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ABSENCE OF THE SINGULAR CONTINUOUS COMPONENT IN THE SPECTRUM OF ANALYTIC DIRECT INTEGRALS

Abstract. We give a simple proof of the absence of the singular continuous component in the spectrum of self-adjoint operators representable as analytic direct integrals.

Dedicated to the memory of O. A. Ladyzhenskaya

1°. In the last decade, there have been a large number of publications aimed to establish the absolute continuity for various classes of periodic differential operators which form an important class of operators representable as direct integrals, see e.g. survey [1] for references. The proofs of this property always use the specific features of the operators involved. On the other hand, if one is interested in the absence of the singular continuous spectrum only, the specific details of the problem become irrelevant. It has always been regarded as received wisdom (see [6, 5]) that any operator representable as a direct integral of an analytic family $A(\mathbf{k})$ with compact resolvent should have the empty singular continuous spectrum. However, a complete proof of this general fact was given only in paper [4]. In this paper the absence of the singular continuous spectrum is a by-product of the Mourre Theory for analytic direct integrals. The aim of the present note is to give a simple direct proof of this result.

We denote the spectral measure, the spectrum, the pure point and singular continuous components of the spectrum of the selfadjoint operator A by E_A, $\sigma(A)$, $\sigma_{\rm pp}(A)$ and $\sigma_{\rm sc}(A)$ respectively. The notation $\operatorname{meas}_n(\Omega)$ is used for the n-dimensional Lebesgue measure of the (measurable) set $\Omega \subset \mathbb{R}^n$.

Theorem 1 (Main Theorem). *Let $\Omega \subset \mathbb{R}^d$ be a connected open set, let*

$\mathcal{H}$ be a separable Hilbert space, and let

$$\mathfrak{H} = \int_{\Omega}^{\oplus} \mathcal{H} d\mathbf{k}. \tag{1}$$

Suppose that

$$A = \int_{\Omega}^{\oplus} A(\mathbf{k}) d\mathbf{k}, \tag{2}$$

where $A(\mathbf{k})$ is a measurable family of operators self-adjoint for each $\mathbf{k} \in \Omega$, such that the resolvent $R(\mathbf{k}) = (A(\mathbf{k}) + i)^{-1}$ is a real-analytic function of $\mathbf{k} \in \Omega$, and that $R(\mathbf{k}), \mathbf{k} \in \Omega$ is compact. Then

(i) *the set $\sigma_{\rm pp}(A)$ is at most discrete, that is it may consist only of isolated points without finite accumulation points, and each eigenvalue $\lambda \in \sigma_{\rm pp}(A)$ is of infinite multiplicity;*
(ii) *$\sigma_{\rm sc}(A) = \varnothing$.*

Remark 2. The domain Ω can be replaced by a real-analytic manifold.

2°. We precede the proof of the Main Theorem with an auxiliary result.

Let $f = f(\mathbf{k}, \lambda)$ be a real-analytic function on the set $U \times I$, where $U \subset \mathbb{R}^d$ is a connected open set and $I \subset \mathbb{R}$ is an open interval. Let us pick a subset $\Lambda \subset I$, and denote

$$M = M_\Lambda(f) = \{\mathbf{k} \in U, \lambda \in \Lambda : f(\mathbf{k}, \lambda) = 0\}.$$

Clearly,

$$PM_\Lambda(f) = \{\mathbf{k} \in U : f(\mathbf{k}, \lambda) = 0 \quad \text{for some} \quad \lambda \in \Lambda\},$$

where $P : \mathbb{R}^{d+1} \to \mathbb{R}^d$ is the orthogonal projection onto the subspace $\mathbb{R}^d$ of the variable $\mathbf{k}$.

Lemma 3. *Let f be as defined above. Suppose that for any $\lambda \in \Lambda$ there exists a point $\mathbf{k} \in U$ such that $f(\mathbf{k}, \lambda) \neq 0$. Then $\operatorname{meas}_d PM_\Lambda(f) = 0$ if $\operatorname{meas}_1 \Lambda = 0$.*

This Lemma appears for example in [5] and [3]. For the sake of completeness we provide the full proof. Let us first make the following elementary observation:

Lemma 4. *Let f be as defined above and let*

$$\widehat{M} = \widehat{M}_{\Lambda,j}(f) = \{(\mathbf{k},\lambda) \in M_\Lambda(f) : \partial_{k_j} f(\mathbf{k},\lambda) \neq 0\},$$

for some $j = 1, 2, \dots, d$. Then $\mathrm{meas}_d\, P\hat{M} = 0$ if $\mathrm{meas}_1 \Lambda = 0$.

Proof. Assume without loss of generality that $j = d$, and denote $\mathbf{k} = (\hat{\mathbf{k}}, k_d)$. Let us pick a point $(\mathbf{k}_0, \lambda_0) \in \widehat{M}$. Since $\partial_{k_d} f(\mathbf{k}_0, \lambda_0) \neq 0$, by the Implicit Function Theorem there exists a parallelepiped $V \ni (\mathbf{k}_0, \lambda_0)$, $V \subset U \times I$ of the form $\times_{j=1}^{d}(a_j, b_j) \times (A, B)$ and a real-analytic function

$$\theta : \widehat{V} \to \mathbb{R}, \quad \widehat{V} = \times_{j=1}^{d-1}(a_j, b_j) \times (A, B),$$

such that

$$V \cap \widehat{M} = V \cap M = \{\big(\hat{\mathbf{k}}, \theta(\hat{\mathbf{k}}, \lambda), \lambda\big),\ (\hat{\mathbf{k}}, \lambda) \in \widehat{V}_\Lambda\},$$
$$\widehat{V}_\Lambda = \times_{j=1}^{d-1}(a_j, b_j) \times \big((A, B) \cap \Lambda\big).$$

Consequently,

$$P(V \cap \widehat{M}) = \{\big(\hat{\mathbf{k}}, \theta(\hat{\mathbf{k}}, \lambda)\big),\ (\hat{\mathbf{k}}, \lambda) \in \widehat{V}_\Lambda\}.$$

Since $\mathrm{meas}_1 \Lambda = 0$, we also have $\mathrm{meas}_d \widehat{V}_\Lambda = 0$, and hence the measure

$$\mathrm{meas}_d\, P(V \cap \widehat{M}) = \int\limits_{\hat{V}_\Lambda} |\partial_\lambda \theta(\hat{\mathbf{k}}, \lambda)| d\lambda d\hat{\mathbf{k}}$$

also equals zero.

The set $\widehat{M}$ can be covered by a countable collection of neighbourhoods V_i of the same type. Noting that

$$P\widehat{M} \subset \cup_i P(V_i \cap \widehat{M}),$$

we obtain the required result. $\square$

Proof of Lemma 3. The function $f(\,\cdot\,, \lambda)$ is not identically zero for $\lambda \in \Lambda$, and hence for each $(\mathbf{k}, \lambda) \in M_\Lambda(f)$ there exist two multi-indices $\boldsymbol{\beta} = (\beta_1, \beta_2, \dots, \beta_d)$ and $\boldsymbol{\mu} = (\mu_1, \mu_2, \dots, \mu_d)$ such that $|\boldsymbol{\beta}| = |\boldsymbol{\mu}| + 1$ and $\partial_{\mathbf{k}}^{\boldsymbol{\beta}} f(\mathbf{k}, \lambda) \neq 0$, $\partial_{\mathbf{k}}^{\boldsymbol{\mu}} f(\mathbf{k}, \lambda) = 0$, so that

$$(\mathbf{k}, \lambda) \in \bigcup_{j=1}^{d} \widehat{M}_{\Lambda,j}(\partial_{\mathbf{k}}^{\boldsymbol{\mu}} f) \subset \bigcup_{j=1}^{d} \bigcup_{|\boldsymbol{\alpha}| \geqslant 0} \widehat{M}_{\Lambda,j}(\partial_{\mathbf{k}}^{\boldsymbol{\alpha}} f).$$

Therefore

$$PM_\Lambda(f) \subset \bigcup_{j=1}^{d} \bigcup_{|\boldsymbol{\alpha}|\geqslant 0} P\widehat{M}_{\Lambda,j}(\partial_{\mathbf{k}}^{\boldsymbol{\alpha}} f).$$

The measure of every set in the r.h.s. is zero by Lemma 4. $\square$

3°. Proof of Theorem 1. It suffices to prove that for any open interval (α, β) the set $\sigma_{\mathrm{pp}}(A) \cap (\alpha, \beta)$ is discrete and $\sigma_{\mathrm{sc}}(A) \cap (\alpha, \beta) = \varnothing$. From now on we assume without further comments that $\lambda \in \Delta$, where Δ is a fixed complex neighbourhood of real interval (α, β). Clearly,

$$\ker(A(\mathbf{k}) - \lambda) = \ker\bigl(I - T(\mathbf{k}, \lambda)\bigr), \quad T(\mathbf{k}, \lambda) = (\lambda + i)R(\mathbf{k}).$$

In view of the self-adjointness, the above kernel is trivial if $\operatorname{Im}\lambda \neq 0$. Since the operator $R(\mathbf{k})$ is compact, in a neighbourhood of each $\mathbf{k}_0 \in \Omega$ the question of invertibility of $I - T(\mathbf{k}, \lambda)$ reduces to the finite-dimensional case.

Indeed, let $\mathbf{k}_0 \in \Omega$. Splitting $R(\mathbf{k}_0)$ in the sum of two operators, let $T(\mathbf{k}_0, \lambda) = (\lambda + i)R(\mathbf{k}_0) = K_0(\lambda) + S_0(\lambda)$ where $K_0(\lambda)$ is finite-dimensional and $\|S_0(\lambda)\| < 1/2$ for all $\lambda \in \Delta$. Assume that the neighbourhood $U \ni \mathbf{k}_0$ is such that $\|T(\mathbf{k}, \lambda) - T(\mathbf{k}_0, \lambda)\| < 1/2$ for all $\mathbf{k} \in U, \lambda \in \Delta$. Define

$$S(\mathbf{k}, \lambda) = S_0(\lambda) + T(\mathbf{k}, \lambda) - T(\mathbf{k}_0, \lambda),$$

so that $\|S(\mathbf{k}, \lambda)\| < 1$, $\mathbf{k} \in U, \lambda \in \Delta$, and hence

$$I - T(\mathbf{k}, \lambda) = I - S(\mathbf{k}, \lambda) - K_0(\lambda) = \bigl(I - K(\mathbf{k}, \lambda)\bigr)\bigl(I - S(\mathbf{k}, \lambda)\bigr),$$
$$K(\mathbf{k}, \lambda) = K_0(\lambda)\bigl(I - S(\mathbf{k}, \lambda)\bigr)^{-1}, \ \mathbf{k} \in U, \lambda \in \Delta,$$

the operator-function $K(\mathbf{k}, \lambda)$ being real-analytic on the domain $V = U \times (\alpha, \beta)$. Consequently, the set

$$\{\mathbf{k} \in U : \ker(A(\mathbf{k}) - \lambda) \neq \{0\}\}$$

coincides with the set where $\ker(I - K(\mathbf{k}, \lambda)) \neq \{0\}$, or, which is the same, where

$$f(\mathbf{k}, \lambda) := \det\bigl(I - K(\mathbf{k}, \lambda)\bigr) = 0.$$

By definition of the finite-dimensional operator $K(\mathbf{k}, \lambda)$ the function f is real-analytic on V. For any $k \in U$ the function $f(\mathbf{k}, \cdot\,)$ is analytic in Δ, and

$$f(\mathbf{k}, \lambda) \neq 0, \quad \operatorname{Im}\lambda \neq 0,$$

so that f is not identically zero.

The above construction can be implemented for a neighbourhood of each point $\mathbf{k} \in \Omega$, and hence, there exists a countable collection of open sets

$$U_m, \ m = 1, 2, \dots,$$

such that $\Omega \subset \cup_m U_m$ and for each m there exists a function f_m, real-analytic on $V_m = U_m \times (\alpha, \beta)$, for which

$$\text{If} \quad (\mathbf{k}, \lambda) \in V_m, \quad \text{then} \quad \lambda \in \sigma(A(\mathbf{k})) \Longleftrightarrow f_m(\mathbf{k}, \lambda) = 0.$$

Proof of (i). Suppose now that $\mu \in (\alpha, \beta)$ is an eigenvalue of A. Then

$$\operatorname{meas}_d\{\mathbf{k} \in \Omega : \mu \in \sigma\big(A(\mathbf{k})\big)\} > 0,$$

and hence for some $m = 1, 2, \dots$ one has

$$\operatorname{meas}_d\{\mathbf{k} \in U_m, \ f_m(\mathbf{k}, \mu) = 0\} > 0.$$

Using the Fubini Theorem one can prove by induction in dimension, that any analytic function which vanishes on a set of positive measure, is identically zero. Since $f_m(\,\cdot\,, \mu)$ is analytic as a function of $\mathbf{k}$, this implies that

$$f_m(\mathbf{k}, \mu) = 0, \quad \forall \mathbf{k} \in U_m.$$

If the neighbourhoods U_m and U_l intersect, then $f_m(\mathbf{k}, \mu) = 0$ for all $\mathbf{k} \in U_m \cap U_l$, and therefore $f_l(\mathbf{k}, \mu) = 0$ for all $\mathbf{k} \in U_l$ as well. Since $\{U_m\}$ is a covering of Ω and Ω is connected, we conclude that $f_m(\mathbf{k}, \mu) = 0$ for all $m = 1, 2, \dots$. This implies that the point spectrum $\sigma_{\mathrm{pp}}(A)$ consists of isolated points, since none of the functions f_m is identically zero.

Let us now prove that μ is an eigenvalue of A of infinite multiplicity. Pick a point $\mathbf{k}_0 \in \Omega$ and a closed ball $B(\mathbf{k}_0, r) \subset \Omega$ of radius $r > 0$ centered at $\mathbf{k}_0$. Denote by $\lambda_j(\mathbf{k})$, $j = 1, 2, \dots$ the eigenvalues of the family $A(\mathbf{k})$ arranged in ascending order. These are continuous functions. Let us select from this collection the eigenvalues such that

$$(\alpha, \beta) \bigcap \{\lambda_j(\mathbf{k}), \mathbf{k} \in B(\mathbf{k}_0, r)\} \neq \varnothing.$$

Since the operators $R(\mathbf{k})$ are compact, the number of such eigenvalues is finite. It is straightforward to show that for suitable $\mathbf{k}_0 \in \Omega$ and $r > 0$ the number μ is an isolated eigenvalue of $A(\mathbf{k})$ of constant multiplicity for all $\mathbf{k} \in B(\mathbf{k}_0, r)$. Let γ be a circular contour in the complex plane, oriented counterclockwise, enclosing only μ, and separated from the other eigenvalues. Now, using the standard formula

$$P(\mathbf{k}) = -\frac{1}{2\pi i}\int_{\gamma}(A(\mathbf{k})-\zeta)^{-1}d\zeta,$$

for the projection $P(\mathbf{k})$ onto the subspace corresponding to μ, one sees that it is norm-continuous in $\mathbf{k}\in B(\mathbf{k}_0, r)$. Let $\psi\in\mathcal{H}$ be a normalized eigenvector of $A(\mathbf{k}_0)$. Then $\psi(\mathbf{k}) = P(\mathbf{k})\psi$ is an eigenvector of $A(\mathbf{k})$. Taking, if necessary, a smaller value of r, one can claim that $\|\psi(\mathbf{k})\|_{\mathcal{H}} > 1/2$ for all $\mathbf{k}\in B(\mathbf{k}_0, r)$. Let $\mathcal{O}_j\subset B(\mathbf{k}_0, r), j = 1, 2, \ldots$ be an infinite set of non-intersecting balls, and let $\chi_j = \chi_j(\mathbf{k})$ be their characteristic functions. Define the sequence of non-zero vectors $\Psi_j\in\mathfrak{H}$ with fibres $\psi(\mathbf{k})\chi_j(\mathbf{k})$. Clearly, each of these vectors is an eigenvector of A, and they are linearly independent.

Proof of (ii). To prove that $\sigma_{\mathrm{sc}}(A) = \varnothing$, suppose that $\Lambda\subset(\alpha,\beta)$ is a non-empty set of measure zero, such that

$$\Lambda\cap\sigma_{\mathrm{pp}}(A) = \varnothing. \tag{3}$$

Define

$$K_m = \{\mathbf{k}\in U_m : f_m(\mathbf{k},\lambda) = 0 \quad \text{for some} \quad \lambda\in\Lambda\}.$$

For each $\mathbf{k}\in U_m\setminus K_m$ we have $f_m(\mathbf{k},\lambda)\neq 0$ for all $\lambda\in\Lambda$. This implies that

$$\Lambda\cap\sigma(A(\mathbf{k})) = \varnothing, \quad E_{A(\mathbf{k})}(\Lambda) = 0, \quad \mathbf{k}\in U_m\setminus K_m.$$

Due to (3), for all $\lambda\in\Lambda$ the function $f_m(\,\cdot\,,\lambda)$ is not identically zero, and hence $\mathrm{meas}_d\, K_m = 0$ by virtue of Lemma 3. Now we can write for any $u\in\mathfrak{H}$ that

$$\|E_A(\Lambda)u\|^2_{\mathfrak{H}} = \int_{\Omega}\|E_{A(\mathbf{k})}(\Lambda)u(\mathbf{k})\|^2_{\mathcal{H}}d\mathbf{k}$$

$$\leqslant\sum_m\int_{U_m\setminus K_m}\|E_{A(\mathbf{k})}(\Lambda)u(\mathbf{k})\|^2_{\mathcal{H}}d\mathbf{k} = 0.$$

This shows that outside the discrete set $\sigma_{\mathrm{pp}}(A)$ the spectrum of A is absolutely continuous, that is $\sigma_{\mathrm{sc}}(A) = \varnothing$. □

4°. Two examples. Here we illustrate the use of Theorem 1 by considering examples of two widely used periodic operators involving magnetic fields. We omit technical details.

Example 5. The first example is the Schrödinger operator with a periodic *magnetic potential* and a *metric*. In the space $\mathsf{L}^2(\mathbb{R}^d)$, $d \geqslant 2$, consider the quadratic form

$$h[u] = \int\limits_{\mathbb{R}^d} \langle \mathbf{G}(-i\nabla - \mathbf{a})u, (-i\nabla - \mathbf{a})u\rangle dx + \int\limits_{\mathbb{R}^d} |u|^2 d\nu(x). \qquad (4)$$

Suppose that the positive matrix-valued function $\mathbf{G}$, vector-function $\mathbf{a}$ and the real-valued charge ν are periodic with respect to some full rank lattice $\Gamma \subset \mathbb{R}^d$, and satisfy the conditions

$$c \leqslant \mathbf{G} \leqslant C, \text{a. a. } \ x \in \mathbb{R}^d,$$
$$\mathbf{a} \in \mathsf{L}^d_{\mathrm{loc}}(\mathbb{R}^d) \text{ for } d \geqslant 3, \quad \mathbf{a} \in \mathsf{L}^q_{\mathrm{loc}}(\mathbb{R}^d), \ q > 2, \text{ for } d = 2,$$

and

$$\int\limits_{\mathcal{O}} |u|^2 |d\nu| \leqslant \varepsilon \int\limits_{\mathcal{O}} |\nabla u|^2 dx + C(\varepsilon) \int\limits_{\mathcal{O}} |u|^2 dx, \quad u \in \mathsf{H}^1(\mathcal{O}), \ \ \forall \varepsilon > 0. \quad (5)$$

Here $\mathcal{O}$ is the cell of the lattice Γ, and H^1 is the standard Sobolev space. The form (4) is closed and semi-bounded from below on the domain $\mathsf{H}^1(\mathbb{R}^d)$, and hence it defines a self-adjoint operator, which we denote by H. The integral with the charge ν can model a real-valued potential and/or jump conditions on some Γ-periodic surfaces in $\mathbb{R}^d$.

Denote by Ω the interior of the fundamental domain of the dual lattice. The Gelfand transform

$$(Uf)(x, \mathbf{k}) = e^{-i\mathbf{k}x} \sum_{\gamma \in \Gamma} e^{-i\mathbf{k}\gamma} f(x + \gamma),$$

is easily shown to be a unitary operator from $\mathsf{L}^2(\mathbb{R}^d)$ onto the direct integral (1) with $\mathcal{H} = \mathsf{L}^2(\mathcal{O})$. The operator $A = UHU^*$ is the direct integral (2) with self-adjoint operators $A(\mathbf{k})$ acting in $\mathcal{H}$ defined by the quadratic form

$$\int\limits_{\mathcal{O}} \langle \mathbf{G}(-i\nabla - \mathbf{a} + \mathbf{k})u, (-i\nabla - \mathbf{a} + \mathbf{k})u\rangle dx + \int\limits_{\mathcal{O}} |u|^2 d\nu(x) \qquad (6)$$

on $\mathsf{H}^1(\mathbb{T}^d)$, $\mathbb{T}^d = \mathbb{R}^d/\Gamma$. Using standard methods one can show that the operator $A(\mathbf{k})$ has a compact resolvent, which is a real-analytic function of $\mathbf{k} \in \Omega$. Consequently, by Theorem 1 the spectrum of H does not have any singular continuous component.

It is interesting to note that for $d \geqslant 3$, $\mathbf{a} = 0, V = 0$ and a suitable non-smooth matrix $\mathbf{G}$, the point spectrum of H may be non-empty, see example constructed in [2].

The absence of the singular continuous spectrum for the operator in the above example was proved in [5] with $\mathbf{G} = \mathbf{I}$, $d\nu = 0$. One can say even more: under certain conditions on $\mathbf{G}, \mathbf{a}$ and ν the spectrum of H is absolutely continuous, see [1, 7] for references. We stress again that the absolute continuity requires more sophisticated techniques, whereas the absence of the singular continuous spectrum is rather elementary.

In the next example, the absolute continuity of the operator is yet to be investigated, but the absence of the singular continuous spectrum follows again from Theorem 1.

Example 6. Consider in $\mathsf{L}^2(\mathbb{R}^2)$ the Schrödinger operator with a constant magnetic field $\mathbf{B} = \operatorname{curl} \mathbf{a}$ with

$$\mathbf{a} = (-Bx_2, 0), \ B = |\mathbf{B}|,$$

defined by the quadratic form (4) with $\mathbf{G} = \mathbf{I}$ and a Γ-periodic charge ν satisfying again the condition (5). In view of the diamagnetic inequality the following inequality also holds:

$$\int\limits_{\mathbb{R}^2} |u|^2 |d\nu| \leqslant \varepsilon \int\limits_{\mathbb{R}^2} |(-i\nabla - \mathbf{a})u|^2 dx + C(\varepsilon) \int\limits_{\mathbb{R}^2} |u|^2 dx, \quad u \in \mathsf{C}_0^\infty(\mathbb{R}^2),$$

for any $\varepsilon > 0$. Consequently the form $h[\,\cdot\,]$ is semi-bounded from below and closable on $\mathsf{C}_0^\infty(\mathbb{R}^2)$. As in the previous example we denote by H the operator defined by the form (4). For simplicity we assume that the lattice is orthogonal, and hence $\mathcal{O} = [0, T_1) \times [0, T_2)$ with some positive T_1, T_2. Denote $\Omega = (0, 2\pi T_1^{-1}) \times (0, 2\pi T_2^{-1})$ the interior of the fundamental domain of the dual lattice $\Gamma^\dagger$.

Assume that the flux of the magnetic field through $\mathcal{O}$ is integer, i.e.

$$\Phi := \frac{B}{2\pi}|\mathcal{O}| = \frac{B}{2\pi} T_1 T_2 \in \mathbf{Z}. \tag{7}$$

Define the *magnetic translations*

$$\begin{cases} (\tau_1 u)(x_1, x_2) = & u(x_1 + T_1, x_2), \\ (\tau_2 u)(x_1, x_2) = & u(x_1, x_2 + T_2) \exp\big(iBT_2 x_1\big). \end{cases}$$

The operator H commutes with τ_1, τ_2. It is straightforward to check that

$$\tau_1 \tau_2 = \tau_2 \tau_1 e^{2\pi i \Phi},$$

and hence under the condition (7) the translations τ_1 and τ_2 commute with each other. The Gelfand transform has now a form which is different from the previous example – it involves magnetic translations:

$$(Uf)(x, \mathbf{k}) = e^{-i\mathbf{k}x} \sum_{l,s \in \mathbf{Z}} e^{-i\mathbf{k}_1 lT_1} e^{-ik_2 sT_2} (\tau_1^l \tau_2^s f)(x)$$

$$= e^{-i\mathbf{k}x} \sum_{l,s \in \mathbf{Z}} e^{-i\mathbf{k}_1 lT_1} e^{-ik_2 sT_2} e^{isBT_2 x_1} f(x_1 + lT_1,\ x_2 + sT_2).$$

The operator U is unitary from $\mathsf{L}^2(\mathbb{R}^2)$ onto the direct integral (1) with $\mathcal{H} = \mathsf{L}^2(\mathcal{O})$. A direct calculation shows that $A = UHU^*$ is the direct integral (2) with self-adjoint operators $A(\mathbf{k})$ acting in $\mathcal{H}$, defined by the quadratic forms (6) with $\mathbf{G} = \mathbf{I}$ and the $\mathbf{k}$-independent domain

$$D(a(\mathbf{k})) = \big\{u \in \mathsf{H}^1(\mathcal{O}) : u(x_1, 0) = e^{iBT_2 x_1} u(x_1, T_2),$$
$$u(0, x_2) = u(T_1, x_2)\big\}.$$

Again, as in the previous example, one easily shows that the operator $A(\mathbf{k})$ has a compact resolvent which is analytic in $\mathbf{k} \in \Omega$. Now Theorem 1 implies that $\sigma_{\mathrm{sc}}(H) = \varnothing$.

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