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S. I. Repin

LOCAL A POSTERIORI ESTIMATES FOR THE STOKES PROBLEM

Abstract. In the paper, computable estimates of the difference between the exact solution of the Stokes problem and an approximation from the respective energy class are obtained. Estimates are presented in terms of local norms and linear functionals. Certain generalizations to some nonlinear problems are discussesd.

Dedicated to the memory of
Olga Aleksandrovna Ladyzhenskaya

1. Introduction

Reliable methods of mathematical modeling in continuum mechanics are based on two keystones: stable and efficient numerical procedures and explicit control of approximation errors. The first gives an approximate solution of a problem considered, while the second one justifies its accuracy. Numerical methods in the theory of fluids started receiving serious attention in the middle of 20-th century. These researches were strongly influenced by the book of O. A. Ladyzhenskaya [8] and her other works. At present, numerical analysis has a wide spectrum of methods intended to solve problems arising in computational fluid dynamics (see, e.g., [4, 7] and the literature cited therein). A posteriori error control of numerical solutions for the Stokes, Navier–Stokes and other flow problems presents another challenging task. We refer readers interested in the history of this question and the respective references to, e.g., [3, 14, 16]. In this note, we focus our attention on a particular but important problem in a posteriori error control, namely on the estimation of errors in local norms and seminorms.

In the majority of cases, a posteriori error estimates are derived in global norms. Their purpose is to estimate the quantity $\| u - v \|$, where u is the exact solution, v is an approximate one and $\| \cdot \|$ is a global (e.g.,

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energy) norm. In this paper, we obtain computable a posteriori estimates of different types intended to provide a more detailed information on the accuracy of a solution computed. They give guaranteed upper bounds for errors associated with a certain subdomain ω or those expressed in terms of a given linear functional ℓ. If ℓ is generated by an integrable function with local support, then the value of $|\langle \ell, u-v\rangle|$ also gives a certain presentation on the local behavior of approximation errors. Besides, having estimates for several different functionals one can easily find an upper bound of the seminorm generated by them.

For elliptic type boundary value problems of the second order local a posteriori error estimates of a new type has been recently obtained in [11]. In the present paper, we derive them for the Stokes problem and some other models related. Our approach is based upon functional type a posteriori estimates that were derived for the problems in question in [12, 14]. In short, the main result can be formulated as follows: *a posteriori estimates in local norms and a posteriori estimates in terms of linear functionals can be obtained by the minimization of functional a posteriori estimates on specially constructed subspaces.*

In Sec. 2, we recall some basic facts concerning a posteriori estimates for the Stokes problem. In Sec. 3, we derive a posteriori estimates in local norms, and in Sec. 4, obtain guaranteed bounds of deviations from exact solution of the Stokes problem expressed in terms of linear functionals. A consise discussion on possible applications of the approach considered to some nonlinear problems is presented in Sec. 5. Section 6 contains some remarks on the practical implementation of the estimates.

2. A posteriori estimates for the Stokes problem in energy norm

Let $\Omega \in \mathbb{R}^d$ $(d=2,3)$ be a bounded domain. Consider the classical Stokes problem. It consists of finding a vector-valued function u (velocity), a tensor-valued function σ (stress), and a scalar-valued function p (pressure) satisfying the system

$$-\operatorname{div} a\sigma + \nabla p = f \text{ in } \Omega, \tag{2.1}$$

$$\sigma = \nu\varepsilon(u) \;\text{ in } \Omega, \tag{2.2}$$

$$\operatorname{div} u = 0 \;\text{ in } \Omega, \tag{2.3}$$

$$u = u_0 \;\text{ on }\; \partial\Omega \tag{2.4}$$

that describes slow motion of an incompressible fluid whose viscosity is characterized by a positive constant ν. Here, $\varepsilon(v) := \frac{1}{2}(\nabla v + (\nabla v)^T)$ is the tensor of small strains and f and u_0 are the given functions. In what follows, $\|\cdot\|$ denotes the L_2-norm of a vector- or tensor-valued function defined in Ω, the closure of smooth solenoidal functions with compact supports in Ω with respect to the norm $\|\nabla v\|$ is denoted by $\overset{\circ}{J}{}^1_2(\Omega)$. Let $V := H^1(\Omega, \mathbb{R}^d)$, and $\Sigma := L_2(\Omega, \mathbb{M}^{d\times d})$, where $\mathbb{M}^{d\times d}$ denotes the space of symmetric $d\times d$ matrices (tensors), whose scalar product is denoted by two dots. V_0 is a subspace of V that consists of functions with zero traces on $\partial\Omega$. The affine set $V_0 + u_0$ is a subset of V containing all functions of the form $w + u_0$, where $w \in V_0$, and u_0 is a given solenoidal vector-valued function. Analogously, $\overset{\circ}{J}{}^1_2(\Omega) + u_0$ consists of the functions $w + u_0$, where $w \in \overset{\circ}{J}{}^1_2(\Omega)$. Also, we define a Hilbert space $\Sigma_{\rm div}(\Omega)$, which is a subspace of Σ containing tensor-valued functions τ such that $\operatorname{div}\tau \in L_2$. The scalar product in this space is defined by the relation

$$(\tau, \eta) := \int\limits_\Omega (\tau : \eta + \operatorname{div}\tau \cdot \operatorname{div}\eta)\, dx\,.$$

The space of square summable functions with zero mean is denoted by $\widetilde{L}_2(\Omega)$. Assume that

$$f \in L_2(\Omega, \mathbb{R}^d), \quad u_0 \in H^1(\Omega, \mathbb{R}^d), \quad \operatorname{div} u_0 = 0.$$

The generalized solution of the Stokes problem is $u \in \overset{\circ}{J}{}^1_2(\Omega) + u_0$ that meets the integral identity

$$\nu \int\limits_\Omega \varepsilon(u) : \varepsilon(v)\, dx = \int\limits_\Omega f \cdot v\, dx \quad v \in \overset{\circ}{J}{}^1_2(\Omega). \tag{2.5}$$

It is well known (see [8]) that this problem is uniquely solvable.

In [12], the following estimate of the difference between u and a vector-valued function $v \in \overset{\circ}{J}{}^1_2(\Omega) + u_0$ was obtained:

$$\begin{aligned}\nu^2\|\varepsilon(u-v)\|^2 \leq \quad & \mathcal{M}_1^{\oplus}(v, \tau, \beta, q) := (1+\beta)\|\nu\varepsilon(v) - \tau\|^2 + \\ & + \frac{(1+\beta)c_F^2}{\beta}\|\operatorname{div}\tau + f - \nabla q\|^2. \end{aligned} \tag{2.6}$$

Here, τ is an arbitrary tensor-valued function from $\Sigma_{\rm div}(\Omega)$, q is a scalar-valued function from $\widetilde{W}(\Omega) := \widetilde{L}_2(\Omega) \cap H^1(\Omega)$, β is any positive number, and c_F is the constant in the Friederichs inequality for the domain Ω

(or any greater constant). We note that $\mathcal{M}_1^{\oplus}$ is a quadratic functional with respect to τ and q what simplifies its minimization over these variables. The parameter β is easy to exclude from (2.6) what gives a simpler estimate

$$\nu\|\varepsilon(u-v)\| \leq \|\nu\varepsilon(v)-\tau\| + c_F\|\operatorname{div}\tau + f - \nabla q\|. \tag{2.7}$$

However, the right-hand side of (2.7) is a convex, but not a quadratic functional. Therefore, for practical purposes the estimate (2.6) is preferable. The meaning of estimates (2.6) and (2.7) is quite obvious: the energy norm of the difference between the exact solution and any function v from the admissible (energy) set is bounded from above by a certain weighted sum of two terms that vanishes if an only if v coincides with u, τ with $\sigma = \nu\varepsilon(u)$, and q with the exact pressure function p.

If the function u is compared with an approximation from a wider functional class, namely with $\widehat{v} \in V_0 + u_0$ (that may not satisfy the incompressibility condition), then the estimate includes an additional term and looks as follows:

$$\begin{aligned}\nu\|\varepsilon(u-\widehat{v})\| \leq & \|\nu\varepsilon(\widehat{v})-\tau\| + \\ & + c_F\|\operatorname{div}\tau + f - \nabla q\| + \frac{2\nu}{C_{LBB}}\|\operatorname{div}\widehat{v}\|,\end{aligned} \tag{2.8}$$

or

$$\begin{aligned}\nu^2\|\varepsilon(u-\widehat{v})\|^2 \leq \quad & \mathcal{M}_2^{\oplus}(v,\tau,\alpha,\beta,\gamma,q) := c_1\|\nu\varepsilon(\widehat{v})-\tau\|^2 \\ & + c_2\|\operatorname{div}\tau + f - \nabla q\|^2 + c_3\|\operatorname{div}(\widehat{v})\|^2,\end{aligned} \tag{2.9}$$

where

$$\begin{aligned} c_1 =& 1+\alpha+\beta; \quad c_2 = c_F^2\left(1+\frac{1}{\alpha}+\gamma\right); \\ c_3 =& \frac{4\nu^2}{C_{LBB}^2}\left(1+\frac{1}{\beta}+\frac{1}{\gamma}\right); \quad \alpha,\ \beta,\ \gamma > 0, \end{aligned}$$

C_{LBB} is the constant in the Ladyzhenskaya–Babuška–Brezzi condition, $\tau \in \Sigma_{\operatorname{div}}(\Omega)$, and $q \in \widetilde{W}(\Omega)$.

Thus, if the constants c_F and C_{LBB} are known (or we know suitable upper bounds for them), then (2.9) gives a computable estimate of the accuracy of $\widehat{v}$. In what follows, we will show that estimates (2.6) and (2.9) also imply computable estimates of the accuracy in local norms and seminorms.

3. A posteriori error estimates in local norms

Let ω be a subdomain of Ω with Lipschitz continuous boundary $\partial\omega$ and $\|\tau\|_\omega^2 := \int\limits_\omega \tau : \tau dx$. Consider first the case $v \in \overset{\circ}{J}{}_2^1(\Omega)$. Assume that φ is a function such that

$$\varphi \subset H_{0\omega} := \{w \in \overset{\circ}{J}{}_2^1(\Omega) \mid \varepsilon(w)(x) = 0 \text{ for a.e. } x \in \omega\}.$$

Since $\widetilde{v} = (v - \varphi)$ can be viewed as an approximation of u in $\overset{\circ}{J}{}_2^1(\Omega)$, we apply (2.6) and obtain

$$\nu^2\|\varepsilon(u-v)\|_\omega^2 \leq \nu^2\|\varepsilon(u-\widetilde{v})\|^2 \leq \mathcal{M}_1^\oplus(v-\varphi,\tau,\beta,q), \qquad (3.1)$$

where $\tau \in \Sigma_{\mathrm{div}}(\Omega)$, $\beta > 0$, and $q \in \widetilde{W}(\Omega)$. To obtain the sharpest upper bound we must minimize the right-hand side of (3.1) with respect to all free functions, what leads to the estimate

$$\nu^2\|\varepsilon(u-v)\|_\omega^2 \leq M_{1\omega}^\oplus(v) \qquad (3.2)$$

where

$$M_{1\omega}^\oplus(v) := \inf_{\substack{\beta>0,\ \varphi\in H_{0\omega} \\ \tau\in\Sigma_{\mathrm{div}},\ q\in\widetilde{L}_2(\Omega)}} \mathcal{M}_1^\oplus(v-\varphi,\tau,\beta,q).$$

An upper estimate for $M_{1\omega}^\oplus(v)$ is obtained if we set $\tau = \nu\varepsilon(u)$, $q = p$, and tend β to $+\infty$. Then, by (3.2), we find that

$$\|\varepsilon(u-v)\|_\omega^2 \leq \inf_{\varphi\in H_{0\omega}} \|\varepsilon(u-v+\varphi)\|^2.$$

Since

$$\|\varepsilon(u-v+\varphi)\|^2 = \|\varepsilon(u-v)\|_\omega^2 + \|\varepsilon(u-v+\varphi)\|_{\Omega\setminus\omega}^2,$$

we arrive at the estimate

$$\nu^2\|\varepsilon(u-v)\|_\omega^2 \ \leq\ M_{1\omega}^\oplus(v) \ \leq\ \nu^2\|\varepsilon(u-v)\|_\omega^2 + I_\omega(v), \qquad (3.3)$$

where

$$I_\omega(v) := \inf_{\varphi\in H_{0\omega}} \nu^2\|\varepsilon(u-v+\varphi)\|_{\Omega\setminus\omega}^2. \qquad (3.4)$$

We observe that the efficiency of the estimate (3.2) depends on the value of $I_\omega(v)$. Let us discuss the meaning of this term. Assume that a vector-valued function v is defined such that $v = u$ on $\partial\omega$ (in the sense of traces).

Then, $I_\omega(v) = 0$. Indeed, take $\varphi = v - u$ in $\Omega \setminus \omega$ and $\varphi \equiv 0$ in ω. Such a function φ belongs to $H_{0\omega}$ and, therefore, by (3.4), we find that $I_\omega(v) = 0$. This result is easy to explain. In general, the local error associated with an approximate solution v given on the domain ω is caused by two reasons: (a) possible violations of the differential equation in ω and (b) incorrectness in the boundary conditions on $\partial\omega$. If v satisfies the same boundary condition on $\partial\omega$ as u, then only the first reason remains and our majorant is exact in the sense that *it is equal to the exact value of the local error*. This equality also holds for a wider class of approximations. Namely, let v be such that

$$v - u = w_\omega \quad \text{on} \quad \partial\omega, \qquad w_\omega \in \mathrm{R}(\omega), \tag{3.5}$$

where $\mathrm{R}(\omega)$ denotes the space of "rigid deflections", i.e.,the space of vector-valued functions having the form $w_0 + \Theta x$, where w_0 is a constant vector, Θ is an antisymmetric tensor in $\mathbb{M}^{d\times d}$ with constant coefficients, and x is the coordinate vector. It is easy to see that $\varepsilon(w_\omega) = 0$ in ω. Therefore, the function

$$\bar{\varphi} := \begin{cases} u - v & \text{in } \Omega \setminus \omega; \\ w_\omega & \text{in } \omega \end{cases}$$

belongs to $H_{0\omega}$ and $I_\omega(v) = 0$. In such a case, $M^{\oplus}_{1\omega}(v)$ gives the exact value of the local error norm.

In other cases, the majorant will contain not only the local error, but also some penalty whose value will be the smaller the closer $v - u$ on $\partial\omega$ to the trace of a function from $\mathrm{R}(\omega)$.

For an approximations $\widehat{v} \in V_0$, local estimates are derived in a similar way. Let $\varphi \in V_{0\omega}$, where

$$V_{0\omega} := \{ w \in V_0 \mid \varepsilon(w)(x) = 0 \text{ for a.e. } x \in \omega \}.$$

Since $\widetilde{v} = (\widehat{v} - \varphi) \in V_0$ can be viewed as an approximation of u, we apply (2.9) and obtain

$$\nu^2 \|\varepsilon(u - \widehat{v})\|^2_\omega \leq \nu^2 \|\varepsilon(u - \widetilde{v})\|^2 \leq \mathcal{M}^{\oplus}_2(\widehat{v} - \varphi, \tau, \alpha, \beta, \gamma, q), \tag{3.6}$$

where α, β, and γ are positive parameters and τ, φ, and q are arbitrary functions. Obviously, they should be chosen such that to minimize the left-hand side of (3.6). Then, we arrive at the estimate

$$\nu^2 \|\varepsilon(u - \widehat{v})\|^2_\omega \leq M^{\oplus}_{2\omega}(v) \tag{3.7}$$

where

$$M_{2\omega}^{\oplus}(v) := \inf_{\substack{\alpha,\beta,\gamma>0,\ \varphi\in V_{0\omega} \\ \tau\in\Sigma_{\mathrm{div}},\ q\in\widetilde{L}_2(\Omega)}} \mathcal{M}_2^{\oplus}(\widehat{v}-\varphi,\tau,\alpha,\beta,\gamma,q).$$

Practical implementation of the estimates (3.2) and (3.7) we discuss in Section 6.

Finally, we note that a posteriori estimates can be also derived for the approximations that does not exactly satisfy the prescribed Dirichlét boundary condition (see [15]). Local error estimates for approximations of such a type can be deduced from the global ones by the approach considered in this section.

4. Error estimates in terms of linear functionals

Functional a posteriori estimates obtained for the energy error norm can be also used to derive estimates of the quantity $|\langle \ell, u-v\rangle|$, where $\ell \in V_0^*$ is a given linear functional and $\langle\cdot,\cdot\rangle$ denotes the duality pairing of the spaces V_0^* and V_0. The functional ℓ is usually taken such that its value characterizes some especially important properties of a solution. In the method exposed below, we obtain computable bounds for such functionals directly from the energy error majorant without using the so-called adjoint problem as in, e.g., [1, 3, 5].

First, we note that

$$|\langle \ell, u-v\rangle| = |\langle \ell, u-v-\varphi\rangle| \qquad \forall\varphi\in V_{0\ell}(\Omega), \tag{4.1}$$

where

$$V_{0\ell}(\Omega) := \{\varphi\in V_0(\Omega) \mid \langle \ell,\varphi\rangle = 0\}.$$

By (4.1) we obtain

$$|\langle \ell, u-v\rangle| \le |\!|\!| \ell |\!|\!| \inf_{\varphi\in V_{0\ell}} \|\varepsilon(u-v-\varphi)\|. \tag{4.2}$$

Here

$$|\!|\!| \ell |\!|\!| := \sup_{\substack{w\in V_0\\ w\neq 0}} \frac{|\langle \ell, w\rangle|}{\|\varepsilon(w)\|}$$

Let us show that (4.2) holds as the equality. Indeed, let $\widehat{u}_\ell \in V_0$ be a function such that

$$\int_\Omega \varepsilon(\widehat{u}_\ell) : \varepsilon(w)\,dx = \langle \ell, w\rangle \qquad \forall w\in V_0. \tag{4.3}$$

Since ℓ is a bounded linear functional on V_0, by standard arguments we conclude that such a function $\widehat{u}_\ell$ exists, unique and $\|\varepsilon(\widehat{u}_\ell)\| \leq |\!|\!| \ell |\!|\!|$. On the other hand,

$$|\!|\!| \ell |\!|\!| := \sup_{\substack{w \in V_0 \\ w \neq 0}} \frac{\int\limits_\Omega \varepsilon(\widehat{u}_\ell) : \varepsilon(w)\, dx}{\|\varepsilon(w)\|} \leq \|\varepsilon(\widehat{u}_\ell)\|,$$

so that $\|\varepsilon(\widehat{u}_\ell)\| = |\!|\!| \ell |\!|\!|$. Take

$$\widehat{\varphi} = u - v - \frac{\langle \ell, u - v\rangle}{|\!|\!| \ell |\!|\!|^2} \widehat{u}_\ell.$$

We observe that

$$\langle \ell, \widehat{\varphi}\rangle = \langle \ell, u - v\rangle \left(1 - \frac{\langle \ell, \widehat{u}_\ell\rangle}{|\!|\!| \ell |\!|\!|^2}\right) = 0,$$

so that $\widehat{\varphi} \in V_{0\ell}$. Therefore,

$$\begin{aligned}\inf_{\varphi \in V_{0\ell}} \|\varepsilon(u - v - \varphi)\| \leq \|\varepsilon(u - v - \widehat{\varphi})\| = \\ = |\langle \ell, u - v\rangle| \frac{\|\varepsilon(\widehat{u}_\ell)\|}{|\!|\!| \ell |\!|\!|^2} = \frac{1}{|\!|\!| \ell |\!|\!|} |\langle \ell, u - v\rangle|\end{aligned}$$

and we see that the left-hand side of (4.2) is equal to the right-hand one.

Now, we use the energy estimate (2.9) and the required estimate comes in the form

$$|\,\langle \ell, u - v\rangle\,|^2 \leq M_\ell^\oplus(v) := \frac{|\!|\!| \ell |\!|\!|^2}{\nu^2} \inf_{\substack{\alpha,\beta,\gamma>0,\quad \varphi \in V_{0\ell} \\ \tau \in \Sigma_{\mathrm{div}},\ q \in \widetilde{L}_2(\Omega)}} \mathcal{M}_2^\oplus(v + \varphi, \tau, \alpha, \beta, \gamma, q) \tag{4.4}$$

If $v \in \overset{\circ}{J}{}^1_2(\Omega) + u_0$, then

$$|\,\langle \ell, u - v\rangle\,|^2 = M_\ell^\oplus(v). \tag{4.5}$$

Indeed, take in (4.4) $\tau = \nu\varepsilon(u)$, $q = p$, and $\varphi = u - v + \frac{\langle \ell, u-v\rangle}{|\!|\!| \ell |\!|\!|^2} u_\ell$, where

$$\int\limits_\Omega \varepsilon(u_\ell) : \varepsilon(w)\, dx = \langle \ell, w\rangle \qquad \forall w \in \overset{\circ}{J}{}^1_2(\Omega). \tag{4.6}$$

Note that the problem (4.6) has a unique solution $u_\ell \in \overset{\circ}{J}{}^1_2(\Omega) + u_0$ and $\|\varepsilon(u_\ell)\| \leq |\!|\!|\ell|\!|\!|$. Then

$$\|\nu\varepsilon(v+\varphi) - \tau\| = \|\nu\varepsilon(v - u + \varphi)\| =$$

$$= \nu\frac{|\langle \ell, u - v\rangle|}{|\!|\!|\ell|\!|\!|^2}\|\varepsilon(u_\ell)\| \leq \nu\frac{|\langle \ell, u - v\rangle|}{|\!|\!|\ell|\!|\!|},$$

$$\|\mathrm{div}\tau + f - \nabla q\| = 0,$$

$$\mathrm{div}(v+\varphi) = \mathrm{div}u + \frac{\langle \ell, u - v\rangle}{|\!|\!|\ell|\!|\!|^2}\mathrm{div}u_\ell = 0.$$

We observe that

$$\mathcal{M}_2^{\oplus}(v+\varphi, \tau, \alpha, \beta, \gamma, q) \leq (1+\alpha+\beta)|\langle \ell, u - v\rangle|^2.$$

Since α and β can be chosen arbitrarily small, we have

$$M_\ell^{\oplus}(v) \leq |\ \langle \ell, u - v\rangle\ |^2\ .$$

Thus, for the divergence-free approximations $M_\ell^{\oplus}(v)$ gives the exact value of $|\ \langle \ell, u - v\rangle\ |^2$.

We end up this section by remarking that a collection of linear functionals ℓ_k, $k = 1, 2, \ldots M$ defines the seminorm

$$|[v]| := \sup_{k=1,\ldots M} |\langle \ell_k, v\rangle|.$$

Therefore, a posteriori estimates for $|\langle \ell_k, u-v\rangle|$ also provide a computable bound for the quantity $|[u-v]|$.

5. Local estimates for some other viscous flow problems

Finally, we shortly discuss how the above presented method can be extended to some nonlinear problems. We consider slow stationary flow of a generalized Newtonian incompressible fluid with nonlinear viscosity. The classical statement of this problem is as follows: find a vector-valued function u (velocity), a scalar-valued function p (pressure) and a tensor-valued function σ (stress deviator) such that

$$-\ \mathrm{div}\sigma = f - \nabla p \qquad \text{in } \Omega \tag{5.1}$$

$$\mathrm{div}u = 0 \text{ in } \Omega \tag{5.2}$$

$$\sigma \in \partial g(\varepsilon(u)) \text{ in } \Omega \tag{5.3}$$

$$u = u_0 \text{ on } \partial\Omega, \tag{5.4}$$

where Ω is a connected bounded domain in $\mathbb{R}^n$ having Lipschitz continuous boundary $\partial\Omega$, f and u_0 are given functions and g is the dissipative potential that defines physical properties of the fluid considered and ∂g denotes the subdifferential of g. The cases $\psi \equiv 0$, and $\psi(\varepsilon) = k_* \left|\varepsilon\right|, \quad k_* > 0$ correspond to the classical Newtonian and Bingham models, respectively. Investigation of mathematical properties of such type models was started by O. A. Ladyzhenskaya (see, e.g., [9, 10]) and continued by other authors. Regularity of weak solutions to problems related to nonlinearly viscous fluids was investigated in [6].

Henceforth, it is assumed that f and u_0 satisfy the same conditions as in Sec. 2 and $g(\varepsilon) = \frac{\nu}{2}\left|\varepsilon\right|^2 + \psi(\varepsilon)$, where $\psi : \mathbb{M}^{d\times d} \to \mathbb{R}_+$ is a convex nonnegative function such that

$$\psi(0) = 0, \quad \psi(\varepsilon) \leq c_1 \left|\varepsilon\right|^2 + c_2, \quad c_1 > 0. \tag{5.5}$$

For the dissipative potentials of such a type, energy estimates of deviations from the exact solution u were obtained in [Re5]. If $v \in \overset{\circ}{J}{}^1_2(\Omega) + u_0$, then the respective estimate is as follows:

$$\frac{\nu}{2}\left\|\varepsilon(v-u)\right\|^2 \leq (1+\beta)D_1(\varepsilon(v),\tau_1) + D_2(\varepsilon(v),\tau_2) + \\ + \left(1+\tfrac{1}{\beta}\right)\tfrac{1}{2\nu}C_F^2\left\|\operatorname{div}(\tau_1+\tau_2) + f - \nabla q\right\|^2, \tag{5.6}$$

where

$$D_1(\varepsilon(v),\tau_1) := \int\limits_\Omega \left(\tfrac{\nu}{2}\left|\varepsilon(v)\right|^2 + \tfrac{1}{2\nu}\left|\tau_1\right|^2 - \varepsilon(v) : \tau_1\right) dx = \tfrac{1}{2\nu}\|\nu\varepsilon(v) - \tau_1\|^2,$$

$$D_2(\varepsilon(v),\tau_2) := \int\limits_\Omega \left(\psi(\varepsilon(v)) + \psi^*(\tau_2) - \varepsilon(v) : \tau_2\right) dx,$$

ψ^* is the functional conjugate to ψ, $\tau_1 + \tau_2 \in \Sigma^*_{\mathrm{div}}(\Omega)$, $q \in \widetilde{W}(\Omega)$, and $\beta > 0$.

Let the right-hand side of (5.6) be equal to zero. This means that

$$-\operatorname{div}(\tau_1+\tau_2) = f - \nabla q$$

and, in addition,

$$\begin{aligned} \tau_1 &= \nu\varepsilon(v) \text{ a.e. in } \Omega, \\ \tau_2 &\in \partial\psi(\varepsilon(v)) \text{ a.e. in } \Omega. \end{aligned}$$

Since $v \in \overset{\circ}{J}{}^1_2(\Omega)+u_0$, we conclude that in such a case v coincides with exact solution u, $\tau_1 = \sigma_1$, and $\tau_2 = \sigma_2$.

Denote the right-hand side of (5.6) by $\mathcal{M}_1^{\oplus}(v, \tau_1, \tau_2, \beta, q)$. It is a functional defined on $\mathbb{R}_+ \times (V_0 + u_0) \times \Sigma \times \Sigma \times \widetilde{L}_2(\Omega)$ and has values in $\mathbb{R}_+$ that majorizes the quantity $\|\varepsilon(v-u)\|^2$. For any $\beta \in \mathbb{R}_+$, infimum of this functional on the set $(\overset{\circ}{J}{}^1_2(\Omega)+u_0) \times \Sigma \times \Sigma \times \widetilde{L}_2(\Omega)$ is equal to zero and it is attained if and only if $v = u$, $\tau_1 = \sigma_1$, $\tau_2 = \sigma_2$, and $q = p$. By means of $\mathcal{M}_1^{\oplus}(v, \tau_1, \tau_2, \beta, q)$ we obtain upper bounds for the local norm

$$\frac{\nu}{2}\|\varepsilon(u-v)\|^2_\omega \leq M^{\oplus}_{1\omega}(v), \tag{5.7}$$

where

$$M^{\oplus}_{1\omega}(v) := \inf_{\substack{\beta\in\mathbb{R},\ \varphi\in H_{0\omega} \\ \tau_1,\tau_2;\tau_1+\tau_2\in\Sigma_{\mathrm{div}},\ q\in\widetilde{L}_2(\Omega)}} \mathcal{M}_1^{\oplus}(v-\varphi, \tau_1, \tau_2, \beta, q).$$

To derive an estimate of $|\langle \ell, u-v\rangle|$, we apply the same arguments as in Sect. 4. The respective estimate is then easily obtained by minimizing the majorant $\mathcal{M}_1^{\oplus}(v-\varphi, \tau_1, \tau_2, \beta, q)$ on the functions φ that are subject to the condition $\langle \ell, \varphi\rangle = 0$.

6. Final remarks

Estimates presented above can be used for explicit error control of approximate solutions of the Stokes problem computed by various methods. In principle, there are two different strategies for applying these estimates. They differ in the way of how "free" functions in the majorants are chosen.

The first (and the simplest) way to apply the estimates is based upon the following idea: use a computed solution to define all "free" functions in the majorant. Let us explain it first on the example of global estimates (2.6) and (2.9). Assume that $\{v_{h_s}\}$ (and also $\{q_{h_s}\}$) $(s = 0, 1, 2, ...)$ are computed by solving finite-dimensional problems on the subspaces $V_{h_s} \subset \overset{\circ}{J}{}^1_2(\Omega) + u_0$ constructed for a sequence of consequently refined meshes $\mathfrak{T}_{h_s}$. In principle, we could simply set $\tau = \tau_{h_s} := \nu\varepsilon(v_{h_s})$ and substitute it in (2.6). However, in general, such a tensor-valued function does not belong to Σ_{div}. Therefore, a certain regularization (post-processing) is required to project τ_{h_s} to the required space. The respective projection operators are known for finite element approximations of various types (see, e.g., [1, 2, 16, 17]). In the majority of cases, they are rather simple so that the computation of $\widetilde{\tau}_{h_s} := G_{h_s}\tau_{h_s}$ by means of a post-processing operator G_{h_s} does not lead to large expenditures. Then, we directly evaluate the accuracy of v_{h_s} by the estimate

$$\nu\|\varepsilon(u - v_{h_s})\| \leq \|\nu\varepsilon(v_{h_s}) - \widetilde{\tau}_{h_s}\| + c_F\|\mathrm{div}\widetilde{\tau}_{h_s} + f - \nabla q_{h_s}\|. \quad (6.1)$$

The estimate (6.1) gives a guaranteed upper bound of the error on the basis of the information contained in the approximate solution computed on $\mathfrak{T}_{h_s}$. Its quality is the better the closer $\widetilde{\tau}_{h_s}$ to σ. To improve it we can use the next step solution, namely

$$\nu\|\varepsilon(u - v_{h_s})\| \leq \|\nu\varepsilon(v_{h_s}) - \widetilde{\tau}_{h_{s+1}}\| + c_F\|\mathrm{div}\widetilde{\tau}_{h_{s+1}} + f - \nabla q_{h_{s+1}}\|, \quad (6.2)$$

where $\widetilde{\tau}_{h_{s+1}} := G_{h_{s+1}}\tau_{h_{s+1}}$. The estimate (6.2) gives a guaranteed error bound by analyzing approximate solutions computed on two neighboring meshes $\mathfrak{T}_{h_s}$ and $\mathfrak{T}_{h_{s+1}}$.

Computation of local a posteriori estimates follows the same lines. For example, we can use (3.1) with $\varphi = v_{h_s} - v_{h_{s+1}}$, $\tau = \nu G_{h_{s+1}}\varepsilon(_{h_{s+1}})$, and $q = q_{h_{s+1}}$.

A more sophisticated method should be used if sharper error bounds are required. In this case, we choose certain subspaces constructed by some mesh $\mathfrak{T}_{H_s}$ (in a particular case $\mathfrak{T}_{h_s} = \mathfrak{T}_{H_s}$). Let us denote them by Σ_{H_s} (it is assumed that $\Sigma_{H_s} \subset \Sigma_{\mathrm{div}}$) and Q_{H_s} ($Q_{H_s} \subset \widetilde{W}(\Omega)$). Then, an upper bound for the error is given by the relation

$$\nu^2\|\varepsilon(u - v_{h_s})\|^2 \leq \inf_{\substack{\beta>0,\ \tau\in\Sigma_{H_s}\\ q\in Q_{H_s}}} \mathcal{M}_1^{\oplus}(v_{h_s}, \tau, \beta, q). \quad (6.3)$$

Computable bounds for local norms are obtained from (3.2) and (3.7) if replace Σ_{div}, $\widetilde{W}(\Omega)$, and $H_{0\omega}$ by some finite dimensional subspaces. For example, if $v_{h_s} \in \overset{\circ}{J}{}_2^1(\Omega) + u_0$, then we have

$$\nu^2\|\varepsilon(u - v_{h_s})\|^2 \leq \inf_{\substack{\beta\in\mathbb{R},\ \tau\in\Sigma_{H_s}\\ q\in Q_{H_s},\ \varphi\in\Phi_{H_s}}} \mathcal{M}_1^{\oplus}(v_{h_s} - \varphi, \tau, \beta, q), \quad (6.4)$$

where $\Phi_{H_s} \subset H_{0\omega}$.

A computable bound of $|\langle \ell, u - v_{h_s}\rangle|$ is obtained if minimization in the right-hand side of (4.4) is fulfilled over $q \in Q_H$, $\varphi \in \Phi_H^{\ell} \subset V_{0\ell}$, $\tau \in \Sigma_H$, and α, β, $\gamma > 0$.

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С.-Петербургское отделение
Математического института
им.В. А. Стеклова РАН

E-mail: repin@pdmi.ras.ru

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