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APPROXIMATION OF CONVOLUTIONS BY ACCOMPANYING LAWS WITHOUT CENTERING

1. Introduction

The aim of this paper is to give a sharpening of a theorem of Le Cam [9]. In his seminal paper, Le Cam estimated the rate of approximation in the Kolmogorov distance for distributions of sums of independent random variables. As approximating distributions he took accompanying compound Poisson distributions.

Let us first introduce some notation. Let $\mathfrak{F}$ denote the set of probability distributions defined on the Borel σ-field of subsets of the real line $\mathbf{R}$ and $\mathcal{L}(\xi) \in \mathfrak{F}$ the distribution of a random variable ξ. For $F \in \mathfrak{F}$ the concentration function is defined by $Q(F, b) = \sup_x F\big\{[x, x+b]\big\}$, $b \geqslant 0$. For $F, H \in \mathfrak{F}$ we denote the corresponding distribution functions by $F(x) = F\{(-\infty, x]\}$, $H(x)$ and the Kolmogorov distance by

$$\rho(F, H) = \sup_x \big| F(x) - H(x) \big|.$$

By c we denote absolute positive constants. Note that constants c can be different in different (or even in the same) formulas. For some positive quantities a and b writing $a \asymp b$ means that $a \leqslant cb$ and $b \leqslant ca$.

Let $X_1, X_2, \dots, X_n$ be independent generally speaking not identically distributed elements of a measurable space $(\mathfrak{X}, \mathcal{S})$ and $f : \mathfrak{X} \to \mathbf{R}$ be a real valued Borel function. Let $F_i = \mathcal{L}(f(X_i))$, $i = 1, 2, \dots, n$, be the distributions of $f(X_i)$. Then the sum

$$S = f(X_1) + f(X_2) + \dots + f(X_n)$$

has the distribution $\prod_{i=1}^n F_i$ (products and powers of measures will be understood in the convolution sense: $FH = F * H$, $H^m = H^{m*}$, $H^0 =$

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$E \stackrel{\mathrm{def}}{=} E_0$, where E_a is the distribution concentrated at a point $a \in \mathbf{R}$). A natural approximating infinitely divisible distribution for $\prod_{i=1}^n F_i$ is the accompanying compound Poisson distribution $\prod_{i=1}^n e(F_i)$, where

$$e(H) \stackrel{\mathrm{def}}{=} e^{-1} \sum_{m=0}^{\infty} \frac{H^m}{m!}, \quad H \in \mathfrak{F},$$

and, more generally,

$$e(\lambda H) \stackrel{\mathrm{def}}{=} e^{-\lambda} \sum_{m=0}^{\infty} \frac{\lambda^m H^m}{m!}, \qquad \lambda > 0.$$

It is easy to see that $\prod_{i=1}^n e(F_i)$ is the distribution of

$$T = \sum_{i=1}^{n} \sum_{j=1}^{\nu_i} f(X_{i,j}),$$

where $X_{i,j}$ and ν_i, $i = 1, \dots, n$, $j = 1, 2, \dots$, are independent in aggregate random elements of the space $\mathfrak{X}$ and random variables respectively with $\mathcal{L}(X_{i,j}) = \mathcal{L}(X_i)$ and $\mathcal{L}(\nu_i) = e(E_1)$. Clearly, $e(E_1)$ is the Poisson distribution with mean 1. Thus, the sum T is defined similarly to S, but the initial sample $\mathbf{X} = (X_1, X_2, \dots, X_n)$ is replaced by its Poissonized version $\mathbf{Y} = \{X_{i,j} : i = 1, \dots, n, \ j = 1, 2, \dots, \nu_i\}$. Poissonization of the sample is known as one of the most powerful tools in studying empirical processes. The random set $\mathbf{Y}$ may be considered as a Poisson point process on the space $\mathfrak{X}$ with intensity measure $\sum_{i=1}^n \mathcal{L}(X_i)$.

At the present time there exists a lot of results on the closeness of the distributions $\prod_{i=1}^n F_i$ and $\prod_{i=1}^n e(F_i)$, see Barbour and Chryssaphinou [3] and Zaitsev [14] and the bibliography therein. However, the most of these results require that distributions F_i are appropriately centered. In general case a better approximation can be obtained for the accompanying compound Poisson approximation of $\prod_{i=1}^n F_i E_{-a_i}$ by $\prod_{i=1}^n e(F_i E_{-a_i})$ with some centering constants a_i. In other words, $\prod_{i=1}^n F_i$ should be approximated by $\prod_{i=1}^n E_{a_i} e(F_i E_{-a_i})$.

The importance of centering in the accompanying approximation was discovered by Le Cam [9]. He has shown that accompanying laws with centering (see the distribution H_6 below) provide the rate of approximation of the form $cn^{-1/3}$ for distributions of sums of i.i.d. random variables with an absolute constant c without any assumptions on the distribution of summands. The same rate was obtained earlier by Kolmogorov [8] for the approximation by infinitely divisible distributions. Le Cam's results said that accompanying laws are sufficient for such a rate. Later Arak [1] obtained the optimal rate $cn^{-2/3}$ for infinitely divisible approximation (see Arak and Zaitsev [2] for the history of the problem). Arak's infinitely divisible approximation is much more complicated than the approximation by accompanying laws. Ibragimov and Presman [6] have shown that the rate $cn^{-1/3}$ is optimal for the accompanying approximations (see Arak and Zaitsev [2], inequality (1.3), p. 178).

Example 1. Let $F_1 = F_2 = \cdots = F_n = F$, $\int |x|^3\, F\{dx\} < \infty$, $a = \int x\, F\{dx\} \neq 0$ and $\sigma^2 = \int (x-a)^2\, F\{dx\} > 0$. It is easy to show that $\mathbf{E}\, S = \mathbf{E}\, T = na$, $\operatorname{Var} S = n\sigma^2$ and $\operatorname{Var} T = n(\sigma^2 + a^2)$. Let Φ_1 and Φ_2 be the normal distributions with the same means and variances as $\mathcal{L}(S)$ and $\mathcal{L}(T)$ respectively. Then, by the Berry–Esseen bound in the CLT, $\rho(\prod_{i=1}^n F_i, \Phi_1) = O(n^{-1/2})$ and $\rho(\prod_{i=1}^n e(F_i), \Phi_2) = O(n^{-1/2})$. Furthermore, $\rho(\Phi_1, \Phi_2) \asymp \min\{1,\, a^2/\sigma^2\}$. Therefore, $\rho(\prod_{i=1}^n F_i, \prod_{i=1}^n e(F_i)) \geqslant c \min\{1,\, a^2/\sigma^2\}$, for sufficiently large $n \geqslant n_0$.

Example 2. In the degenerate case, for example, if $F = E_1$, then $\prod_{i=1}^n F_i = E_n$ and $\prod_{i=1}^n e(F_i) = e(nE_1)$, the Poisson law with parameter n. Clearly, $\rho(E_n, e(nE_1)) \geqslant c$.

Examples 1 and 2 show that the approximation without centering is not always successful. To ensure the validity of such an approximation, one needs additional assumptions such as zero mean, symmetry or a large atom at zero, see Zaitsev [14] and the bibliography therein.

Le Cam [9] considered not only i.i.d. summands, he obtained bounds for the accompanying compound Poisson approximation for distributions of sums of independent non-identically distributed random variables—with and without centering. To formulate his results we need some additional notation.

Let distributions $F_i \in \mathfrak{F}$ be represented as mixtures

$$F_i = (1 - p_i)U_i + p_i V_i,$$

where

$$0 \leqslant p_i \leqslant p = \max_{1 \leqslant i \leqslant n} p_i \leqslant 1,$$

$U_i, V_i \in \mathfrak{F}$ and

$$a_i = \int_{-\infty}^{\infty} x\, U_i\{dx\}, \quad \|\mathbf{a}\|_2^2 = \sum_{i=1}^n a_i^2, \quad \|\mathbf{a}\|_\infty = \max_{1 \leqslant i \leqslant n} |a_i|,$$

$$\sigma_i^2 = (1-p_i)\int_{-\infty}^{\infty}(x-a_i)^2\,U_i\{dx\},\quad B^2=\sum_{i=1}^n \sigma_i^2,$$

Denote

$$H_1=\prod_{i=1}^n F_i,\quad H_2=\prod_{i=1}^n e(F_i),\quad H_3=\prod_{i=1}^n F_iE_{-a_i},$$

$$H_4=\Phi\prod_{i=1}^n e(p_iV_iE_{-a_i}),\quad H_5=\prod_{i=1}^n e(F_iE_{-a_i}),\quad H_6=\prod_{i=1}^n E_{a_i}e(F_iE_{-a_i}),$$

where Φ is the normal distribution with mean zero and variance B^2.

Le Cam [9] considered the rate of approximation by accompanying laws without centering proving the following result (see Le Cam ([10], Theorem 2, p. 431)).

Theorem 1. *Let the above conditions be satisfied and, for some $\tau>0$,*

$$U_i\{\,[-\tau,\,\tau]\,\}=1,\qquad i=1,2,\dots,n. \tag{1}$$

Then

$$\rho(H_1,\,H_2)\leqslant c\,\bigl(p^{1/3}+\bigl((1+\tau^{-2}\,\|\mathbf{a}\|_2^2\,)D^{-2}(\tau)\bigr)^{1/3}\bigr), \tag{2}$$

where

$$D^2(\tau)=\sum_{i=1}^n\int_{-\infty}^{\infty}\min\{\,1,x^2\tau^{-2}\,\}F_i\{dx\}.$$

Applying Theorem 1 in the case $a_i=0$, $i=1,2,\dots,n$, we obtain another result of Le Cam [9] which contains the rate of accompanying approximation with centering, see Le Cam ([10], Theorem 1, p. 429):

Corollary 1. *Let the conditions of Theorem 1 be satisfied. Then*

$$\rho(H_1,\,H_6)=\rho(H_3,\,H_5)\leqslant c\,\bigl(p^{1/3}+D^{-2/3}(\tau)\bigr). \tag{3}$$

For the proof it suffices to notice that the distributions $F_iE_{-a_i}$ satisfy the conditions of Theorem 1 with $a_i=0$ and with replacing τ by 2τ.

The statement of Corollary 1 was slightly improved by Ibragimov and Presman [6]. Zaitsev [12, 13] proved Theorem 2 below which sharpens and generalizes the statement of Corollary 1. To formulate this theorem we need some additional notation.

Following Katz [7] and Petrov [11], we introduce the class $\mathcal{G}$ of real valued functions on $\mathbf{R}$ satisfying the conditions

a) $g(\cdot)$ in is a non-negative even function, which is strictly positive for $x \neq 0$ and does not decrease for $x \geqslant 0$.

b) the function $x/g(x)$ is non-decreasing for $x > 0$.

Theorem 2 (Zaitsev [13]). *Let the conditions of Theorem 1 be satisfied without condition* (1) *and* $g \in \mathcal{G}$. *Denote*

$$\beta_i = \beta_i(g) = (1 - p_i) \int_{-\infty}^{\infty} (x - a_i)^2 g(x - a_i)\, U_i\{dx\} < \infty,$$

$$\beta = \beta(g) = \sum_{i=1}^{n} \beta_i(g).$$

Define $\lambda = \lambda(g) = \min\Big\{B, \dfrac{\beta(g)}{B\, g(B)}\Big\}$, *if* $B^2 > 0$, *and* $\lambda = 0$ *otherwise. Let* $B^2 > 0$. *Then, for* $j = 4, 5$,

$$\rho\big(H_3, H_j\big) \leqslant c\,\big(p + \min_{3 \leqslant k \leqslant 5} Q(H_k, \lambda)\big) \asymp p + \frac{\lambda}{B} Q\Big(\prod_{i=1}^{n} e(p_i V_i E_{-a_i}), B\Big). \tag{4}$$

If $B^2 = 0$, *then* $H_4 = H_5$ *and* $\rho\big(H_3, H_5\big) \leqslant cp$.

It is easy to show that, under condition (1), $\min_{3 \leqslant k \leqslant 5} Q(H_k, \lambda) \leqslant c\,(p + D^{-1}(\tau))$, see inequalities (16) and (17) below. Therefore, Theorem 2 is better with respect to the order than Corollary 1. Moreover, a particular case of Theorem 2 ($p = 0$) gives the bound

$$\rho(H_3, \Phi) \leqslant c \min\Big(1, \frac{\beta}{B^2 g(B)}\Big),$$

coinciding with a classical bound in the CLT of Petrov [11] (see a discussion in Zaitsev [13]). The main result of Zaitsev [12] is a particular case of Theorem 2 with $g(x) \equiv |x|$. Čekanavičius [4] proved an analogue of the result of Zaitsev [12] estimating the rate of approximation of convolutions of identical distributions by signed infinitely divisible measures.

To analyze the optimality of the statement of Theorem 2, let us consider two examples.

Example 3. Let $U_1 = U_2 = \cdots = U_n = E$, $V_1 = V_2 = \cdots = V_n = E_1$ and $p_1 = p_2 = \cdots = p_n = p$. Then $F_1 = F_2 = \cdots = F_n = (1 - p)E + pE_1$,

$\lambda = a_1 = a_2 = \cdots = a_n = 0$, H_3 is the standard binomial distribution with parameters n and p and H_5 is the Poisson distribution with mean np. It is well-known that in this case $\rho\big(H_3,\, H_5\big) \geqslant c\, \min\big\{\, np^2,\, p\,\big\}$ (see, e.g., Arak and Zaitsev [2], Lemma 4.1, p. 197). On the other hand, $Q(H_5, 1) \leqslant c\,(np)^{-1/2}$, see, e.g., Lemma 2 below. This shows the optimality of the first summand in (4), if $(np)^{-1/2} \leqslant p$ or $n \geqslant p^{-3}$.

Example 4. Let $F = F_1 = F_2 = \cdots = F_n = E_{-1}/2 + E_1/2$, $p = 0$. Then $a_1 = a_2 = \cdots = a_n = 0$, $H_3 = F^n$ and $H_5 = e(nF)$. It is easy to show that $\rho\big(H_3,\, H_5\big) \geqslant c\, n^{-1/2}$ (see Arak and Zaitsev [2], p. 198–199, the proof of the lower bound in inequality (1.4), p. 178). Indeed, the distribution H_5 has atoms of order $O(n^{-1/2})$ at points (even or odd) where the distribution H_3 has no atoms. On the other hand, taking $g(x) \equiv |x|$, we have $\lambda = 1$ and $Q(H_3, 1) \asymp n^{-1/2}$. This shows the optimality of the second summand in (4).

Theorem 2 will be the main tool in the proof of the main result of the paper, Theorem 3, which improves Theorem 1.

Theorem 3. *Let the conditions of Theorem 2 be satisfied. Then*

$$\rho\big(H_1,\, H_2\big) \leqslant c\,\big(p + \mathbf{P}\,\{\,|\Delta| > \gamma\,\} + \min_{1 \leqslant k \leqslant 6} Q(H_k, \gamma)\big), \tag{5}$$

for any $\gamma \geqslant \lambda$, where $\Delta = \sum_{i=1}^{n} a_i(\nu_i - 1)$ and ν_i are i.i.d. Poisson with mean 1.

A particular case of Theorem 3 occurring when $a_i = 0$, $i = 1, 2, \ldots, n$, implies inequality (4) with $j = 5$. Theorems 4–6 below are simple consecutive consequences of Theorem 3:

$$\text{Th. } 3 \implies \text{Th. } 4 \implies \text{Th. } 5 \implies \text{Th. } 6.$$

Theorem 4. *Let the conditions of Theorem 2 be satisfied. Then, for any $\varkappa \geqslant \lambda$,*

$$\rho\big(H_1,\, H_2\big) \leqslant c\,\big(p + (1 + \|\mathbf{a}\|_2\, \varkappa^{-1} \sqrt{\delta} + \|\mathbf{a}\|_\infty\, \varkappa^{-1} \delta)\, q\big), \tag{6}$$

where $q = \min\limits_{1 \leqslant k \leqslant 6} Q(H_k, \varkappa)$ and $\delta = \log\big(1 + \varkappa q^{-1}\, \|\mathbf{a}\|_2^{-1}\big)$.

Theorem 5. *Let the conditions of Theorem 1 be satisfied. Then*

$$\rho\big(H_1,\, H_2\big) \leqslant c\,\big(p + (1 + \|\mathbf{a}\|_2\, \tau^{-1} \sqrt{s} + \|\mathbf{a}\|_\infty\, \tau^{-1} s)\, Q\big), \tag{7}$$

where $Q = \min\limits_{1 \leqslant k \leqslant 6} Q(H_k, \tau)$ and $s = \log\big(1 + \tau\, Q^{-1}\, \|\mathbf{a}\|_2^{-1}\big)$.

Theorem 6. *Let the conditions of Theorem 1 be satisfied. Then*

$$\rho\big(H_1,\,H_2\big) \leqslant c\,\big(p + (1 + \|\mathbf{a}\|_2\,\tau^{-1}\sqrt{r} + \|\mathbf{a}\|_\infty\,\tau^{-1} r)\,D^{-1}(\tau)\big),$$

where $r = \log\big(1 + \tau\,D(\tau)\;\|\mathbf{a}\|_2^{-1}\big)$.

Theorem 6 is better with respect to order than Theorem 1 (clearly, $\|\mathbf{a}\|_\infty \leqslant \|\mathbf{a}\|_2$). Theorems 3–5 provide sharper bounds.

As example, consider the case $p = 0$, $F_i = E_{-\tau+a_i}/2 + E_{\tau+a_i}/2$, $|a_i| \leqslant \tau$, $i = 1,\dots,n$. Then $\sigma_i^2 = \tau^2$, $Q \asymp n^{-1/2}$, $D^2(\tau) \asymp n$. Theorems 5 and 6 give (for sufficiently large $n \geqslant n_0$) the bound

$$\rho\big(H_1,\,H_2\big) \leqslant c\,\tau^{-1} n^{-1/2}\Big(\,\|\mathbf{a}\|_2\,\sqrt{\log(1 + \tau n^{1/2}\,\|\mathbf{a}\|_2^{-1})} \\ + \|\mathbf{a}\|_\infty \log(1 + \tau n^{1/2}\,\|\mathbf{a}\|_2^{-1})\Big).$$

Le Cam's Theorem 1 contained the bound

$$\rho\big(H_1,\,H_2\big) \leqslant c\,(\|\mathbf{a}\|_2\,\tau^{-1} n^{-1/2})^{2/3}$$

only. It is easy to show that $\mathbf{E}\,S = \mathbf{E}\,T = \sum_{i=1}^n a_i$, $\operatorname{Var} S = \sum_{i=1}^n \sigma_i^2 = n\tau^2$ and $\operatorname{Var} T = n\tau^2 + \sum_{i=1}^n a_i^2$. Let Φ_1 and Φ_2 be the normal distributions with the same means and variances as $\mathcal{L}(S)$ and $\mathcal{L}(T)$ respectively. Then, by the Berry–Esseen bound in the CLT, $\rho(\prod_{i=1}^n F_i, \Phi_1) \leqslant c\,n^{-1/2}$ and $\rho(\prod_{i=1}^n e(F_i), \Phi_2) \leqslant c\,n^{-1/2}$. Furthermore, $\rho(\Phi_1, \Phi_2) \asymp \min\{1,\,\|\mathbf{a}\|_2^2\,\tau^{-2} n^{-1}\}$. Therefore, $\rho(\prod_{i=1}^n F_i, \prod_{i=1}^n e(F_i)) \geqslant c\,\min\{1,\,\|\mathbf{a}\|_2^2\,\tau^{-2} n^{-1}\}$, for sufficiently large $n \geqslant n_0$ and if $\|\mathbf{a}\|_2 \geqslant c\tau\,n^{1/4}$ with a sufficiently large c.

The question about the optimality of the bounds of Theorems 4–6 with respect to $\|\mathbf{a}\|_2$ remains open. Recall that the optimality of the result in the case $a_i = 0$, $i = 1, 2, \dots, n$ (Theorem 2), was established in Examples 3 and 4.

For the proof of Theorem 3 we need the following simple lemma due to Le Cam [9], see Le Cam ([10], Lemma 2, p. 402).

Lemma 1. *Let Z_1 and Z_2 be two random variables with $\mathcal{L}(Z_1) = H_1$ and $\mathcal{L}(Z_2) = H_2$. Then, for every $\gamma \geqslant 0$,*

$$\rho(H_1,\,H_2) \leqslant \mathbf{P}\,\big\{|Z_1 - Z_2| > \gamma\big\} + \min\big\{\,Q(H_1,\gamma),\;Q(H_2,\gamma)\,\big\}.$$

We need the following well-known simple properties of concentration functions and the Kolmogorov distance, which are valid for any $F, G, H \in$

$\mathfrak{F}$ and $y, y_1, y_2 > 0$:

$$Q(F, y_1) \leqslant \left(1 + \lceil y_1/y_2 \rceil\right) Q(F, y_2); \tag{9}$$

$$Q(F, y) \leqslant 2\rho(F, G) + Q(G, y), \tag{10}$$

where $\lceil\cdot\rceil$ is the integer part of a number (see, e.g., Hengartner and Theodorescu [5]).

Proof of Theorem 3. First we note that

$$\rho\big(H_1,\ H_6\big) = \rho\big(H_3,\ H_5\big) \tag{11}$$

is estimated in Theorem 2.

Next we estimate $\rho\big(H_2,\ H_6\big)$. To this end we note that H_2 is the distribution of

$$T = \sum_{i=1}^{n} \sum_{j=1}^{\nu_i} Y_{i,j},$$

where $Y_{i,j}$ and ν_i, $i = 1, \ldots, n$, $j = 1, 2, \ldots$, are independent in aggregate random variables with $\mathcal{L}(Y_{i,j}) = F_i$, $\mathcal{L}(\nu_i) = e(E_1)$. Moreover, $H_6 = \mathcal{L}(T - \Delta)$, with $\Delta = \sum_{i=1}^{n} a_i(\nu_i - 1)$.

Applying now Lemma 1 with $Z_1 = T$ and $Z_2 = T - \Delta$, we obtain

$$\rho\big(H_2,\ H_6\big) \leqslant \mathbf{P}\left\{|\Delta| > \gamma\right\} + \min\left\{Q(H_2, \gamma),\ Q(H_6, \gamma)\right\}, \tag{12}$$

for any $\gamma > 0$. Fix $\gamma \geqslant \lambda$. It is easy to see that, by (4) and (10),

$$\max_{3\leqslant k\leqslant 5} Q(H_k, \gamma) \leqslant c\left(p + \min_{3\leqslant k\leqslant 5} Q(H_k, \gamma)\right), \tag{13}$$

and, by (10) and (12),

$$\max\left\{Q(H_2, \gamma),\ Q(H_6, \gamma)\right\} \leqslant 2\ \mathbf{P}\left\{|\Delta| > \gamma\right\} + 3\ \min\left\{Q(H_2, \gamma),\ Q(H_6, \gamma)\right\}. \tag{14}$$

Obviously, the distributions shifted by constants have the same concentration functions. Therefore, for any $y \geqslant 0$,

$$Q(H_5, y) = Q(H_6, y) \quad \text{and} \quad Q(H_3, y) = Q(H_1, y). \tag{15}$$

Relations (13)–(15) imply

$$\max_{1\leqslant k\leqslant 6} Q(H_k, \gamma) \leqslant c\left(p + \mathbf{P}\left\{|\Delta| > \gamma\right\} + \min_{1\leqslant k\leqslant 6} Q(H_k, \gamma)\right). \tag{16}$$

It remains to take into account (16) and to apply (4), (11), (12) and the triangle inequality.

Proof of Theorem 4. The random variable Δ has mean zero and variance $\|\mathbf{a}\|_2^2 = \sum_{i=1}^n a_i^2$. It can be easily shown that Δ satisfy the conditions of Bernstein's inequality with parameter

$$\|\mathbf{a}\|_\infty = \max_{1 \leqslant i \leqslant n} |a_i|$$

(see Arak and Zaitsev ([2], pp. 39–41)). Therefore (see Arak and Zaitsev ([2], Theorem 4.1, p. 36)),

$$\mathbf{P}\left\{ |\Delta| > \gamma \right\} \leqslant 2 \max\left\{ e^{-\gamma^2/4\|\mathbf{a}\|_2^2}, e^{-\gamma/4\|\mathbf{a}\|_\infty} \right\},$$

for any $\gamma \geqslant 0$. Fix $\varkappa \geqslant \lambda$ and set $q = \min\limits_{1 \leqslant k \leqslant 6} Q(H_k, \varkappa)$. Choose now

$$\delta = \log\bigl(1 + \varkappa q^{-1} \|\mathbf{a}\|_2^{-1}\bigr) \quad \text{and} \quad \gamma = \max\bigl\{ \varkappa, 2 \|\mathbf{a}\|_2 \sqrt{\delta}, 4 \|\mathbf{a}\|_\infty \delta \bigr\}.$$

Then

$$\mathbf{P}\left\{ |\Delta| > \gamma \right\} \leqslant c\, q\, \varkappa^{-1} \|\mathbf{a}\|_2 .$$

Using (9), we obtain

$$Q(H_k, \gamma) \leqslant (1 + \gamma/\varkappa)\, Q(H_k, \varkappa).$$

Note that $\gamma \geqslant \lambda$. Now the statement of Theorem 4 follows from Theorem 3.

Proof of Theorem 5. It suffices to apply Theorem 4 with $\varkappa = 2\tau$ and $g(x) \equiv |x|$, since now condition (1) implies $\lambda \leqslant 2\tau$.

For the proof of Theorem 6 we need the following lemma which is a particular case of a result of Le Cam [9], see Le Cam ([10], Proposition 1, p. 404) of Arak and Zaitsev ([2], Theorem 2.5, p. 51).

Lemma 2. *Let $F \in \mathfrak{F}$ and $r > 0$. Then, for every $b \geqslant 0$,*

$$Q\bigl(e(rF), b\bigr) \leqslant c \Bigl(r \int_{-\infty}^{\infty} \min\bigl\{ 1, x^2 b^{-2} \bigr\}\, F\{dx\} \Bigr)^{-1/2}.$$

Proof of Theorem 6. It suffices to apply Theorem 5 taking into account that, by Lemma 2,

$$Q(H_2, \tau) \leqslant c\, D^{-1}(\tau). \tag{17}$$

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An improvement of a result of Le Cam [9] about the rate of approximation of distributions of sums of independent random variables by accompanying compound Poisson laws is proved.

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