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ON THE (x,t) ASYMPTOTIC PROPERTIES OF SOLUTIONS OF THE NAVIER–STOKES EQUATIONS IN THE HALF-SPACE

ABSTRACT. We study the space-time asymptotic behavior of classical solutions of the initial boundary value problem for the Navier–Stokes system in the half-space. We construct a (local in time) solution corresponding to an initial data assumed only continuous and decreasing at infinity as $|x|^{-\mu}$, $\mu \in (\frac{1}{2}, n)$. We prove pointwise estimates in the space variable. Moreover, if $\mu \in [1, n)$ and the initial data is suitably small, the above solutions in global (in time) and we prove space-time pointwise estimates.

**Alla memoria di
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1. Introduction

This paper is concerned with the initial boundary value problem for the Navier–Stokes equations in the half-space $\mathbb{R}^n_+$, $n \geq 3$:

$$\begin{aligned} &u_t - \Delta u + u \cdot \nabla u = -\nabla \pi, \quad \nabla \cdot u = 0, \quad \text{on } \mathbb{R}^n_+ \times (0,T), \\ &u(x',t) = 0, \quad \text{on } \mathbb{R}^{n-1} \times (0,T), \quad \lim_{|x| \to \infty} u(x,t) = 0, \\ &u(x,0) = u_\circ(x), \quad \text{on } \mathbb{R}^n_+, \end{aligned} \tag{1}$$

where $u \cdot \nabla u_i = u_k \frac{\partial u_i}{\partial x_k}$, $i = 1, \dots, n$, for any $t > 0$ $u(x',t)$ is the trace of $u(x,t)$ on $\mathbb{R}^{n-1}$, $u_\circ(x)$ is the initial data. We are interested in establishing asymptotic properties in (x,t) of solutions of (1.1) corresponding to suitable initial data. More precisely, we prove that if $|u_\circ(x)| \leq \frac{U_0}{(1+|x|)^\mu}$,

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$\mu \in (\frac{1}{2}, n)$, then there exists a unique smooth solution of (1.1) corresponding to $u_\circ(x)$ such that, for any $\nu \in [0,\mu]$, $|u(x,t)| \le c\frac{U_0}{(1+|x|)^{\mu-\nu}}\frac{1}{(1+t)^{\frac{\nu}{2}}}$, for any $(x,t) \in \mathbb{R}^n_+ \times [0,T)$. The solution $u(x,t)$ exists locally in time $(T<\infty)$ if U_0 is arbitrary. Instead, if U_0 is *sufficiently* small and $\mu \in [1,n)$, then $u(x,t)$ exists for any $t>0$. Moreover, the solution $u(x,t)$ and its derivatives, as well as the associated pressure field, enjoy further properties as stated in Theorem 2.1 and Proposition 2.1.

It is known that Knightly and, more recently, Miyakawa have investigated such a question for the Cauchy problem of the nonstationary Navier–Stokes equations in [4, 12], respectively. The asymptotic properties in (x,t) we obtain for the solution are weaker than the ones obtained in those papers. Of course this is because we are dealing with a boundary value problem.

The results are proved using a representation formula of solutions of the Stokes problem. We appeal to the Green function for the Stokes problem in the half-space given by Solonnikov in [16, 18]. Thus, after introducing some properties of the Green function, we can establish the asymptotic behavior of the solution of the Navier-Stokes equations, starting from its representation as solution to a Stokes system.

The plan of the paper is the following. In Sec. 2, we introduce some notations and state our main results. In Sec. 3, we recall the Green matrix for the Stokes problem and prove some properties necessary for our purposes. In Sec. 4, we consider the initial boundary value problem for the Stokes system and prove pointwise estimates for the vector field solution of the system, for its derivative and for the associated pressure field. Successively, in Sec. 5–7, we obtain similar estimates for the solutions of the Stokes system with body force and zero initial data. Finally, in Sec. 8, on the basis of the results of the previous sections, we prove the main theorem, employing the method of successive approximations.

2. Notations and statement of the main result

For any $x \in \mathbb{R}^n_+$ we set $x = (x', x_n)$, with $x' \in \mathbb{R}^{n-1}$ and $x_n > 0$. If $y' \in \mathbb{R}^{n-1}$, by $(x-y')$ we mean the point $(x'-y', x_n)$. We set $x^* = (x_1, \cdots, x_{n-1}, -x_n)$. We shall denote by $H(x,t)$ the fundamental solution of the heat equation: $H(x,t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$. For any function $g(x)$ defined on $\mathbb{R}^n_+$, $g(x') = g(x',0)$ denotes the trace of $g(x)$ on $x_n = 0$. If $g(x,t)$ is a function defined on $\mathbb{R}^n_+ \times \mathbb{R}$, for any multi-index $h = (h_1, \ldots, h_n)$ and for any $k \in \mathbb{N} \cup \{0\}$ we denote by

$D^{h,k}_{x,t}g(x,t)$ the derivative $\frac{\partial^{|h|+k}g(x,t)}{\partial x_1^{h_1}\partial x_2^{h_2}\cdots\partial x_n^{h_n}\partial t^k}$, $|h|=h_1+\cdots+h_n$, and by $D^{h'}_{x'}g(x,t)$ the derivative $\frac{\partial^{|h'|}g(x,t)}{\partial x_1^{h_1}\partial x_2^{h_2}\cdots\partial x_{n-1}^{h_{n-1}}}$, $|h'|=h_1+\cdots+h_{n-1}$. Sometimes, when there is no danger of confusion, we replace $D^h_x g$ by $D^h g$. We denote by the symbol $\mathbb{R}^n_{+,x_n}$ the layer $\mathbb{R}^{n-1}\times(0,x_n)$. Let $m\in\mathbb{N}\cup\{0,\infty\}$. We denote by $C^m(\overline{\mathbb{R}^n_+})$ the Banach space of all functions g which are bounded and uniformly continuous on $\mathbb{R}^n_+$, together with all their partial derivatives $D^h g$ of orders $|h|\le m$. We denote by $C^{m,\alpha}(\overline{\mathbb{R}^n_+})$, $\alpha\in(0,1)$, the subspace of $C^m(\overline{\mathbb{R}^n_+})$ consisting of all functions g such that, for any $|h|=m$, $\sup\limits_{\substack{x,y\in\mathbb{R}^n_+\\ x\neq y}}\dfrac{|D^h g(x)-D^h g(y)|}{|x-y|^\alpha}<\infty$. Moreover, we denote by $C(a,b;X)$ the Banach space of all functions bounded and continuous on $(a,b)\subseteq\mathbb{R}$ with values in a Banach space X. By $C^{0,\alpha}(a,b;X)$ we denote the subspace of $C(a,b;X)$ consisting of all functions g such that $\sup\limits_{\substack{\bar t,\bar{\bar t}\in(a,b)\\ \bar t\neq\bar{\bar t}}}\dfrac{||g(\bar t)-g(\bar{\bar t})||_X}{|\bar t-\bar{\bar t}|^\alpha}<\infty$. Finally, we denote by $L^p(a,b;X)$, $p\ge 1$, the space of p-summable functions from $(a,b)\subseteq\mathbb{R}$ into a Banach space X. Let us introduce the following sets of functions: $\mathcal{M}(\mathbb{R}^n_+)=\big\{h(x)\in C(\overline{\mathbb{R}^n_+}),h(x')=0,\nabla\cdot h(x)=0\big\}$, and, for any $\mu\in(0,n)$, $\mathcal{M}^\mu_{|x|}(\mathbb{R}^n_+)=$

$$=\left\{h(x)\in\mathcal{M}(\mathbb{R}^n_+)\ \text{ with }\ |h(x)|\le\frac{H_0}{(1+|x|)^\mu}\ \text{ for some }\ H_0>0\right\}.$$

Theorem 2.1. *For any $u_\circ(x)\in\mathcal{M}^\mu_{|x|}(\mathbb{R}^n_+)$, $\mu\in(\frac12,n)$, there exists a unique solution (u,π) of system (1.1) such that, for some $T>0$, for any $\eta\in(0,T)$ and $\beta\in(0,1)$,*

$$u\in C(0,T;C(\overline{\mathbb{R}^n_+})),\quad D^2u,u_t,\nabla\pi\in C^{0,\beta}(\eta,T;C^{0,\beta}(\overline{\mathbb{R}^n_+}));$$

$\forall x_\circ\in\mathbb{R}^n_+$, $\lim\limits_{(x,t)\to(x_\circ,0)}u(x,t)=u_\circ(x_\circ)$. Moreover, there exist two suitable constants b and B, independent of $u_\circ$, such that:

for $T^{\frac12}<(4bBU_0)^{-1}$,

$$\begin{aligned}|u(x,t)|&\le\frac{1-3bt^{\frac12}BU_0}{1-4bt^{\frac12}BU_0}\frac{BU_0}{(1+|x|)^\mu},\\ t^{\frac12}|\nabla u(x,t)|&\le\frac{1-3bt^{\frac12}BU_0}{1-4bt^{\frac12}BU_0}\frac{BU_0}{(1+|x|)^\mu},\end{aligned}\tag{2.1}$$

for any $(x,t)\in\mathbb{R}^n_+\times[0,T)$; *for* $\alpha\in(0,1)$,

$$
\begin{aligned}
&|u(\overline{x},\overline{t})-u(\overline{\overline{x}},\overline{\overline{t}})|\le\frac{1-3bt_1^{\frac12}BU_0}{1-4bt_1^{\frac12}BU_0}BU_0t_0^{-\frac{\alpha}{2}}[|\overline{x}-\overline{\overline{x}}|^2+|\overline{t}-\overline{\overline{t}}|]^{\frac{\alpha}{2}},\\
&t_0^{\frac12}|\nabla u(\overline{x},\overline{t})-\nabla u(\overline{\overline{x}},\overline{\overline{t}})|\le\frac{1-3bt_1^{\frac12}BU_0}{1-4bt_1^{\frac12}BU_0}BU_0t_0^{-\frac{\alpha}{2}}[|\overline{x}-\overline{\overline{x}}|^2+|\overline{t}-\overline{\overline{t}}|]^{\frac{\alpha}{2}},
\end{aligned}
\tag{2.2}
$$

for any $\overline{x}$, $\overline{\overline{x}}\in\mathbb{R}^n_+$, *for any* $\overline{t}$, $\overline{\overline{t}}\in(\eta,T)$;

if $\mu\in[1,n)$ *and* $U_0<(4bB)^{-1}$, *then* $T=\infty$ *and, for any* $\nu\in[0,\mu]$,

$$
\begin{aligned}
|u(x,t)|&\le\frac{1-3bBU_0}{1-4bBU_0}\frac{BU_0}{(1+|x|)^{\mu-\nu}}\frac{1}{(1+t)^{\frac{\nu}{2}}},\\
t^{\frac12}|\nabla u(x,t)|&\le\frac{1-3bBU_0}{1-4bBU_0}\frac{BU_0}{(1+|x|)^{\mu-\nu}}\frac{1}{(1+t)^{\frac{\nu}{2}}},
\end{aligned}
\tag{2.3}
$$

for any $(x,t)\in\mathbb{R}^n_+\times[0,\infty)$; *for* $\alpha\in(0,1)$,

$$
\begin{aligned}
&|u(\overline{x},\overline{t})-u(\overline{\overline{x}},\overline{\overline{t}})|\le\frac{1-3bBU_0}{1-4bBU_0}BU_0t_0^{-\frac{\alpha}{2}}[|\overline{x}-\overline{\overline{x}}|^2+|\overline{t}-\overline{\overline{t}}|]^{\frac{\alpha}{2}},\\
&t_0^{\frac12}|\nabla u(\overline{x},\overline{t})-\nabla u(\overline{\overline{x}},\overline{\overline{t}})|\le\frac{1-3bBU_0}{1-4bBU_0}BU_0t_0^{-\frac{\alpha}{2}}[|\overline{x}-\overline{\overline{x}}|^2+|\overline{t}-\overline{\overline{t}}|]^{\frac{\alpha}{2}},
\end{aligned}
\tag{2.4}
$$

for any $\overline{x}$, $\overline{\overline{x}}\in\mathbb{R}^n_+$, *for any* $\overline{t},\overline{\overline{t}}\in(\eta,T)$;

$$
\begin{aligned}
&|u(\overline{x},t)-u(\overline{\overline{x}},t)|\le\frac{1-3bBU_0}{1-4bBU_0}BU_0t^{-\frac{\alpha}{2}}\frac{|\overline{x}-\overline{\overline{x}}|^{\,\alpha}}{R(|\overline{x}|,|\overline{\overline{x}}|)^{\mu-\nu}}\frac{1}{(1+t)^{\frac{\nu}{2}}},\\
&t^{\frac12}|\nabla u(\overline{x},t)-\nabla u(\overline{\overline{x}},t)|\le\frac{1-3bBU_0}{1-4bBU_0}BU_0t^{-\frac{\alpha}{2}}\frac{|\overline{x}-\overline{\overline{x}}|^{\,\alpha}}{R(|\overline{x}|,|\overline{\overline{x}}|)^{\mu-\nu}}\frac{1}{(1+t)^{\frac{\nu}{2}}},
\end{aligned}
\tag{2.5}
$$

for any $\overline{x}$, $\overline{\overline{x}}\in\mathbb{R}^n_+$, *for any* $t>0$, *with* $R(|\overline{x}|,|\overline{\overline{x}}|)=\min\{(1+|\overline{x}|),(1+|\overline{\overline{x}}|)\}$.

From the representation formula of the pressure field we will have the following

Proposition 2.1. *Let (u,π) be the solution given in Theorem 2.1. For the pressure field π we have, for $T^{\frac12}<(4bBU_0)^{-1}$, uniformly in $(x,t)\in\mathbb{R}^n_+\times(0,T)$:*

if $\mu\in(\frac12,n-1)$, then

$$|\pi(x,t)| \le \Big(\frac{1-3bt^{\frac12}BU_0}{1-4bt^{\frac12}BU_0}\Big)^2 BU_0(1+BU_0)\Big[1+\log\Big(\frac{3|x|}{t^{\frac12}}+1\Big)\Big]\frac{(1+t^{-\frac12})}{(1+|x|)^{\mu}};$$

if $\mu=n-1$, then

$$|\pi(x,t)|\le \Big(\frac{1-3bt^{\frac12}BU_0}{1-4bt^{\frac12}BU_0}\Big)^2 BU_0(1+BU_0) \times\Big[1+\log\Big(\frac{3|x|}{t^{\frac12}}+1\Big)+\log\Big(\frac{|x|}{2}+1\Big)\Big]\frac{(1+t^{-\frac12})}{(1+|x|)^{n-1}};$$

if $\mu\in(n-1,n)$, then

$$|\pi(x,t)|\le \Big(\frac{1-3bt^{\frac12}BU_0}{1-4bt^{\frac12}BU_0}\Big)^2 BU_0(1+BU_0) \times\Big[1+\log\Big(\frac{3|x|}{t^{\frac12}}+1\Big)\Big]\frac{(1+t^{-\frac12})}{(1+|x|)^{n-1}}.$$

Further we have, in the case $U_0<(4bB)^{-1}$, uniformly in $(x,t)\in\mathbb{R}^n_+\times(0,\infty)$:

if $\mu\in[1,n-1)$, then, for any $\nu\in[0,\mu)$ and for $\nu_2=\Big[1-\frac{\nu}{\mu}\Big]$,

$$t^{\frac12}|\pi(x,t)|\le\Big[\Big(\frac{1-3bBU_0}{1-4bBU_0}\Big)^2BU_0+\Big(1+\Big(\log\Big(\frac{3|x|}{t^{\frac12}}+1\Big)\Big)\Big)^{\nu_2}\Big] \times\frac{BU_0}{(1+|x|)^{\mu-\nu}}\frac{1}{(1+t)^{\frac{\nu}{2}}};$$

if $\mu=n-1$, then, for any $\nu\in[0,\mu)$ and for $\nu_2=\Big[1-\frac{\nu}{\mu}\Big]$,

$$t^{\frac12}|\pi(x,t)| \le\Big[\Big(\frac{1-3bBU_0}{1-4bBU_0}\Big)^2BU_0+\Big(1+\Big(\log\Big(\frac{3|x|}{t^{\frac12}}+1\Big)\Big)^{\nu_2}+\log\Big(\frac{|x|}{2}+1\Big)\Big)\Big] \times\frac{BU_0}{(1+|x|)^{n-1-\nu}}\frac{1}{(1+t)^{\frac{\nu}{2}}};$$

if $\mu \in (n-1, n)$, then, for any $\nu_1 \in [0, n-1)$ and for $\nu_3 = \left[1 - \frac{\nu_1}{\mu}\right]$,

$$t^{\frac{1}{2}}|\pi(x,t)| \le \left[\left(\frac{1-3bBU_0}{1-4bBU_0}\right)^2 BU_0 + \left(1 + \left(\log\left(\frac{3|x|}{t^{\frac{1}{2}}} + 1\right)\right)\right)^{\nu_3}\right] \times \frac{BU_0}{(1+|x|)^{n-1-\nu_1}} \frac{1}{(1+t)^{\frac{\nu_1}{2}}}.$$

The constants b and B are independent of $u_\circ$.

Remark 2.1. We would like to stress that, locally in time, an estimate of the kind (2.5) also holds for $\mu \in (\frac{1}{2}, 1)$. However, for the sake of brevity, we focus our efforts in this sense only on the case of global (in time) solutions.

Remark 2.2. We are concerned with estimates of the kind (2.1)–(2.3) in the hypothesis $\mu \in (\frac{1}{2}, n)$. It is quite natural to question about the case where $\mu \in [0, \frac{1}{2}]$. In this regard our technique needs of some changes. Therefore, for the sake of brevity we do not deal with this case. However, as far as the case where $\mu = 0$ is concerned, we recall that it has been studied for the initial boundary value problem in $\mathbb{R}^n_+ \times (0,T)$ in [18], without requiring convergence at infinity for the field $u(x,t)$. Instead the Cauchy problem related to the case $\mu = 0$ has been studied in [5] and subsequently, with a different approach, in [3]. Finally we recall that the problem of asymptotic behaviors has been also studied in the case of exterior domains in [2, 6, 13]. However, we think that such results have to be set in a context slightly different from ours.

3. The Green function and related estimates

Let us consider the Stokes system

$$\begin{aligned} &u_t - \Delta u = -\nabla\pi + f, \quad \nabla \cdot u = 0, \ \text{ on } \mathbb{R}^n_+ \times (0,T), \\ &u(x',t) = 0, \ \text{ on } \mathbb{R}^{n-1} \times (0,T), \quad \lim_{|x|\to\infty} u(x,t) = 0, \\ &u(x,0) = u_\circ(x), \ \text{ on } \mathbb{R}^n_+. \end{aligned} \tag{3.1}$$

A Green function for system (3.1), in a complete form, has been given either by Solonnikov or Ukai in [16–18] and [19], respectively. Both the authors have deduced semigroup properties, by different approaches and assuming nonhomogeneus initial data. Moreover they have proved existence theorems for the nonlinear system in a L^p-theory. In the sequel we employ the representation formula of solutions of system (3.1) given by Solonnikov, since it seems more congenial for our purposes.

Assuming a suitable behavior for large $|x|$ for the data f, it can be represented via the *Helmholtz decomposition* as

$$f = Pf + \nabla\Psi,$$

$$\Psi(x,t) = -\int\limits_{\mathbb{R}^n_+} \nabla_y[\mathcal{E}(x-y) + \mathcal{E}(x-y^*)] \cdot f(y,t)dy,$$

where $\mathcal{E}(z)$ is the fundamental solution of the Laplace equation, that is $\mathcal{E}(z) = -[(n-2)\omega_n]^{-1}|z|^{2-n}$ with ω_n surface area of the unit sphere in $\mathbb{R}^n$, and

$$\nabla \cdot Pf = 0 \quad \text{and} \quad Pf \cdot n_{|x_n=0} = 0.$$

In [16, 18] it has been shown that a solution of problem (3.1) is formally given by

$$\begin{aligned} u(x,t) &= \int\limits_{\mathbb{R}^n_+} \mathcal{G}(x,y,t)u_0(y)dy + \int\limits_0^t \int\limits_{\mathbb{R}^n_+} \mathcal{G}(x,y,t-\tau)Pf(y,\tau)dyd\tau, \\ \pi(x,t) &= \int\limits_{\mathbb{R}^n_+} \mathcal{Q}(x,y,t)\cdot u_0(y)dy + \int\limits_0^t \int\limits_{\mathbb{R}^n_+} \mathcal{Q}(x,y,t-\tau)\cdot Pf(y,\tau)dyd\tau + \Psi(x,t), \end{aligned} \tag{3.2}$$

where $\mathcal{G} = (G_{ij})_{i,j=1,\dots,n}$, $\mathcal{Q} = (Q_j)_{j=1,\dots,n}$ are the functions

$$\begin{aligned} G_{ij}(x,y,t) &= \delta_{ij}(H(x-y,t) - H(x-y^*,t)) \\ &\quad + 4\widehat{\delta}_{rj} D_{x_r} \int\limits_{\mathbb{R}^n_{+,x_m}} D_{x_i}\mathcal{E}(x-z)H(z-y^*,t)dz \\ &= \delta_{ij}(H(x-y,t) - H(x-y^*,t)) + G^*_{ij}(x,y,t),^1 \\ Q_j(x,y,t) &= 4\widehat{\delta}_{rj} D_{x_r}\Bigg[\int\limits_{\mathbb{R}^{n-1}} D_{x_n}\mathcal{E}(x-z')H(z'-y',y_n,t)dz' \\ &\quad + \int\limits_{\mathbb{R}^{n-1}} \mathcal{E}(x-z')D_{y_n}H(z'-y',y_n,t)dz'\Bigg] \end{aligned} \tag{3.3}$$

[1] Throughout the paper, pairs of identical indices mean summation from 1 to n.

where $\widehat{\delta}_{rj} = \delta_{rj}$ if $j \neq n$, $\widehat{\delta}_{rj} = 0$ if $j = n$, G^*_{ij} and Q_j satisfy the following inequalities (cf. [17, 18]):

$$|D^{h,l,k}_{x,y,t} G^*_{ij}(x,y,t)| \leq ct^{-\frac{l_n}{2}-k}(x_n^2+t)^{-\frac{h_n}{2}}(|x-y^*|^2+t)^{-\frac{n+|h'|+|l'|}{2}} e^{-\widehat{a}\frac{y_n^2}{t}}, \tag{3.4}$$

$$|D^{h,l,k}_{x,y,t} Q_j(x,y,t)| \leq ct^{-1-\frac{l_n}{2}-k}(|x-y^*|^2+t)^{-\frac{n-1+|h|+|l'|}{2}} e^{-\widehat{a}\frac{y_n^2}{t}},$$

for any $|h|, |l|, k \in \mathbb{N} \cup \{0\}$ with $h = (h', h_n)$, $l = (l', l_n)$, $\widehat{a} > 0$. Therefore, taking into account the following property of the heat kernel

$$\begin{aligned} |D^{h,l,k}_{x,y,s} H(x-y,t)| &\leq t^{-\frac{n}{2}-\frac{|h|}{2}-\frac{|l|}{2}-k} e^{-a\frac{|x-y|}{t}^2} \\ &\leq c(|x-y|^2+t)^{-\frac{n+|h|+|l|+2k}{2}}, \end{aligned} \tag{3.5}$$

since $H(x-y^*,t)$ also satisfies (3.4), we have

$$\begin{aligned} |D^{h,l,k}_{x,y,t} G_{ij}(x,y,t)| &\leq c(|x-y|^2+t)^{-\frac{n+|h|+|l|+2k}{2}} \\ &+ ct^{-\frac{l_n}{2}-k}(x_n^2+t)^{-\frac{h_n}{2}}(|x-y^*|^2+t)^{-\frac{n+|h'|+|l'|}{2}} e^{-\widehat{a}\frac{y_n^2}{t}}. \end{aligned} \tag{3.6}$$

Set

$$\begin{aligned} \mathfrak{M}^{\mu}_{|x|}(\mathbb{R}^n_+ \times (0,T)) = \Big\{ g(x,t) : \quad &\text{for some} \quad \eta > 0, \\ &g(x,t) \in C^{0,\beta}(\eta, T; C^{0,\beta}(\overline{\mathbb{R}^n_+})) \cap \mathcal{M}^{\mu}_{|x|}(\mathbb{R}^n_+) \Big\}. \end{aligned}$$

We state the following result.

Lemma 3.1. *Let* $(u_\circ(x), t^{\frac{1}{2}} Pf(x,t)) \in \mathcal{M}^{\mu}_{|x|}(\mathbb{R}^n_+) \times \mathfrak{M}^{\mu}_{|x|}(\mathbb{R}^n_+ \times (0,T))$, *for some* $\mu \in (\frac{1}{2}, n)$, *and suppose that* u *and* π *are given by the integrals*

$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}^n_+} \mathcal{G}(x,y,t) u_\circ(y) dy + \int_0^t \int_{\mathbb{R}^n_+} \mathcal{G}(x,y,t-\tau) Pf(y,\tau) dy d\tau, \\ \pi(x,t) &= \int_{\mathbb{R}^n_+} \mathcal{Q}(x,y,t) \cdot u_\circ(y) dy + \int_0^t \int_{\mathbb{R}^n_+} \mathcal{Q}(x,y,t-\tau) \cdot Pf(y,\tau) dy d\tau + \Psi(x,t). \end{aligned} \tag{3.7}$$

Then the pair (u, π), *with* $u \in C(0,T; C(\overline{\mathbb{R}^n_+}))$, $D^2 u, u_t, \nabla\pi \in C^{0,\beta}(\eta, T;$ $C^{0,\beta}(\overline{\mathbb{R}^n_+}))$, *is a solution of the Stokes system* (3.1).

Remark 3.1. We have left out the proof of the previous lemma. However the proof of the case $f = 0$ has been given by Solonnikov in [18], while the proof of the case $u_\circ = 0$ and $f \neq 0$ can be deduced from [17] and the following Secs. 6 and 8, in the particular case of f as in Sec. 5.

In [7, 18] it has been shown that if there exists a tensor F such that

$$f_j = \nabla \cdot F_j \quad \text{and} \quad F_{nj|x_n=0} = 0, \quad j = 1, \dots, n,$$

where by F_j we denote the j^{th}-column of F, then Pf can be written as

$$\mathfrak{f}_j = (Pf)_j = \nabla \cdot \mathcal{F}_j,$$

with

$$\begin{aligned}
&\mathcal{F}_{nj}(x,t) = F_{nj}(x,t) - \delta_{nj} F_{nn}(x,t), \quad j = 1, \dots, n,\\
&\mathcal{F}_{ms}(x,t) = F_{ms}(x,t) - \delta_{ms} F_{nn}(x,t)\\
&+ D_{x_s}\Big[\int\limits_{\mathbb{R}^n_+} \Big(D_{y_q} \mathcal{N}(x,y) F_{mq}(y,t)\\
&+ D_{y_n} \mathcal{N}(x,y) F_{nm}(y,t) - D_{y_m} \mathcal{N}(x,y) F_{nn}(y,t)\Big) dy \Big],\\
&\mathcal{F}_{mn}(x,t) = -\widehat{\delta}_{rs} D_{x_r} \int\limits_{\mathbb{R}^n_+} D_{x_n} \mathcal{N}(x,y) F_{ms}(y,t) dy\\
&+ D_{x_m} \int\limits_{\mathbb{R}^n_+} D_{x_n} \mathcal{N}(x,y) F_{nn}(y,t) dy\\
&- F_{nm}(x,t) - \widehat{\delta}_{rs} D_{x_r} \int\limits_{\mathbb{R}^n_+} D_{y_s} \mathcal{N}^-(x,y)(F_{mn}(y,t) + F_{nm}(y,t)) dy,
\end{aligned}$$

where

$$\begin{aligned}
&\mathcal{N}(x,y) = \mathcal{N}^+(x,y) = \mathcal{E}(x-y) + \mathcal{E}(x-y^*),\\
&\mathcal{N}^-(x,y) = \mathcal{E}(x-y) - \mathcal{E}(x-y^*).
\end{aligned}$$

Thus,

$$\mathfrak{f}_j = D_{x_i} F_{ij} - D_{x_j} F_{nn} - \widehat{\delta}_{rm} \delta_{nj} D_{x_r}(F_{mn} + F_{nm}) + h_j, \tag{3.8}$$

where h_j is a linear combination of terms of the kind

$$D_{x_m x_s} \int\limits_{\mathbb{R}^n_+} D_{y_q} \mathcal{N}^{\pm}(x,y) F_{pr}(y,t) dy, \quad m, s \neq n.$$

Therefore, if we also assume $F_{ij|x_n=0} = 0$, after an integration by parts, we can write the last term on the right-hand side of $(3.2)_1$ in the form

$$\int_0^t \int_{\mathbb{R}^n_+} G_{ij}(x,y,t-\tau)\mathfrak{f}_j(y,\tau)dyd\tau$$

$$= -\int_0^t \int_{\mathbb{R}^n_+} D_{y_p} G_{ij}(x,y,t-\tau)F_{pj}(y,\tau)dyd\tau$$

$$+ \int_0^t \int_{\mathbb{R}^n_+} D_{y_j} G_{ij}(x,y,t-\tau)F_{nn}(y,\tau)dyd\tau$$

$$+ \widehat{\delta}_{rm} \int_0^t \int_{\mathbb{R}^n_+} D_{y_r} G_{in}(x,y,t-\tau)(F_{mn}+F_{nm})(y,\tau)dyd\tau$$

$$+ \int_0^t \int_{\mathbb{R}^n_+} G_{ij}(x,y,t-\tau)h_j(y,\tau)dyd\tau.$$

In order to obtain regularity properties and asymptotic estimates for this term, under suitable assumptions on F, we need to evaluate the following kernel:

$$K_{ijq}(x,z,s) = \int_{\mathbb{R}^n_+} G_{ij}(x,y,s)D_{y_q}\mathcal{N}^{\pm}(y,z)dy.$$

In [18] it has been already shown that, for any multi-index $h = (h', 0) \in \mathbb{N} \cup \{0\}$, the kernel K_{ijq} satisfies the estimate

$$|D^{h'}_{x'} K_{ijq}(x,z,s)| \leq c[|x-z|^2 + s]^{\frac{-n+1-|h'|}{2}}. \tag{3.9}$$

In view of our purposes we need further estimates on K_{ijq}.

Proposition 3.1. *For any multi-index $h = (h', h_n)$, for any $k \in \mathbb{N} \cup \{0\}$*

$$|D^{h,l',k}_{x,z',s} K_{ijq}(x,z,s)| \leq \frac{c}{[|x-z|^2+s]^{\frac{n-1+|h|+|l'|+2k}{2}}}$$

$$+ cs^{-k} \frac{(x_n^2+s)^{-\frac{h_n}{2}}}{[|x-z|^2+s]^{\frac{n-1+|h'|+|l'|}{2}}} + c\frac{((x_n-z_n)^2+s)^{-\frac{h_n+2k}{2}}}{[|x-z|^2+s]^{\frac{n-1+|h'|+|l'|}{2}}},$$

uniformly in x and $z \in \mathbb{R}^n_+$, $s > 0$, provided that $|l'| = 0, 1$.

Proof. First we observe that, by the identity

$$D_{x'} G_{ij}(x, y, s) = -D_{y'} G_{ij}(x, y, s),$$

the regularity of G_{ij} and elementary properties of the fundamental solution of the Laplace equation, there holds

$$D_{x'} K_{ijq}(x, z, s) = -D_{z'} K_{ijq}(x, z, s).$$

This implies that, in our hypotheses, it suffices to estimate the derivatives with respect to x and s. Taking into account the definition of K_{ijq}, we can restrict ourselves to estimate the term with the singular kernel $\mathcal{E}(x-y)$, the same argument works for the other term. Then, setting

$$\begin{aligned} G_{ij}(x, y, s) &= \delta_{ij} H(x-y, s) - \delta_{ij} H(x-y^*, s) + G^*_{ij}(x, y, s) \\ &= \delta_{ij} H(x-y, s) + \widetilde{G}_{ij}(x, y, s), \end{aligned}$$

in order to study $D^{h,k}_{x,s} K_{ijq}(x, z, s)$ it is sufficient to study the following sum

$$\begin{aligned} \delta_{ij}\int\limits_{\mathbb{R}^n_+} D^{h,k}_{x,s} H(x-y, s) D_{y_q}\mathcal{E}(y-z)dy + \int\limits_{\mathbb{R}^n_+} D^{h,k}_{x,s}\widetilde{G}_{ij}(x, y, s) D_{y_q}\mathcal{E}(y-z)dy \\ = I_1(x, z, s) + I_2(x, z, s). \end{aligned}$$

Moreover, since $D_s H(x-y, s) = \Delta_x H(x-y, s)$ and since the estimate for $|h| = 0$ is already known (see (3.9)), we only study I_1 in the case $k = 0$ and $|h| \geq 1$. Set $\rho = \frac{|x-z|}{2} + \frac{\sqrt{s}}{2}$, $B_\rho = \{w \in \mathbb{R}^n_+ : |w| \leq \rho\}$ and $B_{2\rho} = \{w \in \mathbb{R}^n_+ : |w| \leq 2\rho\}$. We write

$$\begin{aligned} &\int\limits_{\mathbb{R}^n_+} D^h_x H(x-y, s) D_{y_q}\mathcal{E}(y-z)dy = \int\limits_{\mathbb{R}^n_+} D^h_x H(x-w-z, s) D_{w_q}\mathcal{E}(w)dw \\ &= \int\limits_{B_\rho} D^h_x H(x-w-z, s) D_{w_q}\mathcal{E}(w)dw + \int\limits_{B_{2\rho}\setminus B_\rho} D^h_x H(x-w-z, s) D_{w_q}\mathcal{E}(w)dw \\ &\quad + \int\limits_{\mathbb{R}^n_+\setminus B_{2\rho}} D^h_x H(x-w-z, s) D_{w_q}\mathcal{E}(w)dw = \sum_{i=1}^{3} J_i(x, z, s). \end{aligned}$$

Taking into account estimate (3.5), we have

$$|J_1| \leq c \int\limits_{B_\rho} \frac{|w|^{-n+1}}{[|x-w-z|^2+s]^{\frac{n+|h|}{2}}} dw$$

$$\leq \frac{c}{[|x-z|^2+s]^{\frac{n+|h|}{2}}} \int\limits_{B_\rho} \frac{dw}{|w|^{n-1}} \leq \frac{c}{[|x-z|^2+s]^{\frac{n-1+|h|}{2}}};$$

$$|J_3| \leq c \int\limits_{\mathbb{R}^n_+ \setminus B_{2\rho}} \frac{|w|^{-n+1}}{[|x-w-z|^2+s]^{\frac{n+|h|}{2}}} dw$$

$$\leq c \int\limits_{\mathbb{R}^n_+ \setminus B_{2\rho}} \frac{dw}{|w|^{2n-1+|h|}} \leq \frac{c}{[|x-z|^2+s]^{\frac{n-1+|h|}{2}}}.$$

Using the identity $D_x H(x-w-z,s) = -D_w H(x-w-z,s)$, we integrate J_2 $|h'|$-times by parts with respect to w' and, subsequently, h_n-times with respect to w_n. From the integration by parts we get $2|h|$ surface integrals given by

$$\int\limits_{\partial B_{2\rho}} D_x^{\widehat{h}} H(x-w-z,s) D_w^{\tilde{h}} D_{w_q} \mathcal{E}(w) \frac{w_d}{|w|} d\sigma_w$$

$$- \int\limits_{\partial B_\rho} D_x^{\widehat{h}} H(x-w-z,s) D_w^{\tilde{h}} D_{w_q} \mathcal{E}(w) \frac{w_d}{|w|} d\sigma_w, \tag{3.10}$$

with $d = 1, \ldots, n$ and $|\tilde{h}| + |\widehat{h}| = |h| - 1$, plus the integral

$$(1-\delta_{qn}) \int\limits_{\mathbb{R}^{n-1} \cap (B_{2\rho} \setminus B_\rho)} D_{x_n}^{h_n - 1} H(x-w'-z,s) D_{w'}^{h'} D_{w_q} \mathcal{E}(w') dw', \tag{3.11}$$

plus the volume integral

$$\int\limits_{B_{2\rho} \setminus B_\rho} H(x-w-z,s) D_w^h D_{w_q} \mathcal{E}(w) dw.$$

Using estimate (3.5), the terms (3.10) are easily seen to be less or equal then $(|x-z|^2+s)^{-\frac{n-1+|h|}{2}}$. Integral (3.11) can be estimated as follows

$$\Bigg| \int\limits_{\mathbb{R}^{n-1} \cap (B_{2\rho} \setminus B_\rho)} D_{x_n}^{h_n - 1} H(x-w'-z,s) D_{w'}^{h'} D_{w_q} \mathcal{E}(w') dw' \Bigg|$$

$$\leq c\frac{s^{-\frac{h_n}{2}}e^{-a\frac{(x_n-z_n)^2}{4s}}}{[|x-z|^2+s]^{\frac{n-1+|h'|}{2}}}\int\limits_{\mathbb{R}^{n-1}\cap(B_{2\rho}\setminus B_\rho)}\frac{e^{-a\frac{|x'-z'-w'|^2}{4s}}}{s^{\frac{n-1}{2}}}dw'$$

$$\leq ce^{-a\frac{(x_n-z_n)^2}{4s}}\frac{((x_n-z_n)^2+s)^{-\frac{h_n}{2}}}{[|x-z|^2+s]^{\frac{n-1+|h'|}{2}}}.$$

Finally, from well known property of the heat kernel,

$$\Big|\int\limits_{B_{2\rho}\setminus B_\rho}H(x-w-z,s)D_w^h D_{w_q}\mathcal{E}(w)dw\Big|$$

$$\leq\frac{c}{[|x-z|^2+s]}^{n-1+|h|}\int\limits_{\mathbb{R}^n_+}|H(x-w-z,s)|dw\leq\frac{c}{[|x-z|^2+s]}^{n-1+|h|}.$$

Therefore

$$|I_1(x,z,s)|\leq\frac{c}{[|x-z|^2+s]^{\frac{n-1+|h|+2k}{2}}}+c\frac{((x_n-z_n)^2+s)^{-\frac{h_n+2k}{2}}}{[|x-z|^2+s]^{\frac{n-1+|h'|}{2}}}.$$

Let us estimate I_2. Set $B_\rho(z)=\{y\in\mathbb{R}^n_+ : |y-z|\leq\rho\}$ and split the integral over $\mathbb{R}^n_+$ in the sum of an integral over $B_\rho(z)$ and an integral over $\mathbb{R}^n_+\setminus B_\rho(z)$. Taking into account that $G^*(x,y,s)$ and also $H(x-y^*,s)$ satisfy estimate (3.4), we have

$$\Big|\int\limits_{B_\rho(z)}D^{h,k}_{x,s}\widetilde{G}_{ij}(x,y,s)D_{y_q}\mathcal{E}(y-z)dy\Big|$$

$$\leq cs^{-k}\int\limits_{B_\rho(z)}\frac{e^{-\widehat{a}\frac{y_n^2}{s}}(x_n^2+s)^{-\frac{h_n}{2}}}{[|x-y^*|^2+s]^{\frac{n+|h'|}{2}}}\frac{dy}{|y-z|^{n-1}}$$

$$\leq cs^{-k}\frac{(x_n^2+s)^{-\frac{h_n}{2}}}{[|x-z|^2+s]^{\frac{n+|h'|}{2}}}\int\limits_{B_\rho(z)}\frac{dy}{|y-z|^{n-1}}\leq cs^{-k}\frac{(x_n^2+s)^{-\frac{h_n}{2}}}{[|x-z|^2+s]^{\frac{n-1+|h'|}{2}}}.$$

Let us integrate the integral over $\mathbb{R}^n_+\setminus B_\rho(z)$ $|h'|$-times by parts. We get $|h'|$ surface integrals extended to $\partial(\mathbb{R}^n_+\setminus B_\rho(z))$. Since the direction cosines on $\partial\mathbb{R}^n_+$ are null because $h'=(h_1,\dots,h_{n-1},0)$, there remain $|h'|$

surface integrals extended to $\partial B_\rho(z)$. Such integrals, at step $|\tilde{h}'|+1$, with $|\tilde{h}'|\in\{0,\dots,|h'|-1\}$, are of the kind

$$\int\limits_{\partial B_\rho(z)} D^{h_n}_{x_n} D^{\widehat{h}',k}_{x',s}\widetilde{G}_{ij}(x,y,s) D^{\tilde{h}'}_{y'} D_{y_q}\mathcal{E}(y-z)\frac{y_d}{|y|}d\sigma_y,\quad d=1,\dots,n-1,$$

where $|\tilde{h}'|+|\widehat{h}'|=|h'|-1$. Using estimate (3.4), these terms are easily seen to be less or equal then $cs^{-k}\dfrac{(x_n^2+s)^{-\frac{h_n}{2}}}{[|x-z|^2+s]^{\frac{n-1+|h'|}{2}}}$. The last volume integral related to the integration by parts is

$$\int\limits_{\mathbb{R}^n_+\setminus B_\rho(z)} D^{h_n,k}_{x_n,s}\widetilde{G}_{ij}(x,y,s) D^{h'}_{y'} D_{y_q}\mathcal{E}(y-z)dy.$$

Thus, using the following estimate

$$\int\limits_{\mathbb{R}^n_+}\frac{e^{-\widehat{a}\frac{y_n^2}{s}}}{[|x-y^*|^2+s]^{\frac{n}{2}}}dy\le c\int\limits_0^\infty\frac{e^{-\widehat{a}\frac{y_n^2}{s}}}{(y_n^2+s)^{\frac12}}dy_n\le c,$$

we get

$$\Bigg|\int\limits_{\mathbb{R}^n_+\setminus B_\rho(z)} D^{h_n,k}_{x_n,s}\widetilde{G}_{ij}(x,y,s) D^{h'}_{y'} D_{y_q}\mathcal{E}(y-z)dy\Bigg|$$

$$\le cs^{-k}\int\limits_{\mathbb{R}^n_+\setminus B_\rho(z)}\frac{e^{-\widehat{a}\frac{y_n^2}{s}}(x_n^2+s)^{-\frac{h_n}{2}}}{[|x-y^*|^2+s]^{\frac{n}{2}}}\frac{dy}{|y-z|^{n-1+|h'|}}$$

$$\le cs^{-k}\frac{(x_n^2+s)^{-\frac{h_n}{2}}}{[|x-z|^2+s]^{\frac{n-1+|h'|}{2}}}\int\limits_{\mathbb{R}^n_+}\frac{e^{-\widehat{a}\frac{y_n^2}{s}}}{[|x-y^*|^2+s]^{\frac{n}{2}}}dy$$

$$\le cs^{-k}\frac{(x_n^2+s)^{-\frac{h_n}{2}}}{[|x-z|^2+s]^{\frac{n-1+|h'|}{2}}}.$$

Therefore

$$|I_2(x,z,s)|\le cs^{-k}\frac{(x_n^2+s)^{-\frac{h_n}{2}}}{[|x-z|^2+s]^{\frac{n-1+|h'|}{2}}},$$

uniformly in x and $z\in\mathbb{R}^n_+$, $s>0$. Combining the results obtained for I_1 and I_2, we arrive at the desired estimate.

4. The nonstationary Stokes system with $f = 0$ and initial data in $\mathcal{M}^{\mu}_{|x|}(\mathbb{R}^n_+)$

Let us consider the Stokes system

$$\begin{aligned}
&u^0_t - \Delta u^0 = -\nabla \pi^0, \quad \nabla \cdot u^0 = 0, \quad \text{on } \mathbb{R}^n_+ \times (0,T),\\
&u^0(x',t) = 0, \quad \text{on } \mathbb{R}^{n-1} \times (0,T), \quad \lim_{|x|\to\infty} u^0(x,t) = 0, \qquad (4.1)\\
&u^0(x,0) = u_\circ(x), \quad \text{on } \mathbb{R}^n_+.
\end{aligned}$$

The existence, uniqueness and regularity of solutions of problem (4.1) have been recently proved in [10, 18]. By virtue of such results, we can state the following

Theorem 4.1. *For any $u_\circ(x) \in \mathcal{M}^{\mu}_{|x|}(\mathbb{R}^n_+)$, $\mu \in (0,n)$, there exists a unique solution (u^0, π^0) of the Stokes system (4.1) such that, for any $T > 0$ and $\eta \in (0,T)$, for $\alpha \in (0,1)$,*

$$u^0 \in C(0,T; C(\overline{\mathbb{R}^n_+})), \quad D^2u^0, u^0_t, \nabla\pi^0 \in C^{0,\alpha}(\eta, T; C^{0,\alpha}(\overline{\mathbb{R}^n_+})),$$

$\forall x_\circ \in \mathbb{R}^n_+$, $\lim\limits_{(x,t)\to(x_\circ,0)} u^0(x,t) = u_\circ(x_\circ)$. Moreover,

$$|u^0(x,t)| + t^{\frac12}|\nabla u^0(x,t)| + t(|D^2u^0(x,t)| + |u^0_t(x,t)| + |\nabla\pi^0(x,t)|) \le cU_0,$$
$$\forall (x,t) \in \mathbb{R}^n_+ \times [0,T),$$

$$|D^2u^0(\overline{x},\overline{t}) - D^2u^0(\overline{\overline{x}},\overline{\overline{t}})| + |u^0_t(\overline{x},\overline{t}) - u^0_t(\overline{\overline{x}},\overline{\overline{t}})| + |\nabla\pi^0(\overline{x},\overline{t}) - \nabla\pi^0(\overline{\overline{x}},\overline{\overline{t}})|$$
$$\le cU_0 t_{\circ}^{-1-\frac{\alpha}{2}}(|\overline{x} - \overline{\overline{x}}|^2 + |\overline{t} - \overline{\overline{t}}|)^{\frac{\alpha}{2}}, \quad \forall \overline{x}, \overline{\overline{x}} \in \mathbb{R}^n_+, \quad \forall \overline{t}, \overline{\overline{t}} \in (0,T),$$

with $t_{\circ} = \min\{\overline{t}, \overline{\overline{t}}\}$. The constant c is independent of $u_\circ$.

Before giving the following Theorem, we stress that, from estimate (3.4), applying the Lagrange theorem and using an interpolation argument, the following estimate holds: for any $\alpha \in (0,1)$, for any multi-index $h = (h', h_n)$

$$|D^h_x[G^*(\overline{x}, y, s) - G^*(\overline{\overline{x}}, y, s)]|$$
$$\le \frac{c}{s^{\frac{h_n}{2}+\frac{\alpha}{2}}} \frac{|\overline{x} - \overline{\overline{x}}|^{\alpha} e^{-\widehat{a}\frac{y_n^2}{s}}}{[\min\{|\overline{x} - y^*|^2, |\overline{\overline{x}} - y^*|^2\} + s]^{\frac{n+|h'|}{2}}}, \qquad (4.2)$$

uniformly in $\overline{x}$, $\overline{\overline{x}}$, $y \in \mathbb{R}^n_+$ and $s > 0$.

We are in position to state the following

Theorem 4.2. *For the pair (u^0,π^0) of solutions given in Theorem 4.1 the following estimates hold: for $\nu\in[0,\mu]$,*

$$\begin{aligned}|u^0(x,t)|&\le \frac{BU_0}{(1+|x|)^{\mu-\nu}}\frac{1}{(1+t)^{\frac{\nu}{2}}},\\ t^{\frac12}|\nabla u^0(x,t)|&\le \frac{BU_0}{(1+|x|)^{\mu-\nu}}\frac{1}{(1+t)^{\frac{\nu}{2}}},\end{aligned}\tag{4.3}$$

for any $(x,t)\in\mathbb{R}^n_+\times[0,T)$;

$$|u^0(\overline{x},t)-u^0(\overline{\overline{x}},t)|\le cBU_0t^{-\frac{\alpha}{2}}\frac{|\overline{x}-\overline{\overline{x}}|^\alpha}{R(|\overline{x}|,|\overline{\overline{x}}|)^{\mu-\nu}}\frac{1}{(1+t)^{\frac{\nu}{2}}},\tag{4.4}$$

$$|\nabla u^0(\overline{x},t)-\nabla u^0(\overline{\overline{x}},t)|\le cBU_0t^{-\frac12-\frac{\alpha}{2}}\frac{|\overline{x}-\overline{\overline{x}}|^\alpha}{R(|\overline{x}|,|\overline{\overline{x}}|)^{\mu-\nu}}\frac{1}{(1+t)^{\frac{\nu}{2}}},\tag{4.5}$$

for any $\overline{x},\overline{\overline{x}}\in\mathbb{R}^n_+$ and $t\in(0,T)$, with $R(|\overline{x}|,|\overline{\overline{x}}|)=\min\{(1+|\overline{x}|),(1+|\overline{\overline{x}}|)\}$;

$$\begin{aligned}&|u^0(\overline{x},\overline{t})-u^0(\overline{\overline{x}},\overline{\overline{t}})|\le cBU_0t_{_0}^{-\frac{\alpha}{2}}[|\overline{x}-\overline{\overline{x}}|^2+|\overline{t}-\overline{\overline{t}}|]^{\frac{\alpha}{2}},\\ &t_{_0}^{\frac12}|\nabla u^0(\overline{x},\overline{t})-\nabla u^0(\overline{\overline{x}},\overline{\overline{t}})|\le cBU_0t_{_0}^{-\frac{\alpha}{2}}[|\overline{x}-\overline{\overline{x}}|^2+|\overline{t}-\overline{\overline{t}}|]^{\frac{\alpha}{2}},\end{aligned}\tag{4.6}$$

for any $\overline{x}$, $\overline{\overline{x}}\in\mathbb{R}^n_+$, for any $\overline{t},\overline{\overline{t}}\in(\eta,T)$; for $\nu\in[0,\mu)$ and $\nu_1\in[0,n-1)$,

$$t^{\frac12}|\pi^0(x,t)|\le BU_0\begin{cases}\dfrac{t^{-\frac{\nu}{2}}}{(1+|x|)^{\mu-\nu}}\left[1+\left(\log\left(\dfrac{3|x|}{t^{\frac12}}+1\right)\right)^{\nu_2}\right], & \\ \text{if } \mu\in(0,n-1), & \\[2ex] \dfrac{t^{-\frac{\nu}{2}}}{(1+|x|)^{n-1-\nu}}\left[1+\log\left(1+\dfrac{|x|}{2}\right)+\left(\log\left(\dfrac{3|x|}{t^{\frac12}}+1\right)\right)^{\nu_2}\right], & \\ \text{if } \mu=n-1, & \\[2ex] \dfrac{t^{-\frac{\nu_1}{2}}}{(1+|x|)^{n-1-\nu_1}}\left[1+\left(\log\left(\dfrac{3|x|}{t^{\frac12}}+1\right)\right)^{\nu_3}\right], & \\ \text{if } \mu\in(n-1,n),\end{cases}\tag{4.7}$$

for any $(x,t)\in\mathbb{R}^n_+\times[0,T)$, with $\nu_2=\left[1-\frac{\nu}{\mu}\right]$ and $\nu_3=\left[1-\frac{\nu_1}{\mu}\right]$. The constant B is independent of $u_\circ$.

Proof. By formula (3.2), a solution of system (4.1) is given by

$$u^0(x,t)=\int\limits_{\mathbb{R}^n_+}\mathcal{G}(x,y,t)u_\circ(y)dy,\quad \pi^0(x,t)=\int\limits_{\mathbb{R}^n_+}\mathcal{Q}(x,y,t)\cdot u_\circ(y)dy,$$

where $\mathcal{G}$ and $\mathcal{Q}$ are defined by (3.3). Estimates of the kind (4.3)–(4.5) are already known for solutions of the Cauchy problem of the heat equation in the case where $\mu\geq 1$. We refer to [4] for an exhaustive treatment of this question. However, it is easy to obtain the same estimates also for $\mu\in(0,1)$. Therefore, in order to obtain $(4.3)_1$, we only evaluate the term

$$\int\limits_{\mathbb{R}^n_+}G^*(x,y,t)u_\circ(y)dy.$$

By the hypothesis on the data and estimate $(3.4)_2$, we have

$$\Bigg|\int\limits_{\mathbb{R}^n_+}G^*(x,y,t)u_\circ(y)dy\Bigg|\leq cU_0\int\limits_{\mathbb{R}^n_+}e^{-\widehat{a}\frac{y_n^2}{t}}\frac{(1+|y|)^{-\mu}}{[|x-y^*|^2+t]^{\frac{n}{2}}}dy.$$

Then, we split $\mathbb{R}^n_+$ into three subdomains relative to the point x:

$$\begin{aligned}\Lambda_1&=\left\{y\in\mathbb{R}^n_+:0\leq|y|\leq\frac{|x|}{2}\right\},\\ \Lambda_2&=\left\{y\in\mathbb{R}^n_+:\frac{|x|}{2}\leq|y|\leq 2|x|\right\},\\ \Lambda_3&=\left\{y\in\mathbb{R}^n_+:|y|\geq 2|x|\right\}.\end{aligned}\tag{4.8}$$

We have, for any $\nu\in[0,\mu]$,

$$\int\limits_{\Lambda_1}e^{-\widehat{a}\frac{y_n^2}{t}}\frac{(1+|y|)^{-\mu}}{[|x-y^*|^2+t]^{\frac{n}{2}}}dy\leq\frac{c}{[|x|+t^{\frac{1}{2}}]^n}\int\limits_{\Lambda_1}(1+|y|)^{-\mu}dy\leq\frac{c}{|x|^{\mu-\nu}}t^{-\frac{\nu}{2}},$$

$$\int\limits_{\Lambda_3}e^{-\widehat{a}\frac{y_n^2}{t}}\frac{(1+|y|)^{-\mu}}{[|x-y^*|^2+t]^{\frac{n}{2}}}dy\leq c\int\limits_{\Lambda_3}\frac{e^{-\widehat{a}\frac{y_n^2}{t}}}{[|y|+t^{\frac{1}{2}}]^{n+\mu}}dy\leq\frac{c}{|x|^{\mu-\nu}}t^{-\frac{\nu}{2}}.$$

Moreover for $\nu=0$

$$\int\limits_{\Lambda_2}e^{-\widehat{a}\frac{y_n^2}{t}}\frac{(1+|y|)^{-\mu}}{[|x-y^*|^2+t]^{\frac{n}{2}}}dy$$

$$\leq \frac{c}{(1+|x|)^{\mu}} \int\limits_{\mathbb{R}^n_+} \frac{e^{-\widehat{a}\frac{y_n^2}{t}}}{[|x-y^*|^2+t]^{\frac{n}{2}}} dy \leq \frac{c}{(1+|x|)^{\mu}},$$

and for any $\nu \in (0, \mu]$

$$\int\limits_{\Lambda_2} e^{-\widehat{a}\frac{y_n^2}{t}} \frac{(1+|y|)^{-\mu}}{[|x-y^*|^2+t]^{\frac{n}{2}}} dy$$

$$\leq \frac{c}{(1+|x|)^{\mu}} t^{-\frac{\nu}{2}} \int\limits_{\Lambda_2} \frac{e^{-\widehat{a}\frac{y_n^2}{t}}}{[|x-y^*|^2+t]^{\frac{n}{2}-\frac{\nu}{2}}} dy \leq \frac{c}{(1+|x|)^{\mu-\nu}} t^{-\frac{\nu}{2}}.$$

Since, in virtue of Theorem 4.1, u^0 is uniformly bounded, for any $\nu \in [0, \mu]$ these inequalities give

$$\Big| \int\limits_{\mathbb{R}^n_+} G^*(x,y,t) u_\circ(y) dy \Big|$$

$$\leq c \frac{U_0}{(1+|x|)^{\mu-\nu}} \frac{1}{(1+t)^{\frac{\nu}{2}}}, \quad \forall (x,t) \in \mathbb{R}^n_+ \times (0,T).$$

Analogous arguments enable us to obtain estimate $(4.3)_2$ on the first derivatives. To get the Hölder property (4.4), we use estimate (4.2). We have

$$\int\limits_{\mathbb{R}^n_+} |G^*(\overline{x},y,t) - G^*(\overline{\overline{x}},y,t)||u_\circ(y)| dy$$

$$\leq c U_0 t^{-\frac{\alpha}{2}} |\overline{x}-\overline{\overline{x}}|^{\alpha} \int\limits_{\mathbb{R}^n_+} e^{-\widehat{a}\frac{y_n^2}{t}} \frac{(1+|y|)^{-\mu}}{[\min\{|\overline{x}-y^*|^2, |\overline{\overline{x}}-y^*|^2\}+t]^{\frac{n}{2}}} dy$$

$$\leq c U_0 t^{-\frac{\alpha}{2}} |\overline{x}-\overline{\overline{x}}|^{\alpha} \Bigg[\int\limits_{\mathbb{R}^n_+} e^{-\widehat{a}\frac{y_n^2}{t}} \frac{(1+|y|)^{-\mu}}{[|\overline{x}-y^*|^2+t]^{\frac{n}{2}}} dy$$

$$+c \int\limits_{\mathbb{R}^n_+} e^{-\widehat{a}\frac{y_n^2}{t}} \frac{(1+|y|)^{-\mu}}{[|\overline{\overline{x}}-y^*|^2+t]^{\frac{n}{2}}} dy \Bigg].$$

These two terms on the right-hand side have been estimated above and give

$$\int_{\mathbb{R}^n_+} |G^*(\overline{x},y,t)-G^*(\overline{\overline{x}},y,t)||u_\circ(y)|dy$$

$$\leq cU_0 t^{-\frac{\alpha}{2}}\frac{|\overline{x}-\overline{\overline{x}}|^\alpha}{(1+|\overline{x}|)^{\mu-\nu}}\frac{1}{(1+t)^{\frac{\nu}{2}}}+cU_0 t^{-\frac{\alpha}{2}}\frac{|\overline{x}-\overline{\overline{x}}|^\alpha}{(1+|\overline{\overline{x}}|)^{\mu-\nu}}\frac{1}{(1+t)^{\frac{\nu}{2}}},$$

which implies (4.4). Estimate (4.5) on the first derivatives can be obtained in the same way, using estimate (4.2) with $|h|=1$. Moreover estimates (4.6) can be easily obtained using the regularity of the kernel $\mathcal{G}$. Therefore, let us proceed to estimate the pressure π^0. By the hypothesis on the data and estimate $(3.4)_2$, we have

$$|\pi^0(x,t)|\leq c\frac{U_0}{t}\int_{\mathbb{R}^n_+} e^{-\widehat{a}\frac{y_n^2}{t}}\frac{(1+|y|)^{-\mu}}{[|x-y^*|^2+t]^{\frac{n-1}{2}}}dy$$

$$\leq c\frac{U_0}{t^{\frac{1}{2}}}\int_0^\infty \frac{e^{-\widehat{a}\frac{y_n^2}{t}}}{t^{\frac{1}{2}}}\int_{\mathbb{R}^{n-1}}\frac{(1+|y'|^2+y_n^2)^{-\frac{\mu}{2}}}{[|x-y^*|^2+t]^{\frac{n-1}{2}}}dy'\,dy_n. \tag{4.9}$$

We divide $\mathbb{R}^{n-1}$ into the following three parts:

$$\Sigma_1=\left\{y'\in\mathbb{R}^{n-1}\colon 0\leq|y'|\leq\frac{|x|}{2}\right\},$$

$$\Sigma_2=\left\{y'\in\mathbb{R}^{n-1}\colon \frac{|x|}{2}\leq|y'|\leq 2|x|\right\},$$

$$\Sigma_3=\left\{y'\in\mathbb{R}^{n-1}: |y'|\geq 2|x|\right\}.$$

In order to prove estimate (4.7), we first show that, for any $t>0$, π^0 is uniformly bounded in $x\in\mathbb{R}^n_+$. Indeed, for any $a\in(0,\mu)$,

$$\int_{\mathbb{R}^{n-1}}\frac{(1+|y'|^2+y_n^2)^{-\frac{\mu}{2}}}{[|x-y^*|^2+t]^{\frac{n-1}{2}}}dy'$$

$$\leq\frac{c}{|x|^{n-1}}\int_{\Sigma_1}dy'+c\frac{t^{-\frac{a}{2}}}{(1+|x|)^\mu}\int_{\Sigma_2}[|x-y^*|+t^{\frac{1}{2}}]^{-n+1+a}dy'$$

$$+c\int\limits_{\Sigma_3}\frac{(1+|y'|^2+y_n^2)^{-\frac{\mu}{2}}}{[|x-y^*|+t^{\frac12}]^{n-1}}dy'\le c(1+t^{-\frac{a}{2}}).$$

Taking into account this estimate and recalling that $\int\limits_0^\infty\frac{e^{-\widehat{a}\frac{y_n^2}{t}}}{t^{\frac12}}dy_n=c$, we obtain that, for some constant $a\in(0,\mu)$,

$$\begin{aligned}&|\pi^0(x,t)|\le BU_0(1+t^{-\frac{a}{2}}),\\&\forall t\in(0,T),\quad\text{uniformly in}\quad x\in\mathbb{R}^n_+.\end{aligned}\tag{4.10}$$

Let us prove estimate (4.7). For $\nu\in[0,\mu]$ and $\nu_1\in[0,n-1]$, we have

$$\int\limits_{\Sigma_1}\frac{(1+|y'|^2+y_n^2)^{-\frac{\mu}{2}}}{[|x-y^*|^2+t]^{\frac{n-1}{2}}}dy'\le\frac{c}{[|x|+t^{\frac12}]^{n-1}}\int\limits_{\Sigma_1}(1+|y'|^2+y_n^2)^{-\frac{\mu}{2}}dy'$$

$$\le c\begin{cases}|x|^{-\mu+\nu}t^{-\frac{\nu}{2}}, & \text{if }\ \mu\in(0,n-1),\\ |x|^{1-n+\nu}t^{-\frac{\nu}{2}}\log\Big(1+\dfrac{|x|}{2}\Big), & \text{if }\ \mu=n-1,\\ |x|^{1-n+\nu_1}t^{-\frac{\nu_1}{2}}, & \text{if }\ \mu\in(n-1,n).\end{cases}$$

Moreover

$$\int\limits_{\Sigma_2}\frac{(1+|y'|^2+y_n^2)^{-\frac{\mu}{2}}}{[|x-y^*|^2+t]^{\frac{n-1}{2}}}dy'$$

$$\le\frac{ct^{-\frac{\nu}{2}}}{(1+|x|)^\mu}\int\limits_{0\le|x-y'|\le3|x|}[|x-y'|^2+y_n^2+t]^{\frac{1-n+\nu}{2}}dy'$$

$$\le c\begin{cases}\dfrac{1}{(1+|x|)^\mu}\log\Big(\dfrac{3|x|}{t^{\frac12}}+1\Big), & \text{for }\ \nu=0,\\ \dfrac{t^{-\frac{\nu}{2}}}{(1+|x|)^{\mu-\nu}}, & \text{if }\ \mu\in(0,n-1],\ \text{ for }\ \nu\in(0,\mu],\\ \dfrac{t^{-\frac{\nu_1}{2}}}{(1+|x|)^{n-1-\nu_1}}, & \text{if }\ \mu\in[n-1,n),\ \text{ for }\ \nu_1\in(0,n-1].\end{cases}$$

Finally, for $\nu\in[0,\mu)$ and $\nu_1\in[0,n-1)$, we have

$$\int\limits_{\Sigma_3}\frac{(1+|y'|^2+y_n^2)^{-\frac{\mu}{2}}}{[|x-y^*|^2+t]^{\frac{n-1}{2}}}dy'\le c\int\limits_{\Sigma_3}\frac{(1+|y'|^2+y_n^2)^{-\frac{\mu}{2}}}{[|y'|^2+y_n^2+t]^{\frac{n-1}{2}}}dy'$$

$$\le c\begin{cases} |x|^{-\mu+\nu}t^{-\frac{\nu}{2}}, & \text{if } \mu\in(0,n-1],\\ |x|^{1-n+\nu_1}t^{-\frac{\nu_1}{2}}, & \text{if } \mu\in[n-1,n). \end{cases}$$

Putting these estimates in (4.9) and recalling (4.10), we arrive at estimate (4.7).

5. Some preliminary results for the Stokes system with $f\neq 0$

Let us consider the Stokes system

$$\begin{aligned} &v_t-\Delta v=-\nabla p+f,\quad \nabla\cdot v=0,\quad \text{on }\ \mathbb{R}^n_+\times(0,T),\\ &v(x',t)=0,\quad \text{on }\ \mathbb{R}^{n-1}\times(0,T),\quad \lim_{|x|\to\infty}v(x,t)=0,\\ &v(x,0)=0,\quad \text{on }\ \mathbb{R}^n_+. \end{aligned}\tag{5.1}$$

On the body force f we make the following

Assumptions

For any $(x,t)\in\mathbb{R}^n_+\times(0,T)$ $f(x,t)=\nabla\cdot F(x,t)=-\nabla\cdot(l\otimes k)(x,t)$, with $l(x,t)$, $k(x,t)\in C^{1,\alpha}(\overline{\mathbb{R}^n_+})$ for any $t\in(0,T)$, $\nabla\cdot l(x,t)=\nabla\cdot k(x,t)=0$ for any $(x,t)\in\mathbb{R}^n_+\times(0,T)$, $l(x',t)=k(x',t)=0$ on $\mathbb{R}^{n-1}\times(0,T)$ and:

(i) for some $\mu\in(\frac12,n)$

$$|l(x,t)|+t^{\frac12}|\nabla l(x,t)|\le\frac{L(t)}{(1+|x|)^{\mu}},$$

$$|k(x,t)|+t^{\frac12}|\nabla k(x,t)|\le\frac{K(t)}{(1+|x|)^{\mu}},$$

for any $(x,t)\in\mathbb{R}^n_+\times[0,T)$, with $L(t)$, $K(t)$ nondecreasing functions of t;

(ii) for some $\mu\in[1,n)$, for any $\nu\in[0,\mu]$

$$|l(x,t)|+t^{\frac12}|\nabla l(x,t)|\le\frac{L(t)}{(1+|x|)^{\mu-\nu}}\frac{1}{(1+t)^{\frac{\nu}{2}}},$$

$$|k(x,t)|+t^{\frac12}|\nabla k(x,t)|\le\frac{K(t)}{(1+|x|)^{\mu-\nu}}\frac{1}{(1+t)^{\frac{\nu}{2}}},$$

for any $(x,t)\in\mathbb{R}^n_+\times[0,T)$, with $L(t)$, $K(t)$ nondecreasing functions of t;

(iii) for $\mu\in(\frac12,n)$ and for $\alpha\in(0,1)$

$$|l(\overline{x},\overline{t})-l(\overline{\overline{x}},\overline{\overline{t}})|\le cL(t_1)t_0^{-\frac{\alpha}{2}}(|\overline{x}-\overline{\overline{x}}|^2+|\overline{t}-\overline{\overline{t}}|)^{\frac{\alpha}{2}},$$

$$|k(\overline{x},\overline{t})-k(\overline{\overline{x}},\overline{\overline{t}})|\leq cK(t_1)t_0^{-\frac{\alpha}{2}}(|\overline{x}-\overline{\overline{x}}|^2+|\overline{t}-\overline{\overline{t}}|)^{\frac{\alpha}{2}},$$

$$|\nabla l(\overline{x},\overline{t})-\nabla l(\overline{\overline{x}},\overline{\overline{t}})|\leq cL(t_1)t_0^{-\frac{1}{2}-\frac{\alpha}{2}}(|\overline{x}-\overline{\overline{x}}|^2+|\overline{t}-\overline{\overline{t}}|)^{\frac{\alpha}{2}},$$

$$|\nabla k(\overline{x},\overline{t})-\nabla k(\overline{\overline{x}},\overline{\overline{t}})|\leq cK(t_1)t_0^{-\frac{1}{2}-\frac{\alpha}{2}}(|\overline{x}-\overline{\overline{x}}|^2+|\overline{t}-\overline{\overline{t}}|)^{\frac{\alpha}{2}},$$

for any $\overline{x}$, $\overline{\overline{x}}\in\mathbb{R}^n_+$ and $\overline{t}$, $\overline{\overline{t}}>0$, with $t_0=\min\{\overline{t},\overline{\overline{t}}\}$ and $t_1=\max\{\overline{t},\overline{\overline{t}}\}$;
(iv) for $\mu\in[1,n)$, for any $\nu\in[0,\mu]$ and for $\alpha\in(0,1)$

$$|l(\overline{x},t)-l(\overline{\overline{x}},t)|\leq cL(t)t^{-\frac{\alpha}{2}}\frac{|\overline{x}-\overline{\overline{x}}|^\alpha}{R(|\overline{x}|,|\overline{\overline{x}}|)^{\mu-\nu}}\frac{1}{(1+t)^{\frac{\nu}{2}}},$$

$$|k(\overline{x},t)-k(\overline{\overline{x}},t)|\leq cK(t)t^{-\frac{\alpha}{2}}\frac{|\overline{x}-\overline{\overline{x}}|^\alpha}{R(|\overline{x}|,|\overline{\overline{x}}|)^{\mu-\nu}}\frac{1}{(1+t)^{\frac{\nu}{2}}},$$

$$|\nabla l(\overline{x},t)-\nabla l(\overline{\overline{x}},t)|\leq cL(t)t^{-\frac{1}{2}-\frac{\alpha}{2}}\frac{|\overline{x}-\overline{\overline{x}}|^\alpha}{R(|\overline{x}|,|\overline{\overline{x}}|)^{\mu-\nu}}\frac{1}{(1+t)^{\frac{\nu}{2}}},$$

$$|\nabla k(\overline{x},t)-\nabla k(\overline{\overline{x}},t)|\leq cK(t)t^{-\frac{1}{2}-\frac{\alpha}{2}}\frac{|\overline{x}-\overline{\overline{x}}|^\alpha}{R(|\overline{x}|,|\overline{\overline{x}}|)^{\mu-\nu}}\frac{1}{(1+t)^{\frac{\nu}{2}}},$$

for any $\overline{x}$, $\overline{\overline{x}}\in\mathbb{R}^n_+$, for any $t>0$, where, we recall, $R(|\overline{x}|,|\overline{\overline{x}}|)=\min\{(1+|\overline{x}|),(1+|\overline{\overline{x}}|)\}$.

In particular we stress that since, by the assumptions, $f=\nabla\cdot F=-l\cdot\nabla k$, with $l(x',t)=k(x',t)=0$, then $\mathfrak{f}=Pf$ can be represented by formula (3.8).

Set

$$\Phi^{\pm}_{qprsm}(x,t)=D_{x_sx_m}\int\limits_{\mathbb{R}^n_+}D_{y_q}\mathcal{N}^{\pm}(x,y)F_{pr}(y,t)dy,\quad m,s\neq n. \tag{5.2}$$

Lemma 5.1. *If $F=-l\otimes k$ satisfies the assumptions then, for any $T>0$ and $\eta\in(0,T)$, $\Phi^{\pm}(x,t)\in C(0,T;C(\overline{\mathbb{R}^n_+}))$. Moreover*
if $\mu\in(\frac{1}{2},n)$ then

$$t^{\frac{1}{2}}|\Phi^{\pm}(x,t)|\leq c\frac{L(t)K(t)}{(1+|x|)^{\mu}},\quad \forall x\in\mathbb{R}^n_+,\forall t>0; \tag{5.3}$$

if $\mu\in[1,n)$ then, setting $\widehat{\mu}=\min\{2\mu,n+1\}$, for any $\widehat{\nu}\in[0,\widehat{\mu}]$

$$t^{\frac{1}{2}}|\Phi^{\pm}(x,t)|\leq c\frac{L(t)K(t)}{(1+|x|)^{\widehat{\mu}-\widehat{\nu}}}\frac{1}{(1+t)^{\frac{\widehat{\nu}}{2}}},\quad \forall x\in\mathbb{R}^n_+,\forall t>0. \tag{5.4}$$

Finally, for $\alpha \in (0,1)$,

$$|\Phi^{\pm}(\overline{x},t) - \Phi^{\pm}(\overline{\overline{x}},t)| \le cL(t)K(t)t^{-\frac{1}{2}-\frac{\alpha}{2}}|\overline{x} - \overline{\overline{x}}|^{\alpha}, \tag{5.5}$$

for $\alpha_1 \in (0,\alpha)$

$$|\Phi^{\pm}(x,\overline{t}) - \Phi^{\pm}(x,\overline{\overline{t}})| \le cL(t_1)K(t_1)t_0^{-\frac{1}{2}-\frac{\alpha_1}{2}}|\overline{t} - \overline{\overline{t}}|^{\frac{\alpha_1}{2}}, \tag{5.6}$$

for any x, $\overline{x}$, $\overline{\overline{x}} \in \mathbb{R}^n_+$, *for any* t, $\overline{t}$, $\overline{\overline{t}} \in (0,T)$, *with* $t_0 = \min\{\overline{t},\overline{\overline{t}}\}$, $t_1 = \max\{\overline{t},\overline{\overline{t}}\}$. *The constant* c *is independent of* F.

Proof. The regularity of $\Phi^{\pm}$ in the hypothesis made on F is a well known result, related to elementary properties of the fundamental solution of the Laplace equation. Since $F = -l \otimes k \in C^{1,\alpha}(\overline{\mathbb{R}^n_+})$ and $l(x',t) = k(x',t) = 0$, we can extend function F to the whole $\mathbb{R}^n$ by setting $F(x,t) = 0$ on $\mathbb{R}^n_- \times (0,T)$, obtaining a function of $C^{1,\alpha}(\mathbb{R}^n)$. Still calling F the function thus obtained, F satisfies the Assumptions. Then, we can write

$$\Phi^{\pm}(x,t) = D_{x_s x_m} \int\limits_{\mathbb{R}^n} D_{y_q}\mathcal{E}(x-y)F_{pr}(y,t)dy$$
$$\pm D_{x_s x_m} \int\limits_{\mathbb{R}^n} D_{y_q}\mathcal{E}(x-y^*)F_{pr}(y,t)dy.$$

For the sake of brevity we only estimate the term with the singular kernel, the same argument lines work for the other term. Let us prove estimates (5.3) and (5.4). For any $\rho > \varepsilon > 0$, $\int\limits_{\varepsilon \le |x-y| \le \rho} D_{x_s x_q}\mathcal{E}(x-y)dy = 0$. Then, integrating by parts and using the Hölder condition on F we have

$$|D_{x_s x_m} \int\limits_{\mathbb{R}^n} D_{y_q}\mathcal{E}(x-y)F_{pr}(y,t)dy|$$

$$= \Bigg| \frac{\delta_{sq}}{n} D_{x_m}F_{pr}(x,t) + \lim_{\varepsilon \to 0} \int\limits_{\varepsilon \le |x-y| \le \sqrt{t}} D_{x_s x_q}\mathcal{E}(x-y)D_{y_m}[F_{pr}(y,t) - F_{pr}(x,t)]dy$$

$$+ \int\limits_{|x-y| \ge \sqrt{t}} D_{x_s x_m x_q}\mathcal{E}(x-y)F_{pr}(y,t)dy - \int\limits_{|x-y| = \sqrt{t}} D_{x_s x_q}\mathcal{E}(x-y)F_{pr}(y,t)\frac{y_m}{|y|}d\sigma_y \Bigg|$$

$$\le cL(t)K(t)t^{-\frac{1}{2}} + cL(t)K(t)t^{-\frac{1}{2}-\frac{\alpha}{2}} \int\limits_{\varepsilon \le |x-y| \le \sqrt{t}} |x-y|^{\alpha-n}dy$$

$$+cL(t)K(t)\int\limits_{|x-y|\geq\sqrt{t}}|x-y|^{-n-1}dy+cL(t)K(t)\int\limits_{|x-y|=\sqrt{t}}|x-y|^{-n}d\sigma_y$$

$$\leq cL(t)K(t)t^{-\frac{1}{2}},\quad \forall x\in\mathbb{R}^n,\forall t>0.$$

This gives, for any $t>0$, a uniform bound in x. Using the above property of $\mathcal{E}$ we have

$$D_{x_s x_m}\int\limits_{\mathbb{R}^n}D_{y_q}\mathcal{E}(x-y)F_{pr}(y,t)dy$$

$$=-\lim_{\varepsilon\to 0}\int\limits_{\varepsilon\leq|w|\leq\frac{|x|}{2}}D_{w_s w_q}\mathcal{E}(w)D_{x_m}[F_{pr}(x-w,t)-F_{pr}(x,t)]dw$$

$$-\int\limits_{\frac{|x|}{2}\leq|w|\leq 2|x|}D_{w_s w_q}\mathcal{E}(w)D_{x_m}F_{pr}(x-w,t)dw$$

$$-\int\limits_{|w|\geq 2|x|}D_{w_s w_q}\mathcal{E}(w)D_{x_m}F_{pr}(x-w,t)dw+\frac{\delta_{sq}}{n}D_{x_m}F_{pr}(x,t)$$

$$=\lim_{\varepsilon\to 0}I_1(\varepsilon,x,t)+I_2(x,t)+I_3(x,t)+I_4(x,t).$$

First assume $\mu\in(\frac{1}{2},n)$. Since, from hypothesis (i),

$$|D_z F(z,t)|\leq\frac{c}{t^{\frac{1}{2}}}\frac{L(t)K(t)}{(1+|z|)^{2\mu}}\leq\frac{c}{t^{\frac{1}{2}}}\frac{L(t)K(t)}{(1+|z|)^{\mu}},$$

we easily get

$$|I_2|+|I_3|+|I_4|\leq\frac{c}{t^{\frac{1}{2}}}\frac{L(t)K(t)}{(1+|x|)^{\mu}},\quad\forall(x,t)\in\mathbb{R}^n\times(0,T).$$

For the term I_1 we distinguish two cases: for $t>0$ we consider $2|x|\leq\sqrt{t}$ and $2|x|>\sqrt{t}$. In the first case, using the Hölder condition (iii) on l and k, we have

$$|I_1|\leq\int\limits_{\varepsilon\leq|w|\leq\frac{|x|}{2}}|w|^{-n}\Big[|l(x-w,t)||\nabla k(x-w,t)-\nabla k(x,t)|$$

$$+|\nabla k(x,t)||l(x-w,t)-l(x,t)|\Big]dw$$

$$\leq cL(t)K(t)t^{-\frac{1}{2}-\frac{\alpha}{2}} \int\limits_{\varepsilon\leq|w|\leq\frac{|x|}{2}} \frac{|w|^{-n+\alpha}}{(1+|x-w|)^{\mu}}\,dw \leq \frac{c}{t^{\frac{1}{2}}}\frac{L(t)K(t)}{(1+|x|)^{\mu}},$$

uniformly in $\varepsilon > 0$. In the second case, $2|x| > \sqrt{t}$, integrating by parts we obtain

$$\begin{aligned} I_1 = &-\int\limits_{\varepsilon\leq|w|\leq\frac{\sqrt{t}}{4}} D_{w_s w_q}\mathcal{E}(w)D_{x_m}[F_{pr}(x-w,t)-F_{pr}(x,t)]dw \\ &- \int\limits_{\frac{\sqrt{t}}{4}\leq|w|\leq\frac{|x|}{2}} D_{w_s w_q w_m}\mathcal{E}(w)F_{pr}(x-w,t)dw \\ &+ \int\limits_{|w|=\frac{\sqrt{t}}{4}} D_{w_s w_q}\mathcal{E}(w)F_{pr}(x-w,t)\frac{w_m}{|w|}d\sigma_w \\ &- \int\limits_{|w|=\frac{|x|}{2}} D_{w_s w_q}\mathcal{E}(w)F_{pr}(x-w,t)\frac{w_m}{|w|}d\sigma_w. \end{aligned}$$

Then, by calculations similar to the ones of case $2|x| \leq \sqrt{t}$, we get

$$\begin{aligned} |I_1| \leq &\int\limits_{\varepsilon\leq|w|\leq\frac{\sqrt{t}}{4}} |w|^{-n}\Big[|l(x-w,t)||\nabla k(x-w,t)-\nabla k(x,t)| \\ &+|\nabla k(x,t)||l(x,t)-l(x-w,t)|\Big]dw \\ &+ \int\limits_{\frac{\sqrt{t}}{4}\leq|w|\leq\frac{|x|}{2}} |w|^{-n-1}|(l\otimes k)(x-w,t)|dw + \int\limits_{|w|=\frac{\sqrt{t}}{4}} |w|^{-n}|(l\otimes k)(x-w,t)|d\sigma_w \\ &+ \int\limits_{|w|=\frac{|x|}{2}} |w|^{-n}|(l\otimes k)(x-w,t)|d\sigma_w \leq \frac{c}{t^{\frac{1}{2}}}\frac{L(t)K(t)}{(1+|x|)^{\mu}}, \end{aligned}$$

uniformly in $\varepsilon > 0$. Combining the estimates thus obtained for each term I_i, we arrive at (5.3).

Assume now $\mu \in [1,n)$. Since, by the hypotheses, for any $\nu \in [0,\mu]$,

$$|D_z F(z,t)| \leq \frac{c}{t^{\frac{1}{2}}}\frac{L(t)K(t)}{(1+|z|)^{2\mu-2\nu}}\frac{1}{(1+t)^{\nu}},$$

we easily get, for any $\nu \in [0,\mu]$,

$$|I_3| + |I_4| \le \frac{c}{t^{\frac{1}{2}}} \frac{L(t)K(t)}{(1+|x|)^{2\mu-2\nu}} \frac{1}{(1+t)^{\nu}}, \quad \forall (x,t) \in \mathbb{R}^n_+ \times (0,T).$$

For the term I_1, for $t>0$, we consider $2|x| \le \sqrt{t}$ and $2|x| > \sqrt{t}$. In the first case, using the Hölder condition (iv) on l and k, we have

$$\begin{aligned} |I_1| \le & \int\limits_{\varepsilon \le |w| \le \frac{|x|}{2}} |w|^{-n} \Big[|l(x-w,t)||\nabla k(x-w,t) - \nabla k(x,t)| \\ & \qquad + |\nabla k(x,t)||l(x-w,t) - l(x,t)| \Big] dw \\ \le & \frac{c}{t^{\frac{1}{2}+\frac{\alpha}{2}}} \frac{L(t)K(t)}{(1+t)^{\nu}} \int\limits_{\varepsilon \le |w| \le \frac{|x|}{2}} \frac{|w|^{-n+\alpha}}{(1+|x-w|)^{\mu-\nu}} \frac{dw}{R(|x-w|,|x|)^{\mu-\nu}} \\ \le & \frac{c}{t^{\frac{1}{2}}} \frac{L(t)K(t)}{(1+|x|)^{2\mu-2\nu}} \frac{1}{(1+t)^{\nu}}, \quad \text{uniformly in } \varepsilon > 0. \end{aligned}$$

In the second case, $2|x| > \sqrt{t}$, integrating by parts we obtain

$$\begin{aligned} I_1 = & -\int\limits_{\varepsilon \le |w| \le \frac{\sqrt{t}}{4}} D_{w_s w_q} \mathcal{E}(w) D_{x_m} [F_{pr}(x-w,t) - F_{pr}(x,t)] dw \\ & - \int\limits_{\frac{\sqrt{t}}{4} \le |w| \le \frac{|x|}{2}} D_{w_s w_q w_m} \mathcal{E}(w) F_{pr}(x-w,t) dw \\ & + \int\limits_{|w| = \frac{\sqrt{t}}{4}} D_{w_s w_q} \mathcal{E}(w) F_{pr}(x-w,t) \frac{w_m}{|w|} d\sigma_w \\ & - \int\limits_{|w| = \frac{|x|}{2}} D_{w_s w_q} \mathcal{E}(w) F_{pr}(x-w,t) \frac{w_m}{|w|} d\sigma_w . \end{aligned}$$

Then, by calculations similar to the case $2|x| \le \sqrt{t}$, we get

$$|I_1| \le \int\limits_{\varepsilon \le |w| \le \frac{\sqrt{t}}{4}} |w|^{-n} \Big[|l(x-w,t)||\nabla k(x-w,t) - \nabla k(x,t)|$$

$$+|\nabla k(x,t)||l(x,t)-l(x-w,t)|\Big]dw$$

$$+\int\limits_{\frac{\sqrt{t}}{4}\le|w|\le\frac{|x|}{2}}|w|^{-n-1}|(l\otimes k)(x-w,t)|dw+\int\limits_{|w|=\frac{\sqrt{t}}{4}}|w|^{-n}|(l\otimes k)(x-w,t)|d\sigma_w$$

$$+\int\limits_{|w|=\frac{|x|}{2}}|w|^{-n}|(l\otimes k)(x-w,t)|d\sigma_w\le\frac{c}{t^{\frac12}}\frac{L(t)K(t)}{(1+|x|)^{2\mu-2\nu}}\frac{1}{(1+t)^{\nu}},$$

uniformly in $\varepsilon>0$. Finally we estimate the term I_2. Integrating by parts we obtain

$$\begin{aligned}|I_2|=\Bigg|&-\int\limits_{\frac{|x|}{2}\le|w|\le2|x|}D_{w_sw_qw_m}\mathcal{E}(w)F_{pr}(x-w,t)dw\\&+\int\limits_{|w|=\frac{|x|}{2}}D_{w_sw_q}\mathcal{E}(w)F_{pr}(x-w,t)\frac{w_m}{|w|}d\sigma_w\\&-\int\limits_{|w|=2|x|}D_{w_sw_q}\mathcal{E}(w)F_{pr}(x-w,t)\frac{w_m}{|w|}d\sigma_w\Bigg|\\&\le c\frac{L(t)K(t)}{(1+t)^{\frac12}}\int\limits_{\frac{|x|}{2}\le|w|\le2|x|}\frac{|w|^{-n-1}}{(1+|x-w|)^{2\mu-1}}dw\\&+c\frac{L(t)K(t)}{(1+t)^{\frac12}}\int\limits_{|w|=\frac{|x|}{2}}\frac{|w|^{-n}}{(1+|x-w|)^{2\mu-1}}d\sigma_w\\&+c\frac{L(t)K(t)}{(1+t)^{\frac12}}\int\limits_{|w|=2|x|}\frac{|w|^{-n}}{(1+|x-w|)^{2\mu-1}}d\sigma_w.\end{aligned}$$

Since

$$\int\limits_{\frac{|x|}{2}\le|w|\le2|x|}\frac{|w|^{-n-1}}{(1+|x-w|)^{2\mu-1}}dw\le c\begin{cases}|x|^{-2\mu}&\text{if }\ 2\mu<n,\\|x|^{-n-1}&\text{if }\ 2\mu>n,\end{cases}$$

we obtain

$$|I_2|\le\frac{c}{t^{\frac12}}\frac{L(t)K(t)}{(1+|x|)^{\widehat{\mu}}},\quad\forall(x,t)\in\mathbb{R}^n_+\times(0,T),$$

with $\widehat{\mu} = \min\{2\mu, n+1\}$. Moreover

$$|I_2| = \Big| \int\limits_{\frac{|x|}{2} \le |w| \le 2|x|} D_{w_s w_q}\mathcal{E}(w) D_{w_m} F_{pr}(x-w,t)dw \Big|$$

$$\le \frac{c}{t^{\frac{1}{2}}} \frac{L(t)K(t)}{(1+t)^{\mu}} \int\limits_{\frac{|x|}{2} \le |w| \le 2|x|} |w|^{-n} dw \le \frac{c}{t^{\frac{1}{2}}} \frac{L(t)K(t)}{(1+t)^{\mu}}, \quad \forall (x,t) \in \mathbb{R}^n_+ \times (0,T).$$

Therefore

$$|I_2| \le \frac{c}{t^{\frac{1}{2}}} \frac{L(t)K(t)}{(1+|x|)^{\widehat{\mu}-\widehat{\nu}}} \frac{1}{(1+t)^{\frac{\widehat{\nu}}{2}}}, \quad \forall (x,t) \in \mathbb{R}^n_+ \times (0,T).$$

Combining the estimates thus obtained for each term I_i, we arrive at (5.4). The Hölder condition (5.5) is a well known result for the second derivatives of a Newtonian potential with density belonging to $C^{0,\alpha}(\mathbb{R}^n)$(cf. [8]). Thus we just prove the Hölder condition (5.6), with respect to the time variable. We assume, without loss of generality, $\overline{t} < \overline{\overline{t}}$. We have

$$D_{x_s x_m} \int\limits_{\mathbb{R}^n} D_{y_q}\mathcal{E}(x-y) F_{pr}(y,\overline{t})dy - D_{x_s x_m} \int\limits_{\mathbb{R}^n} D_{y_q}\mathcal{E}(x-y) F_{pr}(y,\overline{\overline{t}})dy$$

$$= \frac{\delta_{sq}}{n}[D_{x_m}F(x,\overline{t}) - D_{x_m}F(x,\overline{\overline{t}})]$$

$$- \lim_{\varepsilon\to 0} \int\limits_{\varepsilon \le |x-y| \le \sqrt{t}} D_{x_s x_q}\mathcal{E}(x-y) D_{y_m}[F_{pr}(y,\overline{t}) - F_{pr}(x,\overline{t}) + F_{pr}(x,\overline{\overline{t}}) - F_{pr}(y,\overline{\overline{t}})]dy$$

$$- \int\limits_{|x-y| \ge \sqrt{t}} D_{x_s x_m x_q}\mathcal{E}(x-y)[F_{pr}(y,\overline{t}) - F_{pr}(y,\overline{\overline{t}})]dy$$

$$+ \int\limits_{|x-y| = \sqrt{t}} D_{x_s x_q}\mathcal{E}(x-y)[F_{pr}(y,\overline{t}) - F_{pr}(y,\overline{\overline{t}})]\frac{y_m}{|y|}d\sigma_y$$

$$= J_1(x,\overline{t},\overline{\overline{t}}) + \lim_{\varepsilon\to 0} J_2(x,\overline{t},\overline{\overline{t}}) + J_3(x,\overline{t},\overline{\overline{t}}) + J_4(x,\overline{t},\overline{\overline{t}}).$$

By the Hölder condition on F, for $\alpha_1 \in (0,\alpha)$ we immediately have

$$|J_1| + |J_3| + |J_4| \le cL(\overline{\overline{t}})K(\overline{\overline{t}})\overline{t}^{\,-\frac{1}{2}-\frac{\alpha_1}{2}}|\overline{t} - \overline{\overline{t}}|^{\frac{\alpha_1}{2}}.$$

Moreover, for any $a \in (0,1)$, we have

$$|J_2| \le cL(\bar{\bar{t}})K(\bar{\bar{t}}) \int\limits_{\varepsilon \le |x-y| \le \bar{t}^{\frac{1}{2}}} |x-y|^{-n} \left(\frac{|x-y|^{\alpha}}{\bar{t}^{\frac{1}{2}+\frac{\alpha}{2}}} \right)^{a} \left(\frac{|\bar{t}-\bar{\bar{t}}|^{\frac{\alpha}{2}}}{\bar{t}^{\frac{1}{2}+\frac{\alpha}{2}}} \right)^{(1-a)} dy$$

$$\le cL(\bar{\bar{t}})K(\bar{\bar{t}})\bar{t}^{-\frac{1}{2}-\frac{\alpha}{2}(1-a)} |\bar{t}-\bar{\bar{t}}|^{\frac{\alpha}{2}(1-a)},$$

uniformly in $\varepsilon > 0$. Therefore, choosing $a = \frac{\alpha-\alpha_1}{\alpha}$ we get

$$\left| D_{x_s x_m} \int\limits_{\mathbb{R}^n} D_{y_q} \mathcal{E}(x-y) F_{pr}(y,\bar{t}) dy - D_{x_s x_m} \int\limits_{\mathbb{R}^n} D_{y_q} \mathcal{E}(x-y) F_{pr}(y,\bar{\bar{t}}) dy \right|$$

$$\le cL(\bar{\bar{t}})K(\bar{\bar{t}})\bar{t}^{-\frac{1}{2}-\frac{\alpha_1}{2}} |\bar{t}-\bar{\bar{t}}|^{\frac{\alpha_1}{2}}, \quad \forall\, 0 < \bar{t} \le \bar{\bar{t}}.$$

This estimate completes the proof of the lemma.

Finally, we state the following elementary results, that will give the main tools for the proof of the asymptotic properties of the solution. We observe that Lemmas 5.3 and 5.4 are stated just as they come useful.

Lemma 5.2. *Let be*

$$\mathfrak{L}^1(x,t) = \int\limits_0^t \int\limits_{\mathbb{R}^n_+} \frac{(1+|y|)^{-\gamma_1} \tau^{-\gamma_2}}{\left[|x-y| + (t-\tau)^{\frac{1}{2}}\right]^{n+\gamma}} dy d\tau,$$

with $\gamma \in (0,2)$, $\gamma_1 \in [0,n)$, *and* $\gamma_2 \in [0,1)$. *Then*

$$\mathfrak{L}^1(x,t) \le \frac{c}{(1+|x|)^{\gamma_1}} t^{1-\gamma/2-\gamma_2}, \quad \forall (x,t) \in \mathbb{R}^n_+ \times (0,T).$$

Lemma 5.3. *Let be*

$$\mathfrak{L}^2(x,t) = \int\limits_0^{\frac{t}{2}} \int\limits_{\mathbb{R}^n_+} \frac{(1+|y|)^{-\gamma_1} \tau^{-\gamma_2}}{\left[|x-y| + (t-\tau)^{\frac{1}{2}}\right]^{n+\gamma}} dy d\tau,$$

with $\gamma > 0$, $\gamma_1 \in [0,n)$, $\gamma_2 \in [0,1)$. *Then,*

$$\mathfrak{L}^2(x,t) \le ct^{1-\gamma_2-\frac{\gamma}{2}-\frac{\gamma_1}{2}}, \quad \forall (x,t) \in \mathbb{R}^n_+ \times (0,T).$$

Lemma 5.4. *Let be*

$$\mathfrak{L}^3(x,t) = \int_0^{\frac{t}{2}} \int_{\mathbb{R}^n_+} \frac{(1+|y|)^{-\gamma_1} \tau^{-\gamma_2}}{[|x-y^*| + (t-\tau)^{\frac{1}{2}}]^n} e^{-\widehat{a}\frac{y_n^2}{t-\tau}} dy d\tau,$$

with $\gamma_1 \in (0,n)$ *and* $\gamma_2 \in [0,1)$. *Then, for any* $\eta > 0$,

$$\mathfrak{L}^3(x,t) \le ct^{1-\gamma_2-\frac{\gamma_1}{2}}, \quad \forall (x,t) \in \mathbb{R}^n_+ \times (\eta, T).$$

Lemma 5.5. *Let* $g(x,t)$ *be a function defined on* $\mathbb{R}^n_+ \times (0,T)$ *such that, for* $\mu \in [1,n)$, *for any* $\nu \in [0,\mu]$, $|g(x,t)| \le G(1+|x|)^{-2\mu+2\nu}(1+t)^{-\nu}$ *and let be, for any* $\tau \in (0,t)$,

$$\mathfrak{L}^4(x,t,\tau) = \int_{\mathbb{R}^n_+} \frac{|g(y,\tau)|}{[|x-y| + (t-\tau)^{\frac{1}{2}}]^n} dy.$$

Then, for any a *and* $b \in (0,1)$ *with* $a+b=1$, *we have*

$$\mathfrak{L}^4(x,t,\tau) \le cG \begin{cases} \frac{\tau^{-\frac{1}{2}}}{(1+|x|)^{\mu}} + \frac{\tau^{-\frac{a}{2}}}{(1+|x|)^{\mu}} \frac{1}{(t-\tau)^{\frac{b}{2}}}, & \tau \in (0,t), \\ \frac{\tau^{-\frac{1}{2}}}{(t-\tau)^{\frac{\mu}{2}}}, & \tau \in (0, \frac{t}{2}), \\ \tau^{-\frac{1}{2}-\frac{\mu}{2}} + \frac{\tau^{-\frac{1}{2}-\frac{\mu}{2}+\frac{b}{2}}}{(t-\tau)^{\frac{b}{2}}}, & \tau \in (\frac{t}{2}, t), \end{cases} \tag{5.7}$$

uniformly in $x \in \mathbb{R}^n_+$.

Proof. The results can be easily obtained by majorizing function g in a suitable way. For instance, in order to obtain inequality $(5.7)_1$, using the decomposition (4.8) of $\mathbb{R}^n_+$, we write

$$\mathfrak{L}^4(x,t,\tau) \le cG \int_{\Lambda_1 \cup \Lambda_3} \frac{(1+|y|)^{-\mu}(1+\tau)^{-\frac{1}{2}}}{[|x-y| + (t-\tau)^{\frac{1}{2}}]^n} dy$$

$$+cG \int_{\Lambda_2} \frac{(1+|y|)^{a-2\mu}(1+\tau)^{-\frac{a}{2}}}{[|x-y| + (t-\tau)^{\frac{1}{2}}]^{n-b}} \frac{dy}{(t-\tau)^{\frac{b}{2}}}.$$

This by standard calculations gives the desired result.

Let be

$$\mathfrak{L}^5(x,t) = \int_0^t \int_{\mathbb{R}^n_+} \frac{(y_n^2+(t-\tau))^{-\frac{1}{2}}}{[|x-y^*|+(t-\tau)^{\frac{1}{2}}]^{n-1}} \frac{e^{-\widehat{a}\frac{y_n^2}{(t-\tau)}}}{(t-\tau)^{\frac{1}{2}}} |\mathfrak{f}(y,\tau)| dy d\tau,$$

$$x \in \mathbb{R}^n_+, \quad t>0.$$

Lemma 5.6. *Assume that, for some* $\mu \in (0,n)$,

$$|\mathfrak{f}(y,\tau)| \le \frac{c}{\tau^{\frac{1}{2}}} \frac{L(\tau)K(\tau)}{(1+|y|)^{\mu}}, \quad \forall (y,\tau) \in \mathbb{R}^n_+ \times (0,T).$$

Then

$$\mathfrak{L}^5(x,t)$$

$$\le c \begin{cases} \Big(1+\log\Big(\frac{3|x|}{t^{\frac{1}{2}}}+1\Big)\Big)\frac{L(t)K(t)}{(1+|x|)^{\mu}}, & \text{if } \mu \in (0,n-1), \\ \Big(\log\Big(1+\frac{|x|}{2}\Big)+\log\Big(\frac{3|x|}{t^{\frac{1}{2}}}+1\Big)\Big)\frac{L(t)K(t)}{(1+|x|)^{n-1}}, & \text{if } \mu = n-1, \\ \Big(1+\log\Big(\frac{3|x|}{t^{\frac{1}{2}}}+1\Big)\Big)\frac{L(t)K(t)}{(1+|x|)^{n-1}}, & \text{if } \mu \in (n-1,n), \end{cases}$$

for any $(x,t) \in \mathbb{R}^n_+ \times (0,T)$.

Lemma 5.7. *Assume that, for some* $\mu \in [1,n)$ *and* $\widehat{\mu} = \min\{2\mu, n+1\}$, *for any* $\widehat{\nu} \in [0,\widehat{\mu}]$

$$|\mathfrak{f}(y,\tau)| \le \frac{c}{\tau^{\frac{1}{2}}} \frac{L(\tau)K(\tau)}{(1+|y|)^{\widehat{\mu}-\widehat{\nu}}} \frac{1}{(1+\tau)^{\frac{\widehat{\nu}}{2}}}, \quad \forall (y,\tau) \in \mathbb{R}^n_+ \times (0,T).$$

Then, for any $\nu \in [0,\mu]$ *and* $\nu_1 \in [0,n-1]$,

$$\mathfrak{L}^5(x,t) \le c \begin{cases} \frac{L(t)K(t)}{(1+|x|)^{\mu-\nu}} \frac{t^{-\frac{1}{2}}}{(1+t)^{\frac{\nu}{2}}}, & \text{if } \mu \in [1,n-1], \\ \frac{L(t)K(t)}{(1+|x|)^{n-1-\nu_1}} \frac{t^{-\frac{1}{2}}}{(1+t)^{\frac{\nu_1}{2}}}, & \text{if } \mu \in [n-1,n), \end{cases}$$

for any $(x,t) \in \mathbb{R}^n_+ \times (0,T)$.

Lemma 5.8. *Let be*

$$\mathfrak{F}(x) = \int_{\mathbb{R}^d} |x-y|^{-\mu_1}(1+|y|)^{-\mu_2} dy,$$

with $\mu_1 \in [0,d)$, $\mu_2 \neq d$, *and* $\mu_1+\mu_2 > d$. *Then, setting* $\mu_3 = \min\{\mu_1, \mu_1+\mu_2-d\}$, *we have*

$$\mathfrak{F}(x) \le \frac{c}{(1+|x|)^{\mu_3}}, \quad \forall x \in \mathbb{R}^d.$$

6. Existence of solutions of the Stokes system with $f \neq 0$

Theorem 6.1. *If data* $f(x,t)$ *satisfies the assumptions, then there exists a unique solution* (v,p) *of the Stokes system* (5.1) *such that, for any* $\eta \in (0,T)$ *and for* $\beta \in (0,1)$,

$$v \in C(0,T; C(\overline{\mathbb{R}^n_+})), \quad D^2 v, v_t, \nabla p \in C^{0,\beta}(\eta, T; C^{0,\beta}(\overline{\mathbb{R}^n_+}));$$

for any $x_\circ \in \mathbb{R}^n_+$, $\lim\limits_{(x,t)\to(x_\circ,0)} v(x,t) = 0$. *Moreover, if* $\mu \in (\frac{1}{2}, n)$ *then*

$$|v(x,t)| \le c\frac{L(t)K(t)}{(1+|x|)^\mu} t^{\frac{1}{2}}, \quad |\nabla v(x,t)| \le c\frac{L(t)K(t)}{(1+|x|)^\mu}, \tag{6.1}$$

for any $(x,t) \in \mathbb{R}^n_+ \times [0,T)$; *for* $\alpha \in (0,1)$,

$$|v(\overline{x},\overline{t}) - v(\overline{\overline{x}},\overline{t})| \le cL(t_1)K(t_1)t_0^{-\frac{\alpha}{2}}(|\overline{x}-\overline{\overline{x}}|^2 + |\overline{t}-\overline{\overline{t}}|)^{\frac{\alpha}{2}}, \tag{6.2}$$

$$|\nabla v(\overline{x},\overline{t}) - \nabla v(\overline{\overline{x}},\overline{t})| \le cL(t_1)K(t_1)t_0^{-\frac{1}{2}-\frac{\alpha}{2}}(|\overline{x}-\overline{\overline{x}}|^2 + |\overline{t}-\overline{\overline{t}}|)^{\frac{\alpha}{2}}, \tag{6.3}$$

for any $\overline{x}, \overline{\overline{x}} \in \mathbb{R}^n_+$, $\overline{t}, \overline{\overline{t}} \in (\eta, T)$, *with* $t_0 = \min\{\overline{t}, \overline{\overline{t}}\}$, $t_1 = \max\{\overline{t}, \overline{\overline{t}}\}$. *Finally, if* $\mu \in [1,n)$ *then, for any* $\nu \in [0,\mu]$,

$$\begin{aligned} |v(x,t)| &\le c\frac{L(t)K(t)}{(1+|x|)^{\mu-\nu}} \frac{1}{(1+t)^{\frac{\nu}{2}}}, \\ t^{\frac{1}{2}}|\nabla v(x,t)| &\le c\frac{L(t)K(t)}{(1+|x|)^{\mu-\nu}} \frac{1}{(1+t)^{\frac{\nu}{2}}}, \end{aligned} \tag{6.4}$$

for any $(x,t) \in \mathbb{R}^n_+ \times [0,T)$;

$$|v(\overline{x},t) - v(\overline{\overline{x}},t)| \le cL(t)K(t)t^{-\frac{\alpha}{2}} \frac{|\overline{x}-\overline{\overline{x}}|^\alpha}{R(|\overline{x}|, |\overline{\overline{x}}|)^{\mu-\nu}} \frac{1}{(1+t)^{\frac{\nu}{2}}}, \tag{6.5}$$

$$|\nabla v(\overline{x},t)-\nabla v(\overline{\overline{x}},t)|\le cL(t)K(t)t^{-\frac{1}{2}-\frac{\alpha}{2}}\frac{|\overline{x}-\overline{\overline{x}}|^{\alpha}}{R(|\overline{x}|,|\overline{\overline{x}}|)^{\mu-\nu}}\frac{1}{(1+t)^{\frac{\nu}{2}}},\qquad (6.6)$$

for any $\overline{x},\overline{\overline{x}}\in\mathbb{R}^n_+$ and $t\in(\eta,T)$. The constant c is independent of f.

Proof. From Lemma 3.1, a solution of system (5.1) is given by

$$v(x,t)=\int_0^t\int_{\mathbb{R}^n_+}\mathcal{G}(x,y,t-\tau)\mathfrak{f}(y,\tau)dyd\tau,$$

$$p(x,t)=\int_0^t\int_{\mathbb{R}^n_+}\mathcal{Q}(x,y,t-\tau)\cdot\mathfrak{f}(y,\tau)dyd\tau+\Psi(x,t).$$

Recalling the expression of $\mathfrak{f}$ given by (3.8), integrating by parts and using property $l(x',t)=k(x',t)=0$ on $\mathbb{R}^{n-1}\times(0,T)$, we write

$$v_i(x,t)=\int_0^t\int_{\mathbb{R}^n_+}D_{y_p}G_{ij}(x,y,t-\tau)(l_pk_j)(y,\tau)dyd\tau$$

$$-\int_0^t\int_{\mathbb{R}^n_+}D_{y_j}G_{ij}(x,y,t-\tau)(l_nk_n)(y,\tau)dyd\tau$$

$$-\widehat{\delta}_{rm}\int_0^t\int_{\mathbb{R}^n_+}D_{y_r}G_{in}(x,y,t-\tau)(l_mk_n+l_nk_m)(y,\tau)dyd\tau$$

$$+\int_0^t\int_{\mathbb{R}^n_+}G_{ij}(x,y,t-\tau)h_j(y,\tau)dyd\tau,$$

where, after two integration by parts and taking into account that $D_{x'}G(x,y,s)=-D_{y'}G(x,y,s)$, the last integral can be written as a linear combination of the expressions

$$D_{x_sx_m}\int_0^t\int_{\mathbb{R}^n_+}G_{ij}(x,y,t-\tau)\left(\int_{\mathbb{R}^n_+}D_{y_q}\mathcal{N}^{\pm}(y,z)(l_pk_r)(z,\tau)dz\right)dyd\tau$$

$$=\int_0^t\int_{\mathbb{R}^n_+} D_{x_s x_m} K_{ijq}(x,z,t-\tau)(l_p k_r)(z,\tau)dz,$$

where $m, s \neq n$. Then, from estimates (3.6) and (3.9) of the kernels and the assumptions on l, k, we deduce

$$|v(x,t)| \leq c\int_0^t\int_{\mathbb{R}^n_+} \frac{L(\tau)K(\tau)}{[|x-y|+(t-\tau)^{\frac{1}{2}}]^{n+1}}dyd\tau$$

$$+c\int_0^t\int_{\mathbb{R}^n_+}\frac{L(\tau)K(\tau)}{[|x-y^*|+(t-\tau)^{\frac{1}{2}}]^{n}}\frac{e^{-\widehat{a}\frac{y_n^2}{(t-\tau)}}}{(t-\tau)^{\frac{1}{2}}}dyd\tau$$

$$\leq cL(t)K(t)t^{\frac{1}{2}}, \quad \forall (x,t) \in \mathbb{R}^n_+ \times (0,T).$$

This estimate in particular ensures that $\lim\limits_{t\to 0} v(x,t) = 0$, uniformly in $x \in \mathbb{R}^n_+$. Our first goal is to obtain inequalities (6.1) and (6.4). We just prove the inequalities related to ∇v, since the proof is similar to the one for v, but more involved. The gradient of v has the following expression

$$\nabla v_i(x,t) = -\int_0^t\int_{\mathbb{R}^n_+}\nabla_x G_{ij}(x,y,t-\tau)D_{y_p}(l_p k_j)(y,\tau)dyd\tau$$

$$+\int_0^t\int_{\mathbb{R}^n_+} \nabla_x G_{ij}(x,y,t-\tau)D_{y_j}(l_n k_n)(y,\tau)dyd\tau$$

$$+\widehat{\delta}_{rm}\int_0^t\int_{\mathbb{R}^n_+}\nabla_x G_{in}(x,y,t-\tau)D_{y_r}(l_m k_n + l_n k_m)(y,\tau)dyd\tau$$

$$+\int_0^t\int_{\mathbb{R}^n_+}\nabla_x G_{ij}(x,y,t-\tau)h_j(y,\tau)dyd\tau = \sum_{i=1}^4 I_i(x,t),$$

where I_4 is a linear combination of the expressions

$$\widetilde{I}(x,t) = \int_0^t \int_{\mathbb{R}^n_+} \nabla_x G_{ij}(x,y,t-\tau) D_{y_s y_m} \Big(\int_{\mathbb{R}^n_+} D_{y_q} \mathcal{N}^{\pm}(y,z)(l_p k_r)(z,\tau)dz \Big) dy d\tau,$$
$$\text{where} \quad m,\ s \neq n. \tag{6.7}$$

Let us consider the general case where $\mu \in (\frac{1}{2}, n)$. Taking into account Lemma 5.1, since l and k satisfy assumption (i), we have

$$\left| D_{y_s y_m} \int_{\mathbb{R}^n_+} D_{y_q} \mathcal{N}^{\pm}(y,z)(l_p k_r)(z,\tau)dz \right| \leq \frac{c}{\tau^{\frac{1}{2}}} \frac{L(\tau)K(\tau)}{(1+|y|)^{\mu}}.$$

Then, using the previous estimate, assumption (i) and estimate (3.6) of the kernel, we have

$$|\nabla v_i(x,t)| \leq c \int_0^t \int_{\mathbb{R}^n_+} \frac{L(\tau)K(\tau)(1+|y|)^{-\mu}\tau^{-\frac{1}{2}}}{[|x-y| + (t-\tau)^{\frac{1}{2}}]^{n+1}} dy d\tau$$

$$+c \int_0^t \int_{\mathbb{R}^n_+} \frac{L(\tau)K(\tau)(1+|y|)^{-\mu}\tau^{-\frac{1}{2}}}{[|x-y^*| + (t-\tau)^{\frac{1}{2}}]^{n}} \frac{e^{-\widehat{a}\frac{y_n^2}{t-\tau}}}{(t-\tau)^{\frac{1}{2}}} dy d\tau.$$

By applying Lemma 5.2 on the first term and using for the second one the following inequality (obtained in the proof of Theorem 4.2)

$$\int_{\mathbb{R}^n_+} e^{-\widehat{a}\frac{y_n^2}{s}} \frac{(1+|y|)^{-\mu}}{[|x-y^*|^2 + s]^{\frac{n}{2}}} dy \leq c(1+|x|)^{-\mu}, \qquad \text{uniformly in } (x,t) \in \mathbb{R}^n_+ \times (0,T), \tag{6.8}$$

we easily get

$$|\nabla v_i(x,t)| \leq c \frac{L(t)K(t)}{(1+|x|)^{\mu}}, \quad \forall (x,t) \in \mathbb{R}^n_+ \times (0,T),$$

that is estimate $(6.1)_2$.

We now deal with the case $\mu \in [1, n)$.[2] The idea, which often recurs in the sequel, is the following one: we prove the estimate in the limiting cases $\nu = 0$ and $\nu = \mu$, and finally we get $(6.4)_2$ by combining these results. We consider $\sum\limits_{i=1}^{3} I_i$ and, separately, I_4. Integrating by parts in $(0, \frac{t}{2})$ and recalling that $l(x', t) = k(x', t) = 0$ on $\mathbb{R}^{n-1} \times (0, T)$, for the first three terms we get

$$\begin{aligned}
\sum_{i=1}^{3} I_i(x,t) = &\int_0^{\frac{t}{2}}\int_{\mathbb{R}^n_+} \nabla_x D_{y_p} G_{ij}(x,y,t-\tau)(l_p k_j)(y,\tau)dyd\tau \\
&- \int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+} \nabla_x G_{ij}(x,y,t-\tau) D_{y_p}(l_p k_j)(y,\tau)dyd\tau \\
&- \int_0^{\frac{t}{2}}\int_{\mathbb{R}^n_+} \nabla_x D_{y_j} G_{ij}(x,y,t-\tau)(l_n k_n)(y,\tau)dyd\tau \\
&+ \int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+} \nabla_x G_{ij}(x,y,t-\tau) D_{y_j}(l_n k_n)(y,\tau)dyd\tau \\
-\widehat{\delta}_{rm} &\int_0^{\frac{t}{2}}\int_{\mathbb{R}^n_+} \nabla_x D_{y_r} G_{in}(x,y,t-\tau)(l_m k_n + l_n k_m)(y,\tau)dyd\tau \\
+\widehat{\delta}_{rm} &\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+} \nabla_x G_{in}(x,y,t-\tau) D_{y_r}(l_m k_n + l_n k_m)(y,\tau)dyd\tau. \qquad (6.9)
\end{aligned}$$

[2] If the initial data of system (1.1) belongs to $\mathcal{M}^{\mu}_{|x|}(\mathbb{R}^n_+)$ for some $\mu \in (\frac{1}{2}, n)$, we prove the existence locally in time of a vector field of velocities, solution of the system, satisfying pointwise estimates of the kind (6.1), without restrictions on the size of the data. If $\mu \in [1, n)$ and the data is *sufficiently* small, we are able to prove the existence of a solution of the Navier–Stokes system (1.1) global in time, provided that different estimates on the solutions are performed. This fact justifies the double study of the case $\mu \in [1, n)$.

We first show inequality $(6.4)_2$ with $\nu = 0$. Taking into account estimate (3.6) and using assumption (ii) we have

$$\Big|\sum_{i=1}^{3} I_i(x,t)\Big| \leq cL(t)K(t)\Big[\int_0^{\frac{t}{2}}\int_{\mathbb{R}^n_+} \frac{(1+|y|)^{1-2\mu}(1+\tau)^{-\frac{1}{2}}}{\big[|x-y|+(t-\tau)^{\frac{1}{2}}\big]^{n+2}} dyd\tau$$

$$+c\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+} \Big[\frac{(1+|y|)^{1-2\mu}(1+\tau)^{-\frac{1}{2}}\tau^{-\frac{1}{2}}}{\big[|x-y|+(t-\tau)^{\frac{1}{2}}\big]^{n+1}} dyd\tau$$

$$+c\int_0^{\frac{t}{2}}\int_{\mathbb{R}^n_+} \frac{(1+|y|)^{1-2\mu}(1+\tau)^{-\frac{1}{2}}}{\big[|x-y^*|+(t-\tau)^{\frac{1}{2}}\big]^{n}} \frac{e^{-\widehat{a}\frac{y_n^2}{t-\tau}}}{(t-\tau)} dyd\tau$$

$$+c\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+} \frac{(1+|y|)^{1-2\mu}(1+\tau)^{-\frac{1}{2}}\tau^{-\frac{1}{2}}}{\big[|x-y^*|+(t-\tau)^{\frac{1}{2}}\big]^{n}} \frac{e^{-\widehat{a}\frac{y_n^2}{t-\tau}}}{(t-\tau)^{\frac{1}{2}}} dyd\tau\Big]$$

$$\leq \frac{c}{t^{\frac{1}{2}}} L(t)K(t)\int_0^{t}\int_{\mathbb{R}^n_+} \Big[\frac{(1+|y|)^{-\mu}(1+\tau)^{-\frac{1}{2}}}{\big[|x-y|+(t-\tau)^{\frac{1}{2}}\big]^{n+1}}$$

$$+\frac{(1+|y|)^{-\mu}(1+\tau)^{-\frac{1}{2}}}{\big[|x-y^*|+(t-\tau)^{\frac{1}{2}}\big]^{n}} \frac{e^{-\widehat{a}\frac{y_n^2}{t-\tau}}}{(t-\tau)^{\frac{1}{2}}}\Big] dyd\tau.$$

Using Lemma 5.2 for the first integral and estimate (6.8) for the second one, we readily get

$$\Big|\sum_{i=1}^{3} I_i(x,t)\Big| \leq c\frac{L(t)K(t)}{(1+|x|)^{\mu}} t^{-\frac{1}{2}}, \quad \forall (x,t) \in \mathbb{R}^n_+ \times (0,T). \tag{6.10}$$

Therefore estimate $(6.4)_2$ for the terms I_i, $i = 1,2,3$, with $\nu = 0$ is proved. In order to prove $(6.4)_2$ for these terms with $\nu = \mu$, first assume $t \geq 1$. Taking into account (6.9), in virtue of estimate (3.6) and assumptions (ii), we have

$$\Big|\sum_{i=1}^{3} I_i(x,t)\Big| \leq cL(t)K(t)\Big[\int_0^{\frac{t}{2}}\int_{\mathbb{R}^n_+} \frac{(1+|y|)^{1-2\mu}(1+\tau)^{-\frac{1}{2}}}{\big[|x-y|+(t-\tau)^{\frac{1}{2}}\big]^{n+2}} dyd\tau$$

$$+c\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\frac{(1+|y|)^{1-\mu}(1+\tau)^{-\frac{\mu}{2}-\frac{1}{2}}\tau^{-\frac{1}{2}}}{\left[|x-y|+(t-\tau)^{\frac{1}{2}}\right]^{n+1}}dyd\tau$$

$$+c\int_{0}^{\frac{t}{2}}\int_{\mathbb{R}^n_+}\frac{(1+|y|)^{1-2\mu}(1+\tau)^{-\frac{1}{2}}}{\left[|x-y^*|+(t-\tau)^{\frac{1}{2}}\right]^{n}}\frac{e^{-\widehat{a}\frac{y_n^2}{(t-\tau)}}}{(t-\tau)}dyd\tau$$

$$+c\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\frac{(1+|y|)^{1-\mu}(1+\tau)^{-\frac{\mu}{2}-\frac{1}{2}}\tau^{-\frac{1}{2}}}{\left[|x-y^*|+(t-\tau)^{\frac{1}{2}}\right]^{n}}\frac{e^{-\widehat{a}\frac{y_n^2}{(t-\tau)}}}{(t-\tau)^{\frac{1}{2}}}dyd\tau\Bigg]$$

$$=cL(t)K(t)\sum_{i=1}^{4}\widehat{J}_i(x,t). \tag{6.11}$$

For the first term, using the hypothesis $\mu\geq 1$ and then applying Lemma 5.3, we obtain

$$\widehat{J}_1(x,t)\leq c\int_{0}^{\frac{t}{2}}\int_{\mathbb{R}^n_+}\frac{(1+|y|)^{-\mu}(1+\tau)^{-\frac{1}{2}}}{\left[|x-y|+(t-\tau)^{\frac{1}{2}}\right]^{n+2}}dyd\tau\leq c\frac{t^{-\frac{1}{2}}}{(1+t)^{\frac{\mu}{2}}},\quad\forall t\geq 1.$$

For $\widehat{J}_2$, since $\mu\geq 1$, we have

$$\widehat{J}_2(x,t)\leq c\left(\frac{2}{2+t}\right)^{1+\frac{\mu}{2}}\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\frac{dyd\tau}{\left[|x-y|+(t-\tau)^{\frac{1}{2}}\right]^{n+1}}\leq c\frac{t^{-\frac{1}{2}}}{(1+t)^{\frac{\mu}{2}}},\quad\forall t\geq 1.$$

Applying Lemma 5.4 with $\gamma_1=\mu$ and $\gamma_2=\frac{1}{2}$, we get for $\widehat{J}_3$

$$\widehat{J}_3(x,t)\leq\frac{c}{t}\int_{0}^{\frac{t}{2}}\int_{\mathbb{R}^n_+}\frac{(1+|y|)^{-\mu}(1+\tau)^{-\frac{1}{2}}}{\left[|x-y^*|+(t-\tau)^{\frac{1}{2}}\right]^{n}}e^{-\widehat{a}\frac{y_n^2}{(t-\tau)}}dyd\tau$$

$$\leq c\frac{t^{-\frac{1}{2}}}{(1+t)^{\frac{\mu}{2}}},\quad\forall t\geq 1.$$

For $\widehat{J}_4$, using estimate (6.8), we have

$$\widehat{J}_4(x,t)\leq c\left(\frac{2}{2+t}\right)^{1+\frac{\mu}{2}}\int_{\frac{t}{2}}^{t}(t-\tau)^{-\frac{1}{2}}\int_{\mathbb{R}^n_+}\frac{e^{-\widehat{a}\frac{y_n^2}{(t-\tau)}}}{\left[|x-y^*|+(t-\tau)^{\frac{1}{2}}\right]^{n}}dyd\tau$$

$$\leq ct^{-\frac{1}{2}-\frac{\mu}{2}}, \quad \forall t \geq 1.$$

Therefore, recalling inequality (6.11), the estimates obtained for each $\widehat{J}_i$, $i = 1, \ldots, 4$, imply

$$\Big|\sum_{i=1}^{3} I_i(x,t)\Big| \leq \frac{c}{t^{\frac{1}{2}}}\frac{L(t)K(t)}{(1+t)^{\frac{\mu}{2}}}, \quad \forall x \in \mathbb{R}^n_+, \forall t \geq 1.$$

On the other hand, if $t < 1$ from the validity of $(6.1)_2$, we get

$$\Big|\sum_{i=1}^{3} I_i(x,t)\Big| \leq cL(t)K(t) \leq \frac{c}{t^{\frac{1}{2}}}\frac{L(t)K(t)}{(1+t)^{\frac{\mu}{2}}}.$$

Therefore

$$\Big|\sum_{i=1}^{3} I_i(x,t)\Big| \leq \frac{c}{t^{\frac{1}{2}}}\frac{L(t)K(t)}{(1+t)^{\frac{\mu}{2}}}, \quad \forall (x,t) \in \mathbb{R}^n_+ \times (0,T). \tag{6.12}$$

Now, from (6.10) and (6.12) an easy computation gives $(6.4)_2$ for $\sum_{i=1}^{3} I_i(x,t)$. It remains to estimate I_4. Recalling (6.7) and integrating by parts, I_4 can be written as a linear combination of terms of the kind

$$\begin{aligned}\widetilde{I}(x,t) &= \int_0^{\frac{t}{2}}\int_{\mathbb{R}^n_+} \nabla_x D_{x_s x_m} K_{ijq}(x,z,t-\tau)(l_p k_r)(z,\tau)dzd\tau \\ &+ \int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+} \nabla_x D_{x_s} K_{ijq}(x,z,t-\tau) D_{z_m}(l_p k_r)(z,\tau)dzd\tau \\ &= \widetilde{J}_1(x,t) + \widetilde{J}_2(x,t).\end{aligned} \tag{6.13}$$

Recalling the estimate of K_{ijq} found in Proposition 3.1, applying first Lemma 5.2, then Lemma 5.3 and combining the estimates thus obtained, we get, for any $\nu \in [0,\mu]$,

$$\begin{aligned}|\widetilde{J}_1(x,t)| &\leq cL(t)K(t)\int_0^{\frac{t}{2}}\int_{\mathbb{R}^n_+}\frac{(1+|z|)^{-\mu}(1+\tau)^{-\frac{1}{2}}}{\big[|x-z|+(t-\tau)^{\frac{1}{2}}\big]^{n+1}}\frac{dzd\tau}{(t-\tau)^{\frac{1}{2}}} \\ &\leq c\frac{L(t)K(t)}{(1+|x|)^{\mu-\nu}}\frac{t^{-\frac{1}{2}}}{(1+t)^{\frac{\nu}{2}}}, \quad \forall (x,t) \in \mathbb{R}^n_+ \times (0,T).\end{aligned}$$

In order to estimate $\widetilde{J}_2$ we employ Proposition 3.1 and then use Lemma 5.5. Then we get

$$|\widetilde{J}_2(x,t)| \le c\frac{L(t)K(t)}{(1+|x|)^{\mu}}t^{-1}\int\limits_{\frac{t}{2}}^{t}\frac{d\tau}{(t-\tau)^{\frac{1}{2}}}$$

$$+c\frac{L(t)K(t)}{(1+|x|)^{2\mu-a}}t^{-\frac{a}{2}-\frac{1}{2}}\int\limits_{\frac{t}{2}}^{t}\frac{d\tau}{(t-\tau)^{\frac{1}{2}+\frac{b}{2}}} \le c\frac{L(t)K(t)}{(1+|x|)^{\mu}}t^{-\frac{1}{2}},$$

uniformly in $(x,t) \in \mathbb{R}^n_+ \times (0,T)$. Moreover, using Proposition 3.1 and then Lemma 5.5, for $t \ge 1$ we have

$$|\widetilde{J}_2(x,t)| \le cL(t)K(t)t^{-1-\frac{\mu}{2}}\int\limits_{\frac{t}{2}}^{t}\frac{d\tau}{(t-\tau)^{\frac{1}{2}}} + cL(t)K(t)t^{-1-\frac{\mu}{2}+\frac{b}{2}}\int\limits_{\frac{t}{2}}^{t}\frac{d\tau}{(t-\tau)^{\frac{1}{2}+\frac{b}{2}}}$$

$$\le c\frac{L(t)K(t)}{(1+t)^{\frac{\mu}{2}}}t^{-\frac{1}{2}}, \forall x \in \mathbb{R}^n_+,\ \ \forall t \ge 1.$$

If $t < 1$, since we have already shown estimate $(6.1)_2$, we get

$$|\widetilde{J}_2(x,t)| \le cL(t)K(t) \le c\frac{L(t)K(t)}{(1+t)^{\frac{\mu}{2}}}t^{-\frac{1}{2}}, \quad \forall x \in \mathbb{R}^n_+, \quad \forall t < 1.$$

Once we combine these results, we completely prove the estimate $(6.4)_2$ for I_4 and hence for ∇v.

The proof of (6.2), (6.3), (6.5), and (6.6) comes very easily from the previous estimates. Nevertheless, for the sake of completeness, we just sketch the proof of (6.6), that seems the most involved. Taking into account the expression (6.9) and then using the Hölder property of the heat kernel and the Hölder property (4.2) of G^*, recalling hypothesis (ii), we have

$$\sum_{i=1}^{3}|I_i(\overline{x},t) - I_i(\overline{\overline{x}},t)|$$

$$\le cL(t)K(t)|\overline{x}-\overline{\overline{x}}|^{\alpha}\Big[\int\limits_0^{\frac{t}{2}}\int\limits_{\mathbb{R}^n_+}\frac{(1+|y|)^{1-2\mu}(1+\tau)^{-\frac{1}{2}}}{[\min\{|\overline{x}-y|,|\overline{\overline{x}}-y|\}+(t-\tau)^{\frac{1}{2}}]^{n+2+\alpha}}dyd\tau$$

$$+c\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\Big[\frac{(1+|y|)^{1-2\mu}(1+\tau)^{-\frac{1}{2}}\tau^{-\frac{1}{2}}}{[\min\{|\overline{x}-y|,|\overline{\overline{x}}-y|\}+(t-\tau)^{\frac{1}{2}}]^{n+1+\alpha}}dyd\tau$$

$$+c\int_{0}^{\frac{t}{2}}\int_{\mathbb{R}^n_+}\frac{(1+|y|)^{1-2\mu}(1+\tau)^{-\frac{1}{2}}}{[\min\{|\overline{x}-y^*|,|\overline{\overline{x}}-y^*|\}+(t-\tau)^{\frac{1}{2}}]^n}\frac{e^{-\widehat{a}\frac{y_n^2}{t-\tau}}}{(t-\tau)^{1+\frac{\alpha}{2}}}dyd\tau$$

$$+c\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\frac{(1+|y|)^{1-2\mu}(1+\tau)^{-\frac{1}{2}}\tau^{-\frac{1}{2}}}{[\min\{|\overline{x}-y^*|,|\overline{\overline{x}}-y^*|\}+(t-\tau)^{\frac{1}{2}}]^n}\frac{e^{-\widehat{a}\frac{y_n^2}{t-\tau}}}{(t-\tau)^{\frac{1}{2}+\frac{\alpha}{2}}}dyd\tau\Big].$$

Therefore

$$\sum_{i=1}^{3}|I_i(\overline{x},t)-I_i(\overline{\overline{x}},t)|$$

$$\leq ct^{-\frac{1}{2}}L(t)K(t)|\overline{x}-\overline{\overline{x}}|^\alpha\Big[\int_0^t\int_{\mathbb{R}^n_+}\frac{(1+|y|)^{-\mu}(1+\tau)^{-\frac{1}{2}}}{[\min\{|\overline{x}-y|,|\overline{\overline{x}}-y|\}+(t-\tau)^{\frac{1}{2}}]^{n+1+\alpha}}dyd\tau$$

$$+\int_0^t\int_{\mathbb{R}^n_+}\frac{(1+|y|)^{-\mu}(1+\tau)^{-\frac{1}{2}}}{[\min\{|\overline{x}-y^*|,|\overline{\overline{x}}-y^*|\}+(t-\tau)^{\frac{1}{2}}]^n}\frac{e^{-\widehat{a}\frac{y_n^2}{t-\tau}}}{(t-\tau)^{\frac{1}{2}+\frac{\alpha}{2}}}dyd\tau\Big].$$

In view of Lemma 5.2 and estimate (6.8) it is easily seen that

$$\sum_{i=1}^{3}|I_i(\overline{x},t)-I_i(\overline{\overline{x}},t)|\leq ct^{-\frac{1}{2}-\frac{\alpha}{2}}L(t)K(t)\frac{|\overline{x}-\overline{\overline{x}}|^\alpha}{R(|\overline{x}|,|\overline{\overline{x}}|)}\mu.$$

Similarly one can argue in order to obtain the estimate with $\nu=\mu$. Let us estimate I_4. Recalling expression (6.13), using the Hölder property of the kernel K_{ijq}, consequence of Proposition 3.1, and hypothesis (ii), we have

$$|\widetilde{I}(\overline{x},t)-\widetilde{I}(\overline{\overline{x}},t)|$$

$$\leq c|\overline{x}-\overline{\overline{x}}|^\alpha\Big[\int_0^{\frac{t}{2}}\int_{\mathbb{R}^n_+}\frac{|(l\otimes k)(z,\tau)|}{[\min\{|\overline{x}-z|,|\overline{\overline{x}}-z|\}+(t-\tau)^{\frac{1}{2}}]^n}\frac{dzd\tau}{(t-\tau)^{1+\frac{\alpha}{2}}}$$

$$+c\int_{\frac{t}{2}}^{t}\int_{\mathbb{R}^n_+}\frac{|D_z(l\otimes k)(z,\tau)|}{[\min\{|\overline{x}-z|,|\overline{\overline{x}}-z|\}+(t-\tau)^{\frac{1}{2}}]^n}\frac{dzd\tau}{(t-\tau)^{\frac{1}{2}+\frac{\alpha}{2}}}\Big].$$

Using inequality $(5.7)_1$ of Lemma 5.5, with $b \in (0,1)$ such that $\alpha + b < 1$, it is easy to see that

$$|\widetilde{I}(\overline{x},t) - \widetilde{I}(\overline{\overline{x}},t)| \leq ct^{-\frac{1}{2}-\frac{\alpha}{2}} L(t)K(t) \frac{|\overline{x}-\overline{\overline{x}}|^{\alpha}}{R(|\overline{x}|,|\overline{\overline{x}}|)^{\mu}}.$$

Moreover, using inequalities $(5.7)_{2,3}$ of Lemma 5.5 with the same choice of b, it is easy to verify that

$$|\widetilde{I}(\overline{x},t) - \widetilde{I}(\overline{\overline{x}},t)| \leq ct^{-\frac{1}{2}-\frac{\alpha}{2}} L(t)K(t) \frac{|\overline{x}-\overline{\overline{x}}|^{\alpha}}{(1+t)^{\frac{\mu}{2}}}, \quad \forall t \geq 1$$

and, for $t < 1$,

$$|\widetilde{I}(\overline{x},t) - \widetilde{I}(\overline{\overline{x}},t)| \leq ct^{-\frac{1}{2}-\frac{\alpha}{2}} L(t)K(t)|\overline{x}-\overline{\overline{x}}|^{\alpha} \leq ct^{-\frac{1}{2}-\frac{\alpha}{2}} L(t)K(t) \frac{|\overline{x}-\overline{\overline{x}}|^{\alpha}}{(1+t)^{\frac{\mu}{2}}}.$$

Combining the results thus obtained for each term I_i, $i = 1, \dots, 4$, we arrive at

$$|\nabla v(\overline{x},t) - \nabla v(\overline{\overline{x}},t)| \leq ct^{-\frac{1}{2}-\frac{\alpha}{2}} L(t)K(t) \frac{|\overline{x}-\overline{\overline{x}}|^{\alpha}}{R(|\overline{x}|,|\overline{\overline{x}}|)^{\mu-\nu}(1+t)^{\frac{\nu}{2}}},$$

for any $\overline{x}, \overline{\overline{x}} \in \mathbb{R}^n_+$, for any $t > 0$. The Hölder property with respect to the time variable can be obtained with similar estimates, using the Hölder property of the heat kernel and the Hölder property of G^* and K_{ijq}, consequence of the regularities given by estimate (3.4) and Proposition 3.1, and of an application of the Lagrange theorem.

Let us prove the regularity of the second derivatives of v. For $\overline{t} \in (0,T)$, we introduce a smooth cut-off function $\varphi(\frac{\tau}{\overline{t}})$ satisfying the conditions $\varphi(\frac{4\tau}{\overline{t}}) \in [0,1]$, $\varphi(\frac{4\tau}{\overline{t}}) = 1$ for $\tau \leq \frac{\overline{t}}{4}$, $\varphi(\frac{4\tau}{\overline{t}}) = 0$ for $\tau \geq \frac{\overline{t}}{2}$. Introducing $\varphi(\frac{4\tau}{\overline{t}})$ in the expression of v we obtain for any $t \geq \overline{t}$

$$v(x,t) = \int_{\frac{\overline{t}}{4}}^{t} \int_{\mathbb{R}^n_+} \mathcal{G}(x,y,t-\tau)(1-\varphi(4\tau/\overline{t}))\mathfrak{f}(y,\tau)dyd\tau$$

$$+ \int_0^{\frac{\overline{t}}{2}} \int_{\mathbb{R}^n_+} \mathcal{G}(x,y,t-\tau)\varphi(4\tau/\overline{t})\mathfrak{f}(y,\tau)dyd\tau = I_1(x,t) + I_2(x,t).$$

From the regularity of the kernel $\mathcal{G}$ we easily deduce $D^2_x I_2, D_t I_2 \in C^{0,\alpha}(\eta,T;C^{0,\alpha}(\overline{\mathbb{R}^n_+}))$. Setting $\widehat{\mathfrak{f}}(y,s) = \mathfrak{f}(y,s)(1-\varphi(\frac{4s}{\overline{t}}))$, by the definition of φ we have $\widehat{\mathfrak{f}}(y',0,\tau)=0$, for any $\tau\le\frac{\bar{t}}{4}$, and $\widehat{\mathfrak{f}}\in C^{0,\beta}(\frac{\eta}{4},T;C^{0,\beta}(\overline{\mathbb{R}^n_+}))$, with $\beta=\beta(\alpha,\alpha_1)$. Then using a result due to Solonnikov ([16, Theorem 7.1]), we have $D_x^2I_1, D_tI_1\in C^{0,\beta}(\frac{\eta}{4},T;C^{0,\beta}(\overline{\mathbb{R}^n_+}))$. Finally, the regularity of the gradient of pressure p is deduced from the system and the regularity of D^2v and v_t.

7. Estimates on the pressure field solution of the Stokes system with $f\neq 0$

Proposition 7.1. *Let (v,p) be the solution given in Theorem 6.1. For the pressure field we have:*

if $\mu\in(\frac{1}{2},n)$, then

$$|p(x,t)|\le C\begin{cases}\Big(1+\log\Big(\dfrac{3|x|}{t^{\frac12}}+1\Big)\Big)\dfrac{L(t)K(t)}{(1+|x|)^{\mu}}(1+t^{-\frac12}),\\ \qquad\qquad\qquad if\quad \mu\in(\dfrac{1}{2},n-1),\\ \Big(1+\log\Big(1+\dfrac{|x|}{2}\Big)+\log\Big(\dfrac{3|x|}{t^{\frac12}}+1\Big)\Big)\dfrac{L(t)K(t)}{(1+|x|)^{n-1}}(1+t^{-\frac12}),\\ \qquad\qquad\qquad if\ \mu=n-1,\\ \Big(1+\log\Big(\dfrac{3|x|}{t^{\frac12}}+1\Big)\Big)\dfrac{L(t)K(t)}{(1+|x|)^{n-1}}(1+t^{-\frac12}),\\ \qquad\qquad\qquad if\ \mu\in(n-1,n),\end{cases} \tag{7.1}$$

for any $(x,t)\in\mathbb{R}^n_+\times(0,T)$;

if $\mu\in[1,n)$, then for any $\nu\in[0,\mu]$ and $\nu_1\in[0,n-1]$

$$t^{\frac12}|p(x,t)|\le C\begin{cases}\dfrac{L(t)K(t)}{(1+|x|)^{\mu-\nu}}\dfrac{1}{(1+t)^{\frac{\nu}{2}}}, & if\ \mu\in[1,n-1],\\ \dfrac{L(t)K(t)}{(1+|x|)^{n-1-\nu_1}}\dfrac{1}{(1+t)^{\frac{\nu_1}{2}}}, & if\ \mu\in[n-1,n),\end{cases} \tag{7.2}$$

for any $(x,t)\in\mathbb{R}^n_+\times[0,\infty)$.

The constant C is independent of f.

Proof. From Lemma 3.1, we have

$$\begin{aligned}p(x,t)&=\int_0^t\int_{\mathbb{R}^n_+}\mathcal{Q}(x,y,t-\tau)\cdot\mathfrak{f}(y,\tau)dyd\tau-\int_{\mathbb{R}^n_+}\nabla_y\mathcal{N}(x,y)\cdot f(y,t)dy\\&=p_1(x,t)+p_2(x,t),\end{aligned}$$

where $\mathfrak{f}$ is defined by (3.8).

Assume $\mu \in (\frac{1}{2}, n)$. We consider first p_1 and, subsequently, p_2. Using estimate (3.4) we have

$$|p_1(x,t)| \leq \int_0^t \int_{\mathbb{R}^n_+} \frac{(y_n^2+(t-\tau))^{-\frac{1}{2}}}{[|x-y^*|+(t-\tau)^{\frac{1}{2}}]^{n-1}} \frac{e^{-\widehat{a}\frac{y_n^2}{(t-\tau)}}}{(t-\tau)^{\frac{1}{2}}} |\mathfrak{f}(y,\tau)|dyd\tau, \quad (7.3)$$

where, by the definition (3.8), by the assumptions and Lemma 5.1, we have

$$|\mathfrak{f}(y,\tau)| \leq \frac{c}{\tau^{\frac{1}{2}}} \frac{L(\tau)K(\tau)}{(1+|y|)^{\mu}}, \quad \forall (y,\tau) \in \mathbb{R}^n_+ \times (0,T).$$

Then, we can apply Lemma 5.6 and we get

$$|p_1(x,t)| \qquad (7.4)$$

$$\leq c \begin{cases} \Big(1+\log\Big(\dfrac{3|x|}{t^{\frac{1}{2}}}+1\Big)\Big)\dfrac{L(t)K(t)}{(1+|x|)^{\mu}}, \\ \text{if } \mu \in (0, n-1), \\[2ex] \Big(\log\Big(1+\dfrac{|x|}{2}\Big)+\log\Big(\dfrac{3|x|}{t^{\frac{1}{2}}}+1\Big)\Big)\dfrac{L(t)K(t)}{(1+|x|)^{n-1}}, \\ \text{if } \mu = n-1, \\[2ex] \Big(1+\log\Big(\dfrac{3|x|}{t^{\frac{1}{2}}}+1\Big)\Big)\dfrac{L(t)K(t)}{(1+|x|)^{n-1}}, \\ \text{if } \mu \in (n-1, n), \end{cases} \quad \forall (x,t) \in \mathbb{R}^n_+ \times (0,T).$$

We now consider p_2. Since $F = -l \otimes k \in C^{1,\alpha}(\overline{\mathbb{R}^n_+})$ and $l(x',t) = k(x',t) = 0$, we can extend function F to the whole $\mathbb{R}^n$ by setting $F(x,t) = 0$ on $\mathbb{R}^n_- \times (0,T)$, obtaining a function of $C^{1,\alpha}(\mathbb{R}^n)$. Still calling F the function thus obtained, F continues to satisfy the Assumptions. Then, recalling that $f = \nabla \cdot F$, we have

$$p_2(x,t) = -\int_{\mathbb{R}^n} \nabla_y \mathcal{E}(x-y) \cdot f(y,t)dy - \int_{\mathbb{R}^n} \nabla_y \mathcal{E}(x-y^*) \cdot f(y,t)dy = p_2^1 + p_2^2.$$

It is sufficient to study the term with the singular kernel. Integrating by parts we have

$$p_2^1(x,t) = -\int\limits_{|x-y|\le 2} \nabla_y \mathcal{E}(x-y)\cdot f(y,t)dy$$
$$+\int\limits_{|x-y|\ge 2} \nabla_y\nabla_y \mathcal{E}(x-y)\cdot (l\otimes k)(y,t)dy$$
$$+\int\limits_{|x-y|=2} (n\otimes \nabla_y \mathcal{E}(x-y))(l\otimes k)(y,t)d\sigma_y. \tag{7.5}$$

Since $\mu \in (\frac{1}{2}, n)$, l and k satisfy assumption (i). Then it is easy to check that

$$|p_2^1(x,t)| \le cL(t)K(t)(1+t^{-\frac{1}{2}}), \quad \text{uniformly in} \quad x\in \mathbb{R}^n. \tag{7.6}$$

This gives a pointwise estimate with respect to the time. Let us consider the behaviour with respect to the space variable. Integrating by parts, we have

$$\begin{aligned} p_2^1(x,t) = &-\int\limits_{0\le|x-y|\le\frac{|x|}{2}} \nabla_y \mathcal{E}(x-y)\cdot f(y,t)dy \\ &-\int\limits_{\frac{|x|}{2}\le|x-y|\le 2|x|} \nabla_y\nabla_y \mathcal{E}(x-y)\cdot (l\otimes k)(y,t)dy \\ &-\int\limits_{|x-y|\ge 2|x|} \nabla_y\nabla_z \mathcal{E}(x-y)\cdot (l\otimes k)(y,t)dy \\ &+\int\limits_{|x-y|=\frac{|x|}{2}} (n\otimes \nabla_y \mathcal{E}(x-y))\cdot (l\otimes k)(y,t)d\sigma_y \\ &=\sum_{i=1}^{4} I_i(x,t). \end{aligned} \tag{7.7}$$

In order to estimate I_1 we distinguish two cases: $|x|\le 2\sqrt{t}$ and $|x|>2\sqrt{t}$. In the first case, using assumption (i) on f we get

$$\begin{aligned} |I_1| &\le cL(t)K(t)t^{-\frac{1}{2}} \int\limits_{0\le|x-y|\le\frac{|x|}{2}} \frac{|x-y|^{1-n}}{(1+|y|)^{2\mu}}dy \\ &\le cL(t)K(t)t^{-\frac{1}{2}}|x|^{-2\mu+1} \le cL(t)K(t)|x|^{-2\mu}. \end{aligned}$$

In the case $|x| > 2\sqrt{t}$, integrating by parts we obtain

$$\begin{aligned}
|I_1| \le & \int\limits_{\varepsilon \le |x-y| \le \sqrt{t}} |\nabla_y \mathcal{E}(x-y)||f(y,t)|dy \\
& + \int\limits_{\sqrt{t} \le |x-y| \le \frac{|x|}{2}} |\nabla_y \nabla_y \mathcal{E}(x-y)||(l \otimes k)(y,t)|dy \\
& + \int\limits_{|x-y| = \frac{|x|}{2}} |\nabla_y \mathcal{E}(x-y)||(l \otimes k)(y,t)|d\sigma_y \\
& + \int\limits_{|x-y| = \sqrt{t}} |\nabla_y \mathcal{E}(x-y)||(l \otimes k)(y,t)|d\sigma_y \\
& \le c \frac{L(t)K(t)}{(1+|x|)^{2\mu}} \Big(1 + \log\Big(\frac{|x|}{2t^{\frac{1}{2}}}\Big)\Big) \\
& \le c \frac{L(t)K(t)}{(1+|x|)^{2\mu}} \Big(1 + \log\Big(\frac{3|x|}{t^{\frac{1}{2}}} + 1\Big)\Big).
\end{aligned}$$

In addition, it is easy to verify that

$$|I_2| + |I_3| + |I_4| \le c \frac{L(t)K(t)}{(1+|x|)^{2\mu}}.$$

Therefore, recalling (7.6), we get

$$\begin{gathered}
|p_2(x,t)| \le c \frac{L(t)K(t)}{(1+|x|)^{2\mu}} (1 + t^{-\frac{1}{2}}) \Big(1 + \log\Big(\frac{3|x|}{t^{\frac{1}{2}}} + 1\Big)\Big), \\
\forall (x,t) \in \mathbb{R}^n_+ \times (0,T).
\end{gathered} \tag{7.8}$$

Estimates (7.4) and (7.5) imply estimate (7.1) for function p.

We now improve the estimate in the case $\mu \in [1, n)$, by proving (7.2). By definition (3.8), the assumptions on l and k and Lemma 5.3, setting $\widehat{\mu} = \min\{2\mu, n+1\}$, we have, for any $\widehat{\nu} \in [0, \widehat{\mu}]$,

$$|\mathfrak{f}(y,\tau)| \le \frac{c}{\tau^{\frac{1}{2}}} \frac{L(\tau)K(\tau)}{(1+|y|)^{\widehat{\mu} - \widehat{\nu}}} \frac{1}{(1+\tau)^{\frac{\widehat{\nu}}{2}}}, \quad \forall (y,\tau) \in \mathbb{R}^n_+ \times (0,T).$$

Hence, recalling expression (7.3), we can apply for p_1 Lemma 5.7 and obtain estimate (7.2). Therefore, let us pass to consider function p_2. Obviously we can still use decompositions (7.5) and (7.7), since they are independent of μ. Using the assumption (ii) on f with $\nu=1$, from (7.5) it is easy to obtain

$$|p_2^1(x,t)| \le cL(t)K(t)t^{-\frac{1}{2}}, \quad \text{uniformly in } x\in\mathbb{R}^n. \tag{7.9}$$

Recalling the decomposition (7.7) of p_2^1 and assumption (ii) on f it is easy to get

$$\begin{aligned}|p_2^1| &\le \sum_{i=1}^4 |I_i| \le cL(t)K(t)t^{-\frac{1}{2}}\Bigg[\int\limits_{0\le|x-y|\le\frac{|x|}{2}} \frac{|x-y|^{1-n}}{(1+|y|)^{2\mu}}dy\\ &+\int\limits_{\frac{|x|}{2}\le|x-y|\le 2|x|}\frac{|x-y|^{-n}}{(1+|y|)^{2\mu-1}}dy+\int\limits_{|x-y|\ge 2|x|}\frac{|x-y|^{-n}}{(1+|y|)^{2\mu-1}}dy\\ &+\int\limits_{|x-y|=\frac{|x|}{2}}\frac{|x-y|^{-n}}{(1+|y|)^{2\mu-1}}d\sigma_y\Bigg] \le c\frac{L(t)K(t)}{(1+|x|)^\mu}t^{-\frac{1}{2}}, \quad \forall|x|\ge 1.\end{aligned}$$

Hence, we obtain

$$|p_2(x,t)| \le c\frac{L(t)K(t)}{(1+|x|)^\mu}t^{-\frac{1}{2}}, \quad \forall(x,t)\in\mathbb{R}^n_+\times(0,T), \tag{7.10}$$

that is (7.2) for p_2 with $\nu=0$.[3] Let us prove (7.2) for p_2 with $\nu=\mu$.

[3] The previous arguments also hold if $\mu\in(\frac{1}{2},1)$. However in that case we have used a different approach in order to obtain a constant c independent of μ.

Integrating by parts we have

$$
\begin{aligned}
p_2^1(x,t) = &- \int\limits_{0\le|x-y|\le\frac{|x|}{2}} \nabla_y \mathcal{E}(x-y)\cdot f(y,t)dy \\
&- \int\limits_{\frac{|x|}{2}\le|x-y|\le 2|x|} \nabla_y\nabla_y \mathcal{E}(x-y)\cdot (l\otimes k)(y,t)dy \\
&- \int\limits_{|x-y|=2|x|} (n\otimes\nabla_y\mathcal{E}(x-y))\cdot(l\otimes k)(y,t)d\sigma_y \\
&+ \int\limits_{|x-y|=\frac{|x|}{2}} (n\otimes\nabla_y\mathcal{E}(x-y))\cdot(l\otimes k)(y,t)d\sigma_y \\
&- \int\limits_{|x-y|\ge 2|x|} \nabla_y\mathcal{E}(x-y)\cdot f(y,t)dy = \sum_{i=1}^{5} I_i(x,t).
\end{aligned} \tag{7.11}
$$

Recalling the assumption (ii), the integrals I_1–I_4 are easily seen to possess the following property

$$\sum_{i=1}^{4}|I_i| \le c\frac{L(t)K(t)}{(1+t)^{\frac{\mu}{2}}}t^{-\frac{1}{2}}, \quad \forall |x|\ge 1,\ \forall t\ge 1.$$

In order to estimate I_5 we distinguish two cases: $2|x|\le\sqrt{t}$ and $2|x|>\sqrt{t}$. In the first case, using assumption (ii) on f, for some $a\in(0,\mu)$ we get

$$
\begin{aligned}
|I_5| \le c\frac{L(t)K(t)}{(1+t)^{\mu}}t^{-\frac{1}{2}} &\int\limits_{2|x|\le|x-y|\le\sqrt{t}} |x-y|^{1-n}dy \\
+ c\frac{L(t)K(t)}{(1+t)^{\frac{\mu}{2}-\frac{a}{2}}}t^{-\frac{1}{2}} &\int\limits_{|x-y|\ge\sqrt{t}} \frac{|x-y|^{1-n}}{(1+|y|)^{1+a}}dy \le c\frac{L(t)K(t)}{(1+t)^{\frac{\mu}{2}}}t^{-\frac{1}{2}}, \\
&\forall |x|\ge 1,\ \forall t\ge 1.
\end{aligned}
$$

In the case $2|x|>\sqrt{t}$, it is immediate to obtain

$$|I_5| \le cL(t)K(t)t^{-\frac{1}{2}}\int\limits_{|x-y|\ge 2|x|}\frac{|x-y|^{1-n}}{(1+|y|)^{2\mu}}dy$$

$$\leq c\frac{L(t)K(t)}{(1+t)^{\frac{\mu}{2}}}t^{-\frac{1}{2}}, \quad \forall |x|\geq 1,\ \forall t\geq 1.$$

Putting the estimates obtained for each I_i in (7.11), we get

$$|p_2^1(x,t)|\leq c\frac{L(t)K(t)}{(1+t)^{\frac{\mu}{2}}}t^{-\frac{1}{2}}, \quad \forall |x|\geq 1,\ \forall t\geq 1.$$

Therefore, recalling (7.9), we obtain

$$|p_2(x,t)|\leq c\frac{L(t)K(t)}{(1+t)^{\frac{\mu}{2}}}t^{-\frac{1}{2}}, \quad \forall (x,t)\in \mathbb{R}^n_+\times(0,T).$$

This estimate, together with (7.10), implies estimate (7.2) for function p_2. The proposition is completely proved.

8. Proof of Theorem 2.1

We begin the proof of Theorem 2.1 by showing the existence of solutions. We seek the solutions of (1.1) in the form

$$u(x,t)=\int\limits_{\mathbb{R}^n_+}\mathcal{G}(x,y,t)u_\circ(y)dy-\int\limits_0^t\int\limits_{\mathbb{R}^n_+}\mathcal{G}(x,y,t-\tau)P(u\cdot\nabla u)(y,\tau)dyd\tau,$$

$$\begin{aligned}\pi(x,t)=&\int\limits_{\mathbb{R}^n_+}\mathcal{Q}(x,y,t)\cdot u_\circ(y)dy-\int\limits_0^t\int\limits_{\mathbb{R}^n_+}\mathcal{Q}(x,y,t-\tau)\cdot P(u\cdot\nabla u)(y,\tau)dyd\tau\\&+\int\limits_{\mathbb{R}^n_+}\nabla_y\mathcal{N}(x,y)\cdot(u\cdot\nabla u)(y,t)dy.\end{aligned} \tag{8.1}$$

Using the method of successive approximations, we prove that equation (8.1) admits solutions. Let us consider the following equation

$$u^n=\mathcal{G}[u_\circ]-\mathcal{S}[u^{n-1}\cdot\nabla u^{n-1}], \tag{8.2}$$

where

$$\mathcal{G}[u_\circ]=\int\limits_{\mathbb{R}^n_+}\mathcal{G}(x,y,t)u_\circ(y)dy,$$

$$\mathcal{S}[u^{n-1}\cdot\nabla u^{n-1}]=\int\limits_0^t\int\limits_{\mathbb{R}^n_+}\mathcal{G}(x,y,t-\tau)P(u^{n-1}\cdot\nabla u^{n-1})(y,\tau)dyd\tau,$$

with $u^{n-1} \equiv 0$ for $n=0$ and $u^0 = \mathcal{G}[u_\circ]$ solution of system (4.1). First, let us show that equation (8.2) defines a sequence $\{u^n\}_{n\in\mathbb{N}}$ such that for some $T>0$ and for any $\eta\in(0,T)$, for $\beta\in(0,1)$,

$$\begin{aligned}
&u^n \in C(0,T;C(\overline{\mathbb{R}^n_+})),\quad D^2u^n, u^n_t \in C^{0,\beta}(\eta,T;C^{0,\beta}(\overline{\mathbb{R}^n_+}));\\
&\forall x_\circ \in \mathbb{R}^n_+, \lim_{(x,t)\to(x_\circ,0)} u^n(x,t) = u_\circ(x_\circ).
\end{aligned} \tag{8.3}$$

Moreover, if $\mu\in(\frac12,n)$, then, for $\alpha\in(0,1)$,

$$\begin{aligned}
&U^n(t) = \sup_{\substack{x\in\mathbb{R}^n_+\\ t\in(0,T)}} [(1+|x|)^\mu |u^n(x,t)|] \le BU_0 + bt^{\frac12}(U^{n-1}(t))^2;\\
&\sup_{\substack{x\in\mathbb{R}^n_+\\ t\in(0,T)}} [t^{\frac12}(1+|x|)^\mu |\nabla u^n(x,t)|] \le BU_0 + bt^{\frac12}(U^{n-1}(t))^2;\\
&|u^n(\overline{x},\overline{t}) - u^n(\overline{\overline{x}},\overline{\overline{t}})| \le (BU_0 + bt_1^{\frac12}(U^{n-1}(t_1))^2) t_0^{-\frac{\alpha}{2}} [|\overline{x}-\overline{\overline{x}}|^2 + |\overline{t}-\overline{\overline{t}}|]^{\frac{\alpha}{2}};\\
&|\nabla u^n(\overline{x},\overline{t}) - \nabla u^n(\overline{\overline{x}},\overline{\overline{t}})| \le (BU_0 + bt_1^{\frac12}(U^{n-1}(t_1))^2) t_0^{-\frac12-\frac{\alpha}{2}} [|\overline{x}-\overline{\overline{x}}|^2 + |\overline{t}-\overline{\overline{t}}|]^{\frac{\alpha}{2}};
\end{aligned} \tag{8.4}$$

if $\mu\in[1,n)$, then, for $\alpha\in(0,1)$, for any $\nu\in[0,\mu]$,

$$\begin{aligned}
&\overline{U}^n(t) = \sup_{\substack{x\in\mathbb{R}^n_+\\ t>0}} [(1+|x|)^{\mu-\nu}(1+t)^{\frac{\nu}{2}} |u^n(x,t)|] \le BU_0 + b(\overline{U}^{n-1}(t))^2;\\
&\sup_{\substack{x\in\mathbb{R}^n_+\\ t>0}} [t^{\frac12}(1+|x|)^{\mu-\nu}(1+t)^{\frac{\nu}{2}} |\nabla u^n(x,t)|] \le BU_0 + b(\overline{U}^{n-1}(t))^2;\\
&|u^n(\overline{x},\overline{t}) - u^n(\overline{\overline{x}},\overline{\overline{t}})| \le (BU_0 + b(\overline{U}^{n-1}(t_1))^2) t_0^{-\frac{\alpha}{2}} [|\overline{x}-\overline{\overline{x}}|^2 + |\overline{t}-\overline{\overline{t}}|]^{\frac{\alpha}{2}};\\
&|u^n(\overline{x},t) - u^n(\overline{\overline{x}},t)| \le (BU_0 + b(\overline{U}^{n-1}(t))^2) t^{-\frac{\alpha}{2}} \frac{|\overline{x}-\overline{\overline{x}}|^\alpha}{R(|\overline{x}|,|\overline{\overline{x}}|)^{\mu-\nu}} \frac{1}{(1+t)^{\frac{\nu}{2}}};\\
&|\nabla u^n(\overline{x},\overline{t}) - \nabla u^n(\overline{\overline{x}},\overline{\overline{t}})|\\
&\le (BU_0 + b(\overline{U}^{n-1}(t_1))^2) t_0^{-\frac12-\frac{\alpha}{2}} [|\overline{x}-\overline{\overline{x}}|^2 + |\overline{t}-\overline{\overline{t}}|]^{\frac{\alpha}{2}};\\
&|\nabla u^n(\overline{x},t) - \nabla u^n(\overline{\overline{x}},t)|\\
&\le (BU_0 + b(\overline{U}^{n-1}(t))^2) t^{-\frac12-\frac{\alpha}{2}} \frac{|\overline{x}-\overline{\overline{x}}|^\alpha}{R(|\overline{x}|,|\overline{\overline{x}}|)^{\mu-\nu}} \frac{1}{(1+t)^{\frac{\nu}{2}}};
\end{aligned} \tag{8.5}$$

uniformly in n, with $U^{n-1}(t)=0$, for $n=0$, $\overline{U}^{n-1}(t)=0$ for $n=0$, $t_0 = \min\{\overline{t},\overline{\overline{t}}\}$, $t_1 = \max\{\overline{t},\overline{\overline{t}}\}$. We prove properties (8.3)–(8.5) proceeding by induction. For $n=0$, these properties are true by Theorems 4.1 and 4.2. Thus, we have to prove that if u^{n-1} satisfies (8.3)–(8.5) so does u^n. Set $u^n = u^0 + v^n$, where v^n denotes the solution of system (5.1) with $f = -u^{n-1}\cdot\nabla u^{n-1}$. Obviously f satisfies the Assumptions. Indeed f satisfies: (i) with $L(t) = K(t) = BU_0 + bt^{\frac{1}{2}}(U^{n-2}(t))^2$; (ii) with $L(t) = K(t) = BU_0 + b(\overline{U}^{n-2}(t))^2$; (iii) via estimates $(8.4)_{3,4}$ and $(8.5)_{3,4}$; (iv) via estimates $(8.5)_{5,6}$. Therefore, by Theorem 6.1, function v^n satisfies properties (8.3)–(8.5). We now construct a solution of equation $(8.1)_1$ as limit of sequence $\{u^n\}$. We need of a result related to the following inequality:

$$\xi_n \le BU_0 + A(t)\xi_{n-1}^2, \tag{8.6}$$

where $A(t)$, is defined as follows

$$A(t) = \begin{cases} bt^{\frac{1}{2}}, & \text{if } \mu\in(\frac{1}{2},n),\\ b, & \text{if } \mu\in[1,n). \end{cases}$$

The result, due to Solonnikov, [16, Lemma 10.2], can be stated as follows

Lemma 8.1. *Let us consider the inequality $\xi_n \le a\xi_{n-1}^2 + c$, with a, $c>0$, $4ac<1$ and $\xi_0 \le \xi_{\min}$, where $\xi_{\min}$ is the minimal square root of the quadric equation $a\xi^2 - \xi + c = 0$. Then, for any $n>0$, $\xi_n \le \xi_{\min}$ holds.*

With regard to inequality (8.6) we have

$$\xi_{\min} = \frac{2BU_0}{1+\sqrt{1-4A(t)BU_0}} = \frac{1-\sqrt{1-4A(t)BU_0}}{2A(t)}$$

and, if $U_0 \le (4BA(t))^{-1}$, then $U^0(t)\le\xi_{\min}$, $\overline{U}^0(t)\le\xi_{\min}$. Hence, from Lemma 8.1, we get $U^n(t)\le\xi_{\min}$, $\overline{U}^n(t)\le\xi_{\min}$ for any $n\ge 0$. Furthermore, observing that $\xi_{\min}\le 2BU_0$, we deduce that all the $U^n(t)$, $\overline{U}^n(t)$ are bounded as follows

$$\begin{aligned} &\text{if } \mu\in\left(\frac{1}{2},n\right) \quad U^n(t)\le 2BU_0, \quad \forall t\in(0,T), \quad \text{with } T^{\frac{1}{2}} < (4bBU_0)^{-1},\\ &\text{if } \mu\in[1,n) \quad \overline{U}^n(t)\le 2BU_0, \quad \forall t>0, \quad \text{with } U_0 < (4bB)^{-1}, \end{aligned} \tag{8.7}$$

for any $n\ge 0$. Set

$$\begin{aligned} u^n(x,t) &= \sum_{h=1}^{n}[u^h(x,t) - u^{h-1}(x,t)] + u^0(x,t)\\ &= \sum_{h=1}^{n} w^h(x,t) + u^0(x,t), \end{aligned} \tag{8.8}$$

with $w^0 = u^0$. For any $h \in \mathbb{N} \cup \{0\}$, function w^{h+1} is solution of the equation
$$w^{h+1} = -\mathcal{S}[w^h \cdot \nabla u^h + u^{h-1} \cdot \nabla w^h],$$
where, we recall, $u^{h-1} \equiv 0$ for $h = 0$.

Let μ be in $(\frac{1}{2}, n)$. Setting $M(t,h) = 2^{2(h-1)} b^h t^{\frac{h}{2}} B^{h+1} U_0^{h+1}$, in virtue of (8.4), (8.7), and Theorem 6.1 [4] it is easy to verify that,

$$W^h(t) = \sup_{\substack{x \in \mathbb{R}^n_+ \\ t \in (0,T)}} [(1+|x|)^\mu |w^h(x,t)|] \leq M(t,h),$$
$$Z^h(t) = \sup_{\substack{x \in \mathbb{R}^n_+ \\ t \in (0,T)}} [t^{\frac{1}{2}} (1+|x|)^\mu |\nabla w^h(x,t)|] \leq M(t,h),$$
$$|w^h(\overline{x},\overline{t}) - w^h(\overline{\overline{x}},\overline{\overline{t}})| \leq M(t_1,h) t_0^{-\frac{\alpha}{2}} [|\overline{x} - \overline{\overline{x}}|^2 + |\overline{t} - \overline{\overline{t}}|]^{\frac{\alpha}{2}},$$
$$|\nabla w^h(\overline{x},\overline{t}) - \nabla w^h(\overline{\overline{x}},\overline{\overline{t}})| \leq M(t_1,h) t_0^{-\frac{1}{2}-\frac{\alpha}{2}} [|\overline{x} - \overline{\overline{x}}|^2 + |\overline{t} - \overline{\overline{t}}|]^{\frac{\alpha}{2}}.$$

Therefore, in the hypothesis $T^{\frac{1}{2}} < (4bBU_0)^{-1}$, the series $\sum_{h=1}^{\infty} w^h$ and $\sum_{h=1}^{\infty} \nabla w^h$ are absolutely convergent. Set w and ∇w the uniform limits of series $\sum_{h=1}^{\infty} w^h$ and $\sum_{h=1}^{\infty} \nabla w^h$, respectively. Hence, from (8.8), the sequence u^n uniformly converges to $u = w + u^0$, and the sequence ∇u^n uniformly converges to $\nabla u = \nabla w + \nabla u^0$. Further w^h converges to w in $C^{0,\alpha}(\eta, T; C^{0,\alpha}(\overline{\mathbb{R}^n_+}))$, and ∇w^h converges to ∇w in $C^{0,\alpha}(\eta, T; C^{0,\alpha}(\overline{\mathbb{R}^n_+}))$. Therefore, since the estimates (8.4) are uniform with respect to n we also have

$$|u(x,t)| \leq \Big(\frac{bt^{\frac{1}{2}} B^2 U_0^2}{1-4bt^{\frac{1}{2}} BU_0} + BU_0\Big) \frac{1}{(1+|x|)^\mu} \leq \frac{1-3bt^{\frac{1}{2}} BU_0}{1-4bt^{\frac{1}{2}} BU_0} \frac{BU_0}{(1+|x|)^\mu}$$
$$|\nabla u(x,t)| \leq \Big(\frac{bB^2 U_0^2}{1-4bt^{\frac{1}{2}} BU_0} + \frac{BU_0}{t^{\frac{1}{2}}}\Big) \frac{1}{(1+|x|)^\mu} \leq \frac{1-3bt^{\frac{1}{2}} BU_0}{t^{\frac{1}{2}}(1-4bt^{\frac{1}{2}} BU_0)} \frac{BU_0}{(1+|x|)^\mu},$$
for any $(x,t) \in \mathbb{R}^n_+ \times (0,T)$;

$$|u(\overline{x},\overline{t}) - u(\overline{\overline{x}},\overline{\overline{t}})| \leq \frac{1-3bt_1^{\frac{1}{2}} BU_0}{1-4bt_1^{\frac{1}{2}} BU_0} BU_0 t_0^{-\frac{\alpha}{2}} (|\overline{x} - \overline{\overline{x}}|^2 + |\overline{t} - \overline{\overline{t}}|)^{\frac{\alpha}{2}},$$

[4] Of course data f of Theorem 6.1 is now represented by $-w^h \cdot \nabla u^h - u^{h-1} \cdot \nabla w^h$.

$$|\nabla u(\overline{x},\overline{t}) - \nabla u(\overline{\overline{x}},\overline{\overline{t}})| \le \frac{1-3bt_1^{\frac{1}{2}}BU_0}{1-4bt_1^{\frac{1}{2}}BU_0} BU_0 t_0^{-\frac{1}{2}-\frac{\alpha}{2}} (|\overline{x}-\overline{\overline{x}}|^2 + |\overline{t}-\overline{\overline{t}}|)^{\frac{\alpha}{2}},$$

for any $\overline{x}$, $\overline{\overline{x}} \in \mathbb{R}^n_+$, for any $\overline{t}, \overline{\overline{t}} \in (\eta, T)$.

If $\mu \in [1,n)$, setting $M(h) = 2^{2(h-1)} b^h B^{h+1} U_0^{h+1}$, in virtue of (8.5), (8.7), and Theorem 6.1, we have for any $\nu \in [0,\mu]$, we have

$$\overline{W}^h(t) = \sup_{\substack{x\in\mathbb{R}^n_+\\ t\in(0,T)}} [(1+|x|)^{\mu-\nu}(1+t)^{\frac{\nu}{2}}|w^h(x,t)|] \le M(h),$$

$$\overline{Z}^h(t) = \sup_{\substack{x\in\mathbb{R}^n_+\\ t\in(0,T)}} [t^{\frac{1}{2}}(1+|x|)^{\mu-\nu}(1+t)^{\frac{\nu}{2}}|\nabla w^h(x,t)|] \le M(h),$$

$$|w^h(\overline{x},\overline{t}) - w^h(\overline{\overline{x}},\overline{\overline{t}})| \le M(h) t_0^{-\frac{\alpha}{2}} [|\overline{x}-\overline{\overline{x}}|^2 + |\overline{t}-\overline{\overline{t}}|]^{\frac{\alpha}{2}},$$

$$|w^h(\overline{x},t) - w^h(\overline{\overline{x}},t)| \le M(h) t^{-\frac{\alpha}{2}} \frac{|\overline{x}-\overline{\overline{x}}|^{\ \alpha}}{R(|\overline{x}|,|\overline{\overline{x}}|)^{\mu-\nu}} \frac{1}{(1+t)^{\frac{\nu}{2}}},$$

$$|\nabla w^h(\overline{x},\overline{t}) - \nabla w^h(\overline{\overline{x}},\overline{\overline{t}})| \le M(h) t_0^{-\frac{1}{2}-\frac{\alpha}{2}} [|\overline{x}-\overline{\overline{x}}|^2 + |\overline{t}-\overline{\overline{t}}|]^{\frac{\alpha}{2}},$$

$$|\nabla w^h(\overline{x},t) - \nabla w^h(\overline{\overline{x}},t)| \le M(h) t^{-\frac{1}{2}-\frac{\alpha}{2}} \frac{|\overline{x}-\overline{\overline{x}}|^{\ \alpha}}{R(|\overline{x}|,|\overline{\overline{x}}|)^{\mu-\nu}} \frac{1}{(1+t)^{\frac{\nu}{2}}}.$$

Therefore, if $U_0 < (4bB)^{-1}$ the series $\displaystyle\sum_{h=1}^{\infty} w^h$ and $\displaystyle\sum_{h=1}^{\infty} \nabla w^h$ are absolutely convergent. Denoted respectively by w and ∇w the uniform limits of the series, from (8.8) we have that the sequence u^n uniformly converges to the function $u = w + u^0$ and the sequence ∇u^n uniformly converges to the function $\nabla u = \nabla w + \nabla u^0$. Further w^h converges to w in $C^{0,\alpha}(\eta, T; C^{0,\alpha}(\overline{\mathbb{R}^n_+}))$, and ∇w^h converges to ∇w in $C^{0,\alpha}(\eta, T; C^{0,\alpha}(\overline{\mathbb{R}^n_+}))$. Again from (8.8), since the estimates (8.5) are uniform with respect to n we have, for any $\nu \in [0,\mu]$,

$$\begin{aligned}|u(x,t)| &\le \Big(\frac{bB^2U_0^2}{1-4bBU_0} + BU_0\Big)\frac{1}{(1+|x|)^{\mu-\nu}}\frac{1}{(1+t)^{\frac{\nu}{2}}}\\ &\le \frac{1-3bBU_0}{1-4bBU_0}\frac{BU_0}{(1+|x|)^{\mu-\nu}}\frac{1}{(1+t)^{\frac{\nu}{2}}},\end{aligned}$$

$$t^{\frac{1}{2}}|\nabla u(x,t)| \le \Big(\frac{bB^2U_0^2}{1-4bBU_0} + BU_0\Big)\frac{1}{(1+|x|)^{\mu-\nu}}\frac{1}{(1+t)^{\frac{\nu}{2}}}$$

$$\leq \frac{1-3bBU_0}{1-4bBU_0}\frac{BU_0}{(1+|x|)^{\mu-\nu}}\frac{1}{(1+t)^{\frac{\nu}{2}}},$$

for any $(x,t)\in\mathbb{R}^n_+\times[0,\infty)$;

$$|u(\overline{x},\overline{t})-u(\overline{\overline{x}},\overline{\overline{t}})|\leq\frac{1-3bBU_0}{1-4bBU_0}BU_0t_0^{-\frac{\alpha}{2}}[|\overline{x}-\overline{\overline{x}}|^2+|\overline{t}-\overline{\overline{t}}|]^{\frac{\alpha}{2}},$$

$$|\nabla u(\overline{x},\overline{t})-\nabla u(\overline{\overline{x}},\overline{\overline{t}})|\leq\frac{1-3bBU_0}{1-4bBU_0}BU_0t_0^{-\frac{1}{2}-\frac{\alpha}{2}}[|\overline{x}-\overline{\overline{x}}|^2+|\overline{t}-\overline{\overline{t}}|]^{\frac{\alpha}{2}},$$

for any $\overline{x},\,\overline{\overline{x}}\in\mathbb{R}^n_+$, for any $\overline{t},\overline{\overline{t}}\in(\eta,T)$;

$$|u(\overline{x},t)-u(\overline{\overline{x}},t)|\leq\frac{1-3bBU_0}{1-4bBU_0}BU_0t^{-\frac{\alpha}{2}}\frac{|\overline{x}-\overline{\overline{x}}|^{\;\alpha}}{R(|\overline{x}|,|\overline{\overline{x}}|)^{\mu-\nu}}\frac{1}{(1+t)^{\frac{\nu}{2}}},$$

$$|\nabla u(\overline{x},t)-\nabla u(\overline{\overline{x}},t)|\leq\frac{1-3bBU_0}{1-4bBU_0}BU_0t^{-\frac{1}{2}-\frac{\alpha}{2}}\frac{|\overline{x}-\overline{\overline{x}}|^{\;\alpha}}{R(|\overline{x}|,|\overline{\overline{x}}|)^{\mu-\nu}}\frac{1}{(1+t)^{\frac{\nu}{2}}},$$

for any $\overline{x},\,\overline{\overline{x}}\in\mathbb{R}^n_+$, for any $t>0$.

Since the limit function w still satisfies the Assumptions, the regularity of $u(x,t)$ stated in Theorem 2.1 follows from Theorems 4.1 and 6.1.

Taking into account Lemma 3.1, we can associate to function u a function π, solution of equation $(3.7)_2$ with $f=-u\cdot\nabla u$. The estimates on the pressure, given in Proposition 2.1, follow from the representation formula, Theorem 4.2 and Proposition 7.1.

We conclude the proof of the theorem by proving the uniqueness result. We use a duality argument of the Holmegren type. Let u_1, u_2 be two solutions of system (1.1) and $u=u_1-u_2$. Let us consider the initial boundary value problem

$$\begin{aligned}
&\varphi_t-\Delta\varphi+\varphi\cdot(\nabla u_1)^T=u_2\cdot\nabla\varphi-\nabla q+g,\\
&\nabla\cdot\varphi=0,\quad\text{on }\ \mathbb{R}^n_+\times(0,T'),\\
&\varphi(x',t)=0,\quad\text{on }\ \mathbb{R}^{n-1}\times(0,T),\quad \lim_{|x|\to\infty}\varphi(x,t)=0,\\
&\varphi(x,0)=0,\quad\text{on }\ \mathbb{R}^n_+,
\end{aligned}$$

with $T'<T$ and $g\in C_0(\mathbb{R}^n_+\times(0,T'))$. The existence of a solution (φ,q) with data $g\in C_0(\mathbb{R}^n_+\times(0,T'))$, $u_2\in L^\infty(\mathbb{R}^n_+\times(0,T'))$, $\nabla u_1\in L^r(0,T';L^\infty(\mathbb{R}^n_+))$, $r<2$, has been given by Solonnikov in [16, Theorem 4.2]):

$$\varphi\in L^p(0,T;W^{2,p}(\mathbb{R}^n_+)),\quad\nabla\varphi,\varphi_t\in L^p(0,T;L^p(\mathbb{R}^n_+)),\ \ \forall p>1.$$

Since $\mu > 0$, for some $p' > 1$ we have, for any $\varepsilon > 0$, $u(x,t) \in L^{p'}(\varepsilon, T; W^{1,p}(\mathbb{R}^n_+) \cap C^{2,\alpha}(\overline{\mathbb{R}^n_+}))$. Multiplying equation $(1.1)_1$ by $\varphi(x, T' - \tau)$ and integrating by parts on $\mathbb{R}^n_+ \times (\varepsilon, T')$ we arrive at

$$\int_{\varepsilon}^{T'} \int_{\mathbb{R}^n_+} u(x,\tau) \cdot g(x,\tau) dx d\tau + (\varphi(T' - \varepsilon), u(\varepsilon)) = 0.$$

Since $\lim\limits_{\varepsilon \to 0} (\varphi(T' - \varepsilon), u(\varepsilon)) = 0$, performing the limit for $\varepsilon \to 0$ we obtain $\int_0^{T'} \int_{\mathbb{R}^n_+} u(x,\tau) \cdot g(x,\tau) dx d\tau = 0$. Taking into account that g is arbitrary, the above relation ensures that $u = 0$.

References

1. E. B. Fabes, B. F. Jones, N. M. Riviere, *The Initial Value Problem for the Navier–Stokes Equations with Data in L^p.* — Arch. Rational Mech. Anal. **45** (1972), 222–240.
2. G. P. Galdi, H. Sohr, *Existence and Uniqueness of Time-Periodic Physically Reasonable Navier–Stokes Flow Past a Body.* — (To appear).
3. Y. Giga, K. Inui, Sh. Matsui, *On the Cauchy problem for the Navier-Stokes equations with nondecaying initial data.* — Quad. Mat. **4** (1999), 27–68.
4. G. H. Knightly, *On a class of global solutions of the Navier–Stokes equations.* — Arch. Rational Mech. Anal. **21** (1966), 211–245.
5. G. H. Knightly, *A Cauchy problem for the Navier–Stokes equations in $\mathbb{R}^n$.* — SIAM J. Math. Anal. **3** (1972), 506–511.
6. G. H. Knightly, *Some decay properties of solutions of the Navier–Stokes equations.* — Approximations methods for Navier–Stokes problems, Lect. Notes Math., Springer **771** (1980), 287–298.
7. H. Koch, V. A. Solonnikov, *L_q-Estimates of the first-order derivatives of solutions of the nonstationary Navier–Stokes equations.* — in: Nonlinear Problems of Mathematical Physics and Related Topics I In Honor of Professor O. A. Ladyzhenskaya, International Mathematical Series **1**, Kluwer Academic/Plenum Publishers (2002), pp. 203–218.
8. O. A. Ladyzhenskaya, N. N. Ural'tzeva, *Linear and quasi-linear equations of elliptic type.* — Academic Press (1968).
9. O. A. Ladyzhenskaya, V. A. Solonnikov, N. N. Ural'tzeva, *Linear and quasi-linear equations of parabolic type.* — Translation of Math. Monographs AMS 23 (1968).
10. P. Maremonti, G. Starita, *On the nonstationary Stokes equations in half-space with continuous initial data.* — Zap. Nauch. Sem. POMI **295** (2003), 118–167.
11. C. Miranda, *Istituzioni di analisi funzionale lineare.* — U.M.I., Oderisi, Gubbio (1978).

12. T. Miyakawa, *Notes on space-time decay properties of nonstationary incompressible Navier–Stokes flows in* $\mathbb{R}^n$. — Funkcial. Ekvac, **45** (2002), 271–289.
13. R. Mizumachi, *On the asymptotic behavior of incompressible viscous fluid motions past bodies.* — J. Math. Soc. Japan **36** (1984), 497–522.
14. C. W. Oseen, *Sur les formules de Green généralisées qui se presentent dans l'hydrodynamique et sur quelquesunes de leurs applications.* — Acta Math. **34** (1910), 205–284.
15. C. W. Oseen, *Neuere Methoden und Ergebnisse in der Hydrodynamik.* — Akademische Verlagsgesellschaft m.b.h., Leipzig (1927).
16 V. A. Solonnikov, *Estimates for solutions of nonstationary Navier–Stokes equations.* — J. Soviet Math. **8** (1977), 467–528.
17. V. A. Solonnikov, *Estimates for solutions of the nonstationary Stokes problem in anisotropic Sobolev spaces and estimates for the resolvent of the Stokes operator.* — Uspekhi Mat. Nauk **58** (2003), 123–156.
18. V. A. Solonnikov, *On nonstationary Stokes problem and Navier–Stokes problem in half-space with initial data nondecreasing at infinity.* — J. Math. Sciences **114** (2003), 1726–1740.
19. S. Ukai, *A solution formula for the Stokes equation in* $\mathbb{R}^n_+$. — Comm. Pure Appl. Math. **40** (1987), 611–621.

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