + \Big(\frac{2}{p} M_1^2 + \frac{2M_2^2}{\lambda p}\Big)\|w\|^2.$$

Since this inequality gives us a lower bound of $\Psi^{''}(t)$ in terms of $E(t)$, $-\Psi(t)$ and $-\Psi^{'}(t)$, we next set out to estimate $E(t)$ from below using (23).

We proceed to estimate the terms on the right hand side of (23) similar to (24) and (25) with $q = q' = 2$, $\beta = \frac{M_2 \lambda^2}{4}$, and $b = \|w\| e^{-\lambda t}$ in the first case and with $\beta = 1$, $q = \frac{p+2}{p+1}$, and $b = \|w\|_{p+2} e^{-\frac{2\lambda t}{p+2}}$ in the second case to get

$$|\lambda e^{-\lambda t} \langle v, \Delta w \rangle| \leq \frac{M_2 \lambda^2}{4} \|v\|^2 + \frac{1}{M_2} \|\Delta w\|^2 e^{-2\lambda t}, \tag{29}$$

$$|\lambda e^{-\lambda (p-1) t} \langle |v|^p v, w \rangle| \leq G(t, v) + C(p) e^{-2\lambda p t} \int\limits_\Omega |w|^{p+2} dx. \tag{30}$$

By using the Young's inequality we get also the estimates

$$e^{-\lambda t} |\langle \widehat{b}(t, v), v_t \rangle| \leq M_1 \|\nabla v\| \|v_t\| + M_2 \|v\| \|v_t\| \leq$$

$$M_1 \left[\frac{1}{2} \||v_t\|^2 + \frac{1}{2} \||\nabla v\|^2 \right] + M_2 \left[\frac{\lambda^2}{4} \||v\|^2 + \frac{1}{\lambda^2} \|v_t\|^2 \right], \tag{31}$$

$$\lambda e^{-2\lambda t} |\langle \widehat{b}(t, v), w \rangle| \leq$$

$$\frac{M_1}{2} \|\nabla v\|^2 + \frac{M_1 \lambda^2}{2} e^{-2\lambda t} \|w\|^2 + \frac{M_2 \lambda^2}{2} \|v\|^2 + e^{-2\lambda t} \frac{M_2}{2} \|w\|^2. \tag{32}$$

Rewriting (23) and using (29)–(32)

$$\frac{d}{dt} E(t) \geq (M_1 + M_2) E(t) +$$
$$+ \Big(\frac{\lambda p}{p+2} - M_1 - M_2 - 1 \Big) G(t, v) + \Big(2\lambda - \frac{M_2}{\lambda^2} \Big) \|v_t\|^2 - D_1(t), \tag{33}$$

where

$$D_1(t) = \frac{1}{M_2} \|\Delta w\|^2 e^{-2\lambda t} + C(p) e^{-2\lambda p t} \int\limits_\Omega |w|^{p+2} dx +$$

$$+ \left(\frac{M_1 \lambda^2}{2} + \frac{M_2}{2} \right) \|w\|^2 e^{-4\lambda t}.$$

Next we choose

$$\lambda = \max \left\{ \frac{p+2}{p} (M_1 + M_2 + 1); \left(\frac{M_2}{2} \right)^{\frac{1}{3}} \right\}.$$

Then (33) implies

$$\frac{d}{dt} E(t) \geq (M_1 + M_2) E(t) - D_1(t).$$

Integrating the last inequality we obtain

$$E(t) \geq E(0) - D_1, \tag{34}$$

where

$$D_1 = \frac{\|\Delta w\|^2}{M_2(M_1 + M_2 + 2\lambda)} + C(p)\frac{\|w\|_{p+2}^{p+2}}{M_1 + M_2 + 2\lambda p} + $$
$$+\frac{M_1\lambda^2 + M_2}{2M_1 + 2M_2 + 8\lambda}\|w\|^2. \tag{35}$$

Due to the condition (A4) $E(0) \geq D_1 + \frac{2B_1}{p+4}$. Hence

$$E(t) \geq \frac{2B_1}{p+4}.$$

By using the last inequality, we obtain from (28)

$$\langle v_{tt}, v\rangle \geq -(M_1 + \frac{2}{p}M_2^2)\|v\|^2 + (1 + \frac{p}{4})\|v_t\|^2 - 2\lambda\langle v_t, v\rangle.$$

Therefore we have

$$\Psi^{''}(t) = 2\frac{d}{dt}\langle v_t(t), v(t)\rangle = 2\langle v_{tt}(t), v(t)\rangle + 2\|v_t(t)\|^2 \geq$$
$$-2(M_1 + \frac{2}{p}M_2^2)\Psi(t) - 2\lambda\Psi^{'}(t) + 4(1 + \frac{p}{8})\|v_t\|^2.$$

By using the last inequality finally we get

$$\Psi^{''}(t)\Psi(t) - \left(1 + \frac{p}{8}\right)\left[\Psi^{'}(t)\right]^2 \geq -2\left(M_1 + \frac{2}{p}M_2^2\right)\Psi^2(t) - 2\lambda\Psi^{'}(t)\Psi(t).$$

Remark 1. Condition (4) can be relaxed to

$$\int u(x,t)w(x)dx = \phi(t),$$

where $\phi(t)$ is twice differentiable function with $0 < a < \phi(t) < b$, $0 < c < \phi^{'}(t) < d$ for some real constants a, b, c, d.

Remark 2. Similarly we can prove a global nonexistence theorem for the equation

$$u_{tt} - \Delta u - |u|^p u + au_t + b(x,t,u,\nabla u) = F(t)w(x), \quad x \in \Omega, \quad t > 0,$$

where $a \in R$ is any real number.

3. Stability problem and the exponential attractor

In this section, we consider the following inverse source problem

$$u_{tt} - \Delta u + au_t + |u|^p u - bu = F(t)w(x), \quad x \in \Omega, \quad t > 0, \tag{36}$$

$$u(x,t) = 0, \quad x \in \partial\Omega, \quad t > 0, \tag{37}$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{38}$$

$$\int_\Omega u(x,t)w(x)dx = \phi(t), \quad t > 0, \tag{39}$$

where Ω is a bounded domain of R^n with sufficiently smooth boundary $\partial\Omega$, w, u_0, and $\phi(t)$ are given functions and p is a given positive number. We assume that w, u_0 and u_1 satisfy conditions

$$w, u_0 \in H_0^1(\Omega) \cap L^{p+2}(\Omega), \quad u_1 \in L^2(\Omega), \quad \text{and} \int_\Omega u_0(x)w(x)dx = \phi(0). \tag{A5}$$

Theorem 2. *Assume that conditions* (A1) *and* (A5) *are satisfied, the damping coefficient $b \in (0, \lambda_1)$ and suppose that ϕ, $\phi^{'}$, and $\phi^{''}$ are continuous functions defined on $[0, \infty)$ such that $\phi^{''}$ is a bounded function and ϕ, $\phi^{'}$ tend to 0 as $t \to \infty$. Then*

$$\lim_{t\to\infty} \left[\|u_t\|^2 + \|\nabla u(t)\|^2 + \|u(t)\|_{p+2}^{p+2} \right] = 0.$$

Multiplying (36) by w, integrating over Ω and using (39), we obtain

$$F(t) = \phi^{''}(t) + \langle \nabla u, \nabla w\rangle + a\phi^{'}(t) + \langle |u|^p u, w\rangle - b\phi(t). \tag{40}$$

Substituting (40) in (36) we get

$$u_{tt} - \Delta u + au_t + |u|^p u - bu =$$

$$\left[\phi^{''}(t) + \langle \nabla u, \nabla w\rangle + a\phi^{'}(t) + \langle |u|^p u, w\rangle - b\phi(t)\right] w(x), \; x \in \Omega, \; t > 0, \tag{41}$$

Multiplying (41) by $u_t + \eta u$ and integrating over Ω:

$$\frac{d}{dt}\left[\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 + \frac{1}{p+2}\|u\|_{p+2}^{p+2} - \frac{b}{2}\|u\|^2 + \eta\frac{a}{2}\|u\|^2 + \eta\langle u_t, u\rangle\right] +$$

$$+(a-\eta)\|u_t\|^2 + \eta\|\nabla u\|^2 + \eta\|u\|_{p+2}^{p+2} - \eta b\|u\|^2 \leq$$

$$\leq |\phi^{''}(t)\phi^{'}(t)| + (b+\eta a)|\phi^{'}(t)\phi(t)| + \|\nabla u\|\|\nabla w\|(|\phi^{'}(t)|+$$

$$+\eta|\phi(t)|) + |\langle |u|^p u, w\rangle|(|\phi^{'}(t)| + \eta|\phi(t)|).$$

By using the Cauchy inequality with ϵ and the Young inequality with ϵ in the right-hand side of the last inequality we obtain

$$\frac{d}{dt}E_\eta(t) + (a-\eta)\|u_t\|^2 + (\eta-\epsilon)\|\nabla u\|^2+$$

$$+(\eta-\epsilon)\|u\|_{p+2}^{p+2} - \eta b\|u\|^2 \leq H(t), \tag{42}$$

Here

$$E_\eta(t) \equiv \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 + \frac{1}{p+2}\|u\|_{p+2}^{p+2} - \frac{b}{2}\|u\|^2 + \eta\frac{a}{2}\|u\|^2 + \eta\langle u_t, u\rangle,$$

$$H(t) \equiv |\phi^{''}(t)\phi^{'}(t)| + (b+\eta a)|\phi^{'}(t)\phi(t)| + \frac{1}{4\epsilon}\|\nabla w\|^2(|\phi^{'}(t)|+$$

$$\eta|\phi(t)|)^2 + C(\epsilon,p)\|w\|_{p+2}^{p+2}(|\phi^{'}(t)| + \eta|\phi(t)|)^{p+2}.$$

We assume that $\eta < a$ and $\epsilon < \eta$ are positive numbers. We can rewrite (42) as follows:

$$\frac{d}{dt}E_\eta + \delta E_\eta + (a-\eta)\|u_t\|^2 + (\eta-\epsilon)\|\nabla u\|^2+$$

$$+\frac{a}{4}\|u\|_{p+2}^{p+2} - \frac{b}{2}\|u\|^2 - \delta E_\eta \leq H(t),$$

or

$$\frac{d}{dt}E_\eta(t) + \delta E_\eta(t) \leq -\Big(a-\eta-\frac{\delta}{2}\Big)\|u_t\|^2 - \Big(\eta-\epsilon-\frac{\delta}{2}\Big)\|\nabla u\|^2$$

$$-\Big(\eta-\epsilon-\frac{\delta}{p+2}\Big)\|u\|_{p+2}^{p+2} + \eta b\|u\|^2 + \delta\eta\frac{a}{2}\|u\|^2 + \delta\eta\langle u_t, u\rangle + H(t). \tag{43}$$

Since $0 < b < \lambda_1$, we have

$$\|\nabla u\|^2 - b\|u\|^2 \leq d_0\|\nabla u\|^2, \tag{44}$$

where $d_0 = 1 - \frac{b}{\lambda_1} > 0$. Due to the Poincaré–Friedrichs inequality and the Cauchy inequality, we have

$$\begin{aligned} &\delta\eta\frac{a}{2}\|u\|^2 \leq \delta\eta\frac{a}{2\lambda_1}\|\nabla u\|^2, \\ &\delta\eta|\langle u_t, u\rangle| \leq \delta\eta\frac{1}{2\lambda_1}\|\nabla u\|^2 + +\delta\eta\frac{1}{2}\|u_t\|^2, \end{aligned} \tag{45}$$

By using (44) and (45), we obtain from (43):

$$\frac{d}{dt}E_\eta(t) + \delta E_\eta(t) \le -\left(a - \eta - \frac{\delta}{2} - \frac{\delta\eta}{2}\right)\|u_t\|^2 +$$

$$\left(\eta d_0 - \epsilon - \frac{\delta}{2} - \frac{\delta\eta}{2\lambda_1}(1+a)\right)\|\nabla u\|^2 - (\eta - \epsilon - \frac{\delta}{p+2})\|u\|_{p+2}^{p+2} + H(t).$$

Taking $\eta < a$, $\epsilon < d_0\eta$, and $\delta > 0$ sufficiently small, we obtain from the last inequality:

$$\frac{d}{dt}E_\eta(t) + \delta E_\eta(t) \le H(t).$$

Since $H(t) \to 0$ as $t \to \infty$ by the conditions imposed on $\phi(t)$ the result of the theorem follows from the estimate

$$\|u_t(t)\|^2 + \|\nabla u(t)\|^2 + \|u(t)\|_{p+2}^{p+2} \le C_0 E_\eta(t),$$

which we derived using $0 < b < \lambda_1$ for $\eta < \frac{a}{2}$.

Remark 1. When a can be taken large enough, so that $\frac{a^2}{2} > b$, then no condition like $0 < b < \lambda_1$ is required.

Remark 2. Similar stability theorem can be obtained for the semi-linear wave equation with strong damping, $-a\Delta u_t$ instead of au_t.

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