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N. M. Ivochkina

SYMMETRY AND GEOMETRIC EVOLUTION EQUATIONS

Abstract. The paper contains analytic description of geometric flows generated by nonlinear curvature functions in terms of invariant symmetric curvature matrix. Such approach provides the most efficient application of ideas and methods of the modern theory of fully nonlinear second order partial differential equations to the relevant geometric problems.

Dedicated to the memory of
Olga Aleksandrovna Ladyzhenskaya

1. Introduction

The development of the modern theory of fully nonlinear second order partial differential equations has brought out impressive progress in the geometric problems of evolution of closed surfaces. Although it looks like speciality of approach in this theory require modification of some traditional points of view in differential geometry. To describe some necessary reconsiderations we analyze in this paper the application of famous Evans–Krylov estimates to geometric evolution equations. Actually these estimates had been constracted for totally elliptic (parabolic) equations and the approach should be slightly modified for geometric equations which could not be reduced to totally elliptic (parabolic) equations in the most interesting cases. The reality of the gap has been actually discovered by O. A. Ladyzhenskaya during joint work with the author in mid-ninties. The author will never forget those happy times.

Speaking about geometric evolution equations we mean the following. Let $\Gamma_0 \subset R^{n+1}$ be closed hypersurface, then any set of hypersurfaces $\{\Gamma_t, t > 0\}$ smooth in t will be an evolution of Γ_0. Let also $X = X(M)$, $M \in \Gamma_t$ be the position vector, $\nu = \nu(M)$ be the unit normal. Denote $v = (X_t, \nu)(M)$, $X_t = \partial X/\partial t$, i.e., v is the normal velocity of our evolution. It will always be $v \neq 0$ in our considerations and we keep to agreement

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$v > 0$, that is ν will be turned in the direction of motion. Let $\mathbf{k} = (k_1, ..., k_n)(M)$ be the vector of principal curvatures $k_i, i = 1, ..., n$ of Γ_t at M and the choice of orientation provides $k_i > 0$ for a sphere. On the set of geometric evolutions we determine operators $h[\Gamma_t]$ as $h[\Gamma_t] = h(v, \mathbf{k})[\Gamma_t]$, where h is sufficiently smooth function. Now goes geometric evolution equation

$$h[\Gamma_t] = c, \quad c \geq 0. \tag{1.1}$$

The theory of fully nonlinear parabolic equations deals with functions $u \in C^{4,2}(Q_T)$, $Q_T = \Omega \times (0; T)$, where $\Omega \subset R^n$ is a bounded domain. In order to introduce fully nonlinear operator in suitable way we consider functions H defined in some domain $D \subset R^1 \times \mathrm{Sym(n)}$, where Sym(n) is the space of $n \times n$ symmetric matrices. The relevant fully nonlinear equation reads as

$$H[u] := H(s, S)[u] = c, s[u] = -a_0 u_t, S[u] = C u_{xx} C^T + W, \tag{1.2}$$

where $C = C(x, t, u, u_x)$ is given $n \times n$ positive definite matrix, $W = W(x, t, u, u_x) \in \mathrm{Sym(n)}, a_0 = a_0(x, t, u, u_x) > 0$, u_x, u_{xx} are gradient and Hesse matrix of u respectively.

The application of the methods of the theory of parabolic second order partial differential equations to (1.1) assumes relevant choice of u, C and W in (1.2), what in turn depends on the choice of parametrisation in (1.1). One of the possible approaches will be suggested in Section 4. In Section 3 we describe traditional geometric approach through Weingarten mapping and show that it brings out non symmetric matrix in capacity of substitute for S. In Section 2 we try to explain that symmetry of S is highly desireable for non totally parabolic (elliptic) equations.

This paper does not aim to prove any existence theorems. One can find many such theorems in literature. We present here two typical examples for completness.

Let $f_m(\mathbf{k}) := S_m^{1/m}(\mathbf{k}$, where $S_m(\mathbf{k})$ is the m-th elementary symmetric function of $k_i, i = 1, ..., n$. We indicate the surface Γ as m-convex if

$$S_l(\mathbf{k}(M)) > 0, \quad l = 1, ..., m, \quad M \in \Gamma.$$

Consider the following evolution equations

$$v = f_m^q(\mathbf{k})[\Gamma_t]. \tag{1.3}$$

Theorem 1.1. *Let Γ_0 be strictly convex closed C^4-hypersurface and $q \in (0; \infty)$. Then there exists $T < \infty$ such that there is the unique solution to (1.3) $\{\Gamma_t, t \in [0; T)\}$, where Γ_t are strictly convex C^4-hypersurfaces. Moreover, the surfaces $\{\Gamma_t\}$ contract to a point when $t \to T$.*

The assertion of Theorem 1.1 follows from considerations in [1] for $m = 1$, $q = 1$, from [2] for $m = n$, $q = 1, n$ and has been proved in [3] for all values of q.

Theorem 1.2. *Let Γ_0 be m-strictly convex starshaped closed C^4-hypersurface and $-1 < q < 0$. Then there exists the unique solution to (1.3) $\{\Gamma_t, t \in [0; \infty)\}$, where Γ_t are strictly m-convex starshaped C^4-hypersurfaces. Moreover, Γ_t uniformely expand to infinity.*

Theorem 1.2 follows from the results in [4]

The principal concern of the specialists in differential geometry dealing with similar problems is asymptotic behaviour of evolutions. In this paper, we just remark that asymptotic convergence to a sphere has been proved in [1] for $m = 1$ and $q = 1$, in [5] for $m = n$ and $q = 1$ when $t \to T$. For the other values of q in Theorem 1.1 it stays an open problem. As it has been proved in [4] evolutions from Theorem 1.2 asymptotically converge to spheres when $t \to \infty$.

2. Second derivatives Holder estimates and symmetry of $S[u]$

We consider equation (1.2) with the following properties of function H. Let $\mathbf{S} := (s, S)$ and $D = \{\mathbf{S} \in R^1 \times \mathrm{Sym(n)} : H(\mathbf{S}) > 0\}$. Assume the following.

D1. The domain D is convex, $\mathbf{0} \in \partial D$;

D2. If $(s, S) \in D$ then $(s+\sigma, S+\xi \times \xi) \in D$ for arbitrary $\sigma \geq 0, \xi \in R^n$;

D3. If $(s, S) \in D$ then $(s, BSB^T) \in D$, where B is an arbitrary orthogonal matrix.

Assume also that H satisfies in D the following requirements.

H1. H is positive monotone, i.e.,

$$H(s + \sigma, S + \xi \times \xi) > H(s, S), \quad \sigma \geq 0, \quad |\xi| \neq 0;$$

H2. H is concave in D;

H3. $H(s, S) = H(s, BSB^T)$.

Now, we introduce the following characters.

$$D_c = \{\mathbf{S} \in D : H(\mathbf{S}) \geq c\}, \quad H^0 = \frac{\partial H}{\partial s}, \quad H^{ij} = \frac{\partial H}{\partial s_{ij}}.$$

$$\mathbf{S}^\tau = \tau\mathbf{S}_1 + (1-\tau)\mathbf{S}_2, \quad \tau \in [0;1].$$

Also we denote

$$M_{12} = \max_{i=1,2} |\mathbf{S}_i|, \quad |\mathbf{S}|^2 := s^2 + \mathrm{tr} S^2.$$

The following proposition is crucial in this Section.

Lemma 2.1. *Let* $\mathbf{S}_i \in D_c, i = 1,2, c \geq c_0 > 0$. *Then there exist* $\lambda_0 = \lambda(c_0, M_{12}) > 0, \Lambda_0 = \Lambda(c_0, M_{12})$ *such that*

$$\lambda_0 < \int_0^1 H^0(\mathbf{S}^\tau) d\tau < \Lambda_0, \quad \lambda_0 \xi^2 < \int_0^1 H^{ij}(\mathbf{S}^\tau) d\tau \xi^i \xi^j < \Lambda_0 \xi^2. \tag{2.1}$$

Indeed, inequalities (2.1) are an obvious consequence to assumptions D1, D2, H1, H2. On the other hand they are not valid in general for nonconvex D even in presence of D2, H1 and locally concave functions H.

Under assumption H1 the operator $H[u]$ generates the set of H-admissible functions in the following sense.

Definition 2.2. *Function* $u \in C^{2,1}(Q_T)$ *is* H*-admissible iff* $\mathbf{S}[u](x,t) \in D$, *for* $(x,t) \in Q_T$ *and solution* u *to* (1.2) *is admissible iff it is* H*-admissible function.*

If $||u||_{C^{2,1}}$ of admissible solution is under control one can consider equation (1.2) as uniformely parabolic and assume that Krylov's argument to estimate Holder constants of first time and second space derivatives of solutions to totally uniformely parabolic equations should work [6]. Namely, having $|u_{xx}|$ estimated it looks reasonable to get focused on the dependence of fully nonlinear equation on matrix variable u_{xx}, i.e., consider $F(-u_t, u_{xx}, ...) = c$ instead of (1.2). Actually this equation is subject to Krylov's construction if it is uniformely parabolic either concave (convex) on the segment $\tau u_{xx}(z_1) + (1-\tau)u_{xx}(z_2), z_1, z_2 \in Q_T, \tau \in [0;1]$. It is definitely true for totally parabolic equations and not so, for instance, for nontotally parabolic curvature equations. We show that equation (1.2), H1, H2 also suits Krylov's argument.

Denote $z = (x,t)$, $r(z_1, z_2) = \max\{|x_1 - x_2|^2; |t_1 - t_2\}$. Let $\{\gamma_I, I = 1, ..., N\}$ be the set of unit vectors and

$$u_i := \frac{\partial u}{\partial x^i}, \quad i = 1, ..., n, \quad u_I = u_i \gamma_I^i.$$

Lemma 2.3. *Let u be H-admissible function, $H[u] \geq c_0, ||u_{C^{2,1}}|| \leq M_2$. Then there exist $N, b^0, \mathbf{b} = (b^1, ..., b^N)(z), z \in Q_T$ satisfying the following*

(i) *the set $\{\gamma_I\}$ includes coordinate orts $\mathbf{e}_1, ..., \mathbf{e}_n$ and also $(1/\sqrt{2})$ $(\mathbf{e}_i \pm \mathbf{e}_j)$, $i \neq j$;*

(ii) *there is $\nu = \nu(c_0, M_2) > 0$ such that*

$$\nu \leq b_0 \leq \frac{1}{\nu}, \quad \nu \leq b^I \leq \frac{1}{\nu}, \quad I = 1, ..., N,$$

and the following relation

$$H[u](z_1) - H[u](z_2) = \tag{2.2}$$

$$-b^0(u_t(z_1) - u_t(z_2)) + \sum_{I=1}^{N} b^I(u_{II}(z_1) - u_{II}(z_2)) + br^\beta(z_1, z_2)$$

is valid with some $\beta \in (0;1)$ and function $b = b(z_1, z_2, u, u_x, u_{xx})$ under control.

In essence, presentation (2.2) coincides with the relevant piece for totally uniformely parabolic (elliptic) equations [7], [6], and follows in the same way from Motzkin–Wasov Lemma. We present it here as it reads in [8], Lemma 17.13, for completeness.

Lemma 2.3. *Let $S[\lambda; \Lambda]$ denote the set of positive $n \times n$ matrices with eigenvalues lying in the interval $[\lambda; \Lambda]$, where $0 < \lambda < \Lambda$. Then there exists a fimite set of the unit vectors, $\gamma_1, ..., \gamma_N \in R^n$ and positive numbers $\lambda^* < \Lambda^*$, depending only on n, λ, Λ such that any matrix $A = (a^{ij} \in S[\lambda; \Lambda]$ can be written in the form*

$$A = \sum_{I=1}^{N} b^I \gamma_I \times \gamma_I, \quad a^{ij} = \sum_{I=1}^{N} b^I \gamma_I^i \gamma_I^j,$$

where $\lambda^* < b^I < \Lambda^*, I = 1, ..., N$. Furthermore the directions $\gamma_I, ..., \gamma_N$ can be chosen to include the coordinate directions $\mathbf{e}_i, i = 1, ..., n$, together with the directions $(1/\sqrt{2})(\mathbf{e}_i \pm \mathbf{e}_j)$, $i \neq j, i, j = 1, ..., n$.

Appointing in Lemma 2.3

$$a^{ij} = \int_0^1 H^{kl}(\mathbf{S}^\tau) d\tau c_k^i(z_1) c_l^j(z_1), \quad b^0 = \int_0^1 H^0(\mathbf{S}^\tau),$$

we conclude presentation (2.2) by direct calculations.

The above reasoning is collected from the paper [9], where one can also find the analysis of other parts of Krylov's deduction of second derivatives Holder estimates. We resume this Section with

Theorem 2.4. *Let $H \in C^2, c \geq c_0 > 0$ and u be an admissible solution to (1.2). Assume that D, H satisfy D1, D2, H1, H2 and $C_r(z) = B_r(x) \times (t - r^2; t) \subset Q_T$, $c, C, W \in C^{2+\alpha, 1+\alpha/2}$ in $C_r(z)$ with some given $0 < \alpha < 1$. Then there exists $\beta = \beta(c_0, n) \in (0; \alpha]$ such that*

$$||u||_{C^{2+\beta, 1+\frac{\beta}{2}}(C_{r/3}(z))} \leq \mu,$$

where μ depends on $n, c_0, \beta, ||u||_{C^{2,1}(C_r)}$ and on relevant norms of given functions and matrices.

3. Weingarten hypersurfaces

In order to treat equation (1.1), it is necessary to rewrite it in terms of a function defined on the set $\{E\}$ of $n \times n$ matrices with real eigenvalues. Since h in (1.1) is always supposed to be symmetric in $k_1, ..., k_n$ it is possible and we outline the relevant reduction from [10]. Let $S_m(\mathbf{k})$ be the m-th elementary symmetric function of $k_1, ..., k_n$. Then we have uniquely defined function $f = f(S_1, ..., S_n) := h(k_1, ..., k_n)$. Since $S_m(\mathbf{k}) = \mathrm{tr}_m E(k_1, ..., k_n)$, where $S_m(E) := \mathrm{tr}_m(E)$ is the sum of all m-th order principal minors of E and E has indicated in the brackets eigenvalues, we arrive to $F(E) := f(S_1, ..., S_n)(E)$. As it is shown in [10], F inherits smoothness, monotonicity and concavity (convexity) of h pointwise.

In [11], the Weingarten matrices of hypersurfaces were taken in capacity of $\{E\}$ and this is also the point of this Section. In further proceeding we more or less keep to the argument from [5] concerning representation of Weingarten matrix in terms of distance function related to Γ.

Let $\Gamma \subset R^{n+1}$ be a closed starshaped hypersurface, X its position vector and $\{\theta^\alpha,\ \alpha = 1, ..., n\}$ the set of parameters such that vectors $X = X(\theta^1, ..., \theta^n)$ run in a neighborhood of $M \in \Gamma$. If not said the contrary greece subskripts will mean differentiation. For instance,

$$X_\alpha = \frac{\partial X}{\partial \theta^\alpha}, \quad X_{\alpha\beta} = \frac{\partial^2 X}{\partial \theta^\alpha \partial \theta^\beta}.$$

It is not exactly so for the components of the metric tensor:

$$G = (G_{\alpha\beta}), \quad G_{\alpha\beta} = (X_\alpha, X_\beta).$$

Here and below (,) means the Euclidian scalar product. Let $\mathbf{n}$ be the exterior unit normal to Γ at M. Then the following line determines Weingarten matrix (mapping) $\mathbf{W} = (W_\alpha^\beta)$.

$$\mathbf{n}_\alpha = W_\alpha^\beta X_\beta, \quad \alpha = 1, ..., n$$

and equation (1.1) is equivalent in the above sense to

$$F(v, \mathbf{W})[\Gamma_t] = c \tag{3.1}$$

For further proceeding we prefer another expression for $\mathbf{W}$, namely, we will work with

$$W_\alpha^\beta = -G^{\beta\gamma}(X_{\gamma\alpha}, \mathbf{n}). \tag{3.2}$$

To describe Γ as the graph over unit sphere we lokate the origin inside Γ and introduce distance function $\rho = |X|$. Then $X = \rho\mathbf{z}$, where $\mathbf{z}$ is the ort of X. Let $g = (g_{\alpha\beta})$ be the metric tensor of unit sphere, i.e., $g_{\alpha\beta} = (\mathbf{z}_\alpha, \mathbf{z}_\beta)$ and ∇_α means standard differentiation on the sphere. In terms of distance function ρ and $\mathbf{z}$ the characters from the righthand side of (3.2) look as follows.

$$G_{\alpha\beta} = \rho^2(g_{\alpha\beta} + \frac{1}{\rho^2}\rho_\alpha\rho_\beta), \quad G^{\alpha\beta} = \frac{1}{\rho^2}(g^{\alpha\beta} - \frac{g^{\alpha\gamma}g^{\beta\sigma}\rho_\gamma\rho_\sigma}{w^2}), \tag{3.3}$$

where $w^2 = \rho^2 + g^{\alpha\beta}\rho_\alpha\rho_\beta$,

$$\mathbf{n} = \frac{\rho}{w}(\mathbf{z} + \frac{1}{\rho}g^{\alpha\beta}\rho_\alpha\mathbf{z}_\beta), \tag{3.4}$$

$$(X_{\alpha\beta}, \mathbf{n}) = \frac{1}{w}(\rho\nabla_\alpha\nabla_\beta\rho - G_{\alpha\beta} - \rho_\alpha\rho_\beta). \tag{3.5}$$

Invariant formulas (3.3)–(3.5) lead to the following presentation of Weingarten tensor.

Lemma 3.1. *In terms of distance function Weingarten tensor of the hypersurface Γ reads as*

$$\mathbf{W} = -\frac{\rho}{w}G^{-1}\nabla\nabla\rho + \frac{1}{w}(I + G^{-1}\nabla\rho \times \nabla\rho). \tag{3.6}$$

The choice of $\mathbf{z}$ generates natural parametrization of the hypersurfaces $\{\Gamma_t\}$ by $\theta = (\theta^1, ..., \theta^n)$ independent of t, i.e., $\mathbf{z}_t = 0$. From the point of view of geometric invariance it is higly desireable to enlarge the set of admissible parametrizations admitting $\theta = \theta(t, \theta_0)$ with θ_0 independent on time. Say, θ_0 is generated by the field of unit normals to the initial surface Γ_0. Presentation (3.6) does not depend on such exchange and we show that it is also true in a way for the normal velocity v. So, let $\mathbf{z} = \mathbf{z}(\theta), \rho = \rho(t, \theta), \theta = \theta(t, \theta_0)$. Then

$$\frac{dX}{dt} = \frac{d\rho}{dt}\mathbf{z} + \rho \mathbf{z}_\alpha \theta_t^\alpha, \quad \frac{d\rho_t}{dt} = \rho_t + \rho_\alpha \theta_t^\alpha. \tag{3.7}$$

Lemma 3.2. *The relation*

$$v := (X_t, \mathbf{n}) = \frac{\rho}{w}\rho_t \tag{3.8}$$

is inependent on the choice of parametrization.

The line (3.8) follows from (3.4), (3.7) in obvious way. Notice that the factor ρ/w in (3.6), (3.7) has simple geometric meaning. It is the projection of $\mathbf{n}$ on $\mathbf{z}$, which controls starshapedness of the surface over prescribed origin.

Eventually we arrive to the partial differential equation of this Section.

Theorem 3.3. *The geometric equation (3.1) is locally in t equivalent to the second order partial differential equation on the sphere*

$$F(\frac{\rho}{w}\rho_t, -\frac{\rho}{w}G^{-1} \bigtriangledown \bigtriangledown \rho + \frac{1}{w}(I + G^{-1} \bigtriangledown \rho \times \bigtriangledown \rho)) = c. \tag{3.9}$$

There is principal difference between equations (3.1) and (3.9). The latter essentially depends on the choice of the origin. That is why we can not guarantee global equivalence in t of (3.1), (3.9). We also don't know how to apply Evans–Krylov argument to solutions of equation (3.9) because matrix variable in (3.9) is ultimately non symmetric. Equation (3.9) formally looks as second order partial differential equation on the sphere either $\mathbf{z}$ depends on t or does not and it is rather misleading. Indeed, ρ_t from (3.8), when $\mathbf{z}_t \neq 0$, is not common partial derivative of function ρ in t.

4. Symmetrization

From the point of view of the theory of second order partial differential equations it would be more common to work with positive monotone in $\rho_{\theta\theta}$ functions F, what is not the case in (3.9). Although it would be right this way if one keeps to $\nu = -\mathbf{n}$ and in further reasoning we work with ν, i.e., with interior normal. Then the line (3.2) reads as follows.

$$\mathbf{W} = G^{-1}(X_{\theta\theta}, \nu). \tag{4.1}$$

Equation (1.1) contains only the eigenvalues of the matrix (4.1), whereas the lines

$$\mathbf{W}\zeta_\alpha = k_\alpha\zeta_\alpha, \quad \alpha = 1, ..., n \tag{4.2}$$

also provide eigenvectors ζ_α. The desireable symmetrization of $\mathbf{W}$ can be attained at the expense of relevant transformation of the eigenvectors in (4.2). Indeed, there are positive definite matrices

$$\eta = (\eta_\alpha^\beta), \quad \tau = (\tau_\beta^\alpha), \quad \eta\tau = I \tag{4.3}$$

such that

$$G = \eta^T\eta, \quad G^{-1} = \tau\tau^T. \tag{4.4}$$

Furthermore, if we have constructed some η_0 then the whole set $\{\eta = B\eta_0, BB^T = I\}$ will suit (4.4) what provides degree of freedom in the choice (4.4). Let $a = (a^1, ..., a^n), b = (b^1, ..., b^n)$ be vector and covector in respect to G. Denote

$$a^{(\alpha)} = a^\sigma\eta_\sigma^\alpha \quad b_{(\alpha)} = b_\sigma\tau_\alpha^\sigma$$

and expand these notations to tensors framing relevant group of indices with parenthesis. For instance,

$$G_{(\alpha\beta)} = G_{\gamma\sigma}\tau_\alpha^\gamma\tau_\beta^\sigma.$$

Such approach substitutes the tangential to Γ at M basis $\{X_\alpha, \alpha = 1, ..., n\}$ on the orthogonal in Euclidian sense collection $\{X_{(\alpha)}, \alpha = 1, ..., n\}$ since $G_{(\alpha\beta)} := (X_{(\alpha)}, X_{(\beta)}) = \delta_{\alpha\beta}$. As to (4.2) it reduces to

$$(X_{(\theta\theta)}, \nu)\hat{\zeta_\alpha} = k_\alpha\hat{\zeta_\alpha}, \quad \hat{\zeta_\alpha} = (\zeta_\alpha^{(1)}, ..., \zeta_\alpha^{(n)}), \quad \alpha = 1, ..., n.$$

We have arrived to the following geometric evolution equation with symmetric matrix variable

$$H(-(X_t, \nu), (X_{(\theta\theta)}, \nu)) = c \tag{4.5}$$

Lemma 4.1. *There is one to one correspondence between invariant under permutations of $\lambda_1, ..., \lambda_n$ functions h and satisfying* H3 *functions H.*

Indeed, on the one hand the argument to expand h on functions with matrix variable described in the beginning of Section 3 works here as well. On the other hand we take in capacity of h function H with matrix variable restricted to the intersection of D with subspace of diagonal matrices $\{\Lambda\}$ with invariance $H(B(I)\Lambda B(I)) = H(\Lambda)$, what is a particular case of H3. Here the set $\{B(I)\}$ consists of all matrices, which can be reduced to identity matrix by permutation of columns. The latter means that h is invariant under permutations of $\lambda_1, ..., \lambda_n$.

It follows from Lemma 4.1 that geometric equation (1.1) is equivalent to (4.5) although one may say that unlike (1.1) equation (4.5) contains the position vector X and hence extra information of some point as the origin besides the hypersurfaces. But (4.5) is invariant on a choice of this point. Perhaps would be credible to speak about immersions instead of position vector similarly to [5].

In order to reformulate equation (4.5) in terms of distance function we represent spheric metric g similarly to (4.3) (4.4) as the product $g = \tilde{\eta}^T \tilde{\eta}$ and denote

$$\tilde{\rho}_{(\alpha)} = \rho_\beta \tilde{\tau}^\beta_\alpha, \quad \alpha, \beta = 1, ..., n.$$

Now one can verify by direct computation that τ in (4.3), (4.4) may be fixed as

$$\tau^\alpha_\beta = \frac{1}{\rho} \tilde{\tau}^\alpha_\sigma (\delta^\sigma_\beta - \frac{\tilde{\rho}_\sigma \tilde{\rho}_\beta}{w(\rho + w)}). \tag{4.6}$$

Adapting (3.5) to equation (4.5) we conclude the following analog ot Theorem 3.3

Theorem 4.2. *Let the origin be located inside Γ_0. Then there exists $\delta > 0$ such that for $t \in (0; \delta)$ equation (4.5) is equivalent to*

$$H(-\frac{\rho}{w}\rho_t, \frac{\rho}{w}\tau^T \bigtriangledown \bigtriangledown \rho \tau - \frac{1}{w}(I + (\tau^T \bigtriangledown \rho) \times (\tau^T \bigtriangledown \rho)) = c, \tag{4.7}$$

where $\rho, \bigtriangledown$ means the same as in Theorem 3.3 and τ is given by (4.6).

The proof of existence of admissible solutions to geometric parabolic equation (1.1) is based on continuity method, which implies the proof of local in t solvability and construction of a priori $C^{2+\alpha, 1+\alpha/2}$ estimates for solution $\{\Gamma_t\}$. Partial differential equation (3.9) as well as (4.7) provides local solvability. From Theorem 2.4 follows that only (4.7) can provide Holder estimates for Weingarten matrix. But both equations (3.9), (4.7) are rather bulky and all the same are not equivalent to (1.1) in general. We see as highly reasonable to use different reductions of (1.1), (4.5) to partial differential equations depending on what we are after in some particular case of argument. And these reductions should be as simple as possible. As the example we describe one of the local reductions of (4.5), which has been exploited to deduce a priori estimates for principal curvatures and normal velocity in [3, 4], Let $M_0 \in \Gamma_{t_0}$ be a point of interest. We relate to M_0 Cartesian frame $\{e_i, i = 1, ..., n+1, \mathbf{0} = M_0, e_{n+1} = \nu\}$ and let $Y = (y^1, ..., y^{n+1})$ be a vector in this system, $X = X_0 + Y$ be a position vector of Γ_t. There is a cylinder neighborhood Q_0 of M_0 such that $y = (y^1, ..., y^n)$ is independent on t and the set of hypersurfaces $\{\Gamma_t \cap Q_0\}$ appears as graphs of function $u \in C^{2,1} : y^{n+1} = u(y,t)$. Appointing in (4.5) $\theta = y$ we obtain locally equivalent to (4.5) partial differential equation for u:

$$H\Big(-\frac{1}{\sqrt{1+u_y^2}}u_t, \frac{1}{\sqrt{1+u_y^2}}u_{(yy)}\Big) = c, \tag{4.8}$$

where $u_{(yy)} = \tau^T u_{yy} \tau$ and τ may be chosen as

$$\tau = (\tau_\alpha^\beta), \quad \tau_\alpha^\beta = \delta_\alpha^\beta - \frac{u_\alpha u_\beta}{w(1+w)}, \quad w = \sqrt{1+u_y^2}.$$

Equation (4.8) with concave (convex) H also suits Theorem 2.4. Notice that matrix variable in (4.8) is not invariant under nonlinear change of parameters $\{y^1, ..., y^n\}$ but it does not matter.

References

1. G. Huisken, *Flow by mean curvature of convex hypersurfaces into spheres.* — J. Diff. Geom. **20** (1984), 237–268.
2. K. Tso, *Deforming a hypersurface by its Gauss–Kronecker curvature.* — Comm. Pure Appl. Math. **38** (1985), 867–882.
3. N. M. Ivochkina, *On evolution laws forcing convex surfaces to shrink to a point.* — Nonlinear Problems in Mathematical Physics and Related Topics I. In: in Honor of Professor O. A. Ladyzhenskaya Kluwer Academic/Plenum Publishers, New York, Boston, London, Moskow (2002), 167–186.
4. N. M. Ivochkina, Th. Nehring, F. Tomi, *Evolution of starshaped surfaces by non-homogeneous curvature functions.* — Algebra Anal. **12**, No. 1 (2000), 185–203. English transl.: St.-Petersburg Math. J. **12** (2001), 145–160.
5. B. Andrews, *Contraction of convex hypersurfaces in Euclidian space.* — Calc. Var. Partial Differ. Equ. **2** (1994), 151–171.
6. N. V. Krylov, *Nonlinear elliptic and parabolic second order equations.* — Nauka, Moskow (1985) (Russian); English transl., Reidel, Dordrecht (1987).
7. L. S. Evans, *Classical solutions of fully nonlinear, convex, second-order elliptic equations.* — Comm. Pure Appl. Math. **25** (1987), 333–363.
8. D. Gilbarg, N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order.* Springer, Berlin, Heidelberg, New York (1983).
9. N. M. Ivochkina, S. I. Prokof'eva, *A priori estimate for the second-order derivatives of solutions to the Dirichlet problem for the fully nonlinear parabolic equations.* — Problemy Mat. Anal. **16** (1997), 112–133 (Russian); English transl.: J. Math. Sci. **92** (1998), 4302–4315.

10. L. Caffarelli, L. Nirenberg, J. Spruck *Nonlinear second-order elliptic equations* IV: *Star-shaped compact Weingarten hypersurfaces.* — Current topics in partial differential equations Kinokunize Co., Tokyo (1986), 1–26.

St.-Petersburg State University
of Architecture and Civil Engineering
E-mail: ninaiv@NI1570.spb.edu

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