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RETRACT EXTENSIONS OF ORDERED SETS

ABSTRACT. Ideal extensions of ordered semigroups have been considered by Kehayopulu and Tsingelis in [9]. Ideal extensions of ordered sets have been considered by Kehayopulu in [10]. Equivalent extensions of ordered sets have been considered by Kehayopulu and Shum in [11]. In the present paper we introduce and study the concept of the retract extensions of ordered sets. There are given an example of an extension which is retract and an example of an extension which is not retract.

1. INTRODUCTION-PREREQUISITES

The extension problem for groups is as follows: Given two groups H and K, construct all groups G which have a normal subgroup N such that N is isomorphic to H (in symbol, $N \approx H$) and $G/N \approx K$ (where G/N is the quotient of G by N). G is called the Schreier's extension or simply the extension of H by K. Given a semigroup S and a semigroup Q with zero, a semigroup V is called an (ideal) extension (or simply an extension) of P by Q if there exists an ideal P' of V such that P' is isomorphic to P and the Rees quotient V/P' is isomorphic to Q. The extension problem for semigroups (or ordered semigroups) is as follows. Given a semigroup S and a semigroup Q with zero (S and Q are disjoint), construct all the semigroups V which are extensions of S by Q. For the definition of the Rees quotient for semigroups and ordered semigroups, we refer to [12] and [9], respectively. Ideal extensions of semigroups have been considered in [3] with a detailed exposition of the theory appearing in [4, 12]. Extensions of weakly reductive semigroups, strict and pure extensions, retract extensions, dense extensions, equivalent extensions have been also considered in [12]. Ideal extensions of totally ordered semigroups have been studied in [6, 7], of topological semigroups in [2, 5]. For the ideal extensions of lattices we refer to [8]. For the ideal extensions of ordered semigroups we refer to [9]. Inspired by semigroups, ideal extensions of partially ordered sets have been studied in [10]. If P and Q are two disjoint ordered sets, an ordered set V is called an extension of P by Q if there is an ideal P' of V such that P' is isomorphic to P and the complement $V \setminus P'$ of P' to V is isomorphic to Q. The ideal extension problem for ordered sets is as follows: Given two disjoint ordered sets P and Q, construct (all) the ordered sets V which are (ideal) extensions of P by Q. We are often interested in building more complex semigroups, lattices, ordered sets, ordered or topological semigroups out of some of "simpler" structure and this can be sometimes achieved by constructing the ideal extensions. Equivalent extensions of ordered sets have been considered by Kehayopulu and Shum in [11]. In this paper we introduce the concepts of retract extensions of ordered sets and we characterize the retract extensions. As illustrative examples, we give an example of an extension which is retract and an example of an extension which is not retract. In fact, the Example 1 considered in [10] is an extention of an ordered set P by an ordered set Q (P and Q are disjoint) which is not retract. The Hasse diagram of that extension has been given in [10].

Let $(V, \leqslant)$ be an ordered set. A non-empty subset P' of V is called an ideal of V if $a \in P'$ and $V \ni b \leqslant a$ implies $b \in P'$ [1]. Each non-empty subset P' of an ordered set $(V, \leqslant_V)$ with the relation "$\leqslant_{P'}$" on P' defined by $\leqslant_{p'} := \leqslant_V \cap (P' \times P')$ is an ordered set. In the following, each subset P' of an ordered set $(V, \leqslant_V)$ is considered as an ordered set endowed with the order $\leqslant_{P'} := \leqslant_V \cap (P' \times P')$. We denote by $V \setminus P'$ the complement of P' in V.

Definition. *If $(P, \leqslant_P)$ and $(Q, \leqslant_Q)$ are two disjoint ordered sets, an ordered set $(V, \leqslant_V)$ is called an ideal extension (or just an extension) of P by Q if there exists an ideal P' of V such that*

$$(P', \leqslant_{P'}) \approx (P, \leqslant_P) \quad \text{and} \quad (V \setminus P', \leqslant_{V \setminus P'}) \approx (Q, \leqslant_Q),$$

where $\leqslant_{P'} := \leqslant_V \cap (P' \times P')$ and $\leqslant_{V \setminus P'} := \leqslant_V \cap ((V \setminus P') \times (V \setminus P'))$ [10].

Throughout the paper we use the following notations.

Notation 1. If $(V, \leqslant_V)$ is an extension of P by Q, unless otherwise is mentioned, we always denote by φ and ψ the isomorphisms
$\varphi : (P, \leqslant_P) \to (P', \leqslant_V \cap (P' \times P'))$ and
$\psi : (Q, \leqslant_Q) \to (V \setminus P', \leqslant_V \cap ((V \setminus P') \times (V \setminus P'))$,
respectively.

An extension V of P by Q is also denoted by

$$V(P, Q, \varphi : P \to P', \psi : Q \to V \setminus P').$$

We denote by i_P the identity mapping on P.

Notation 2. For every $r \subseteq P \times Q$, we always denote by $\bar{r}$ the set defined by:

$$\bar{r} := \{(a,b) \in P \times Q | \exists (a',b') \in r \quad \text{such that} \quad a \leqslant_P a', b' \leqslant_Q b\}.$$

Clearly, $r \subseteq \bar{r}$.

The main Theorem of the ideal extensions of ordered sets given in [10] is the following.

Theorem (Cf. [10; the Theorem]). *Let $(P, \leqslant_P)$, $(Q, \leqslant_Q)$ be ordered sets such that $P \cap Q = \varnothing$. Let $r \subseteq P \times Q$ and $V := P \cup Q$. We define a relation "$\leqslant_V$" on V as follows: $\leqslant_V := \leqslant_P \cup \leqslant_Q \cup \bar{r}$. Then, $(V, \leqslant_V)$ is an ordered set, P is an ideal of V, and*

$$V(P, Q, i_P : P \to P, i_Q : Q \to Q)$$

is an extension of P by Q.

Conversely, let $(V, \leqslant_V)$ be an extension of P by Q. Suppose that there exists an $r \subseteq P \times Q$ such that for the set $\bar{r}$ defined above, we have:

$$\bar{r} = \{(a,b) \in P \times Q | \, \varphi(a) \leqslant_V \psi(b)\}.$$

Then, the set $P \cup Q$ endowed with the relation "$\leqslant$" mentioned on the first part of the Theorem is an ordered set and $(P \cup Q, \leqslant) \approx (V, \leqslant_V)$.

In the following, we consider extensions $V(P, Q, \varphi : P \to P', \psi : Q \to V \setminus P')$ of P by Q for which there is an $r \subseteq P \times Q$ such that for the set $\bar{r}$ defined in Notation 2, we have

$$\bar{r} = \{(a,b) \in P \times Q | \, \varphi(a) \leqslant_V \psi(b)\}.$$

Such extensions are the retract extensions (also the equivalent extensions), denoted by $V(P, Q, \varphi : P \to P', \psi : Q \to V \setminus P', r)$.

Note (Cf. [10; Proposition 1]. If $V(P, Q, \varphi : P \to P', \psi : Q \to V \setminus P', r, \leqslant_V)$ is an extension of P by Q and $r := \{(a,b) \in P \times Q | \, \varphi(a) \leqslant_V \psi(b)\}$, then $\overline{r} = r$.

2. The main result

Definition 1. *An extension $V(P, Q, \varphi : P \to P', \psi : Q \to V \setminus P', r)$ of P by Q is called retract if there is an isotone mapping*

$$\eta : Q \to P \quad \textit{such that} \quad (a,b) \in r \quad \textit{implies} \quad a \leqslant_P \eta(b).$$

Theorem 1. *An extension $V(P, Q, \varphi : P \to P', \psi : Q \to V \setminus P', r)$ of P by Q is retract if and only if there is an isotone mapping*

$$g : V \to P \quad \textit{such that} \quad g(x) = \varphi^{-1}(x) \quad \textit{for every} \quad x \in P'.$$

Proof. $\Rightarrow$. Let $\eta : Q \to P$ be an isotone mapping such that $(a, b) \in r \Rightarrow a \leqslant_P \eta(b)$. We consider the mapping

$$g : V \to P | a \to \begin{cases} \varphi^{-1}(a) & \text{if} \quad a \in P' \\ \eta(\psi^{-1}(a)) & \text{if} \quad a \in V \setminus P'. \end{cases}$$

1) The mapping g is isotone.

Let $a, b \in V$, $a \leqslant_V b$. Then $g(a) \leqslant_P (b)$. In fact:

1.1. Let $b \in P'$. Since $V \ni a \leqslant_V b \in P'$ and P' and ideal of V, we have $a \in P'$. Since $a, b \in P'$, we have $g(a) := \varphi^{-1}(a)$, $g(b) := \varphi^{-1}(b)$. Since $a, b \in P'$ and $a \leqslant_V b$, we have $(a, b) \in \leqslant_V \cap (P' \times P') = \leqslant_P$. Since φ^{-1} is isotone, we get $\varphi^{-1}(a) \leqslant_P \varphi^{-1}(b)$. Hence $g(a) \leqslant_P g(b)$.

1.2. Let $b \in V \setminus P'$. Then $g(b) := \eta(\psi^{-1}(b))$. In fact: let $a \in P'$. Then $g(a) := \varphi^{-1}(a)$. On the other hand, $\varphi^{-1}(a) \leqslant_P \eta(\psi^{-1}(b))$. Indeed we have $(\varphi^{-1}(a), (\psi^{-1}(b)) \in P \times Q$. Moreover, since $a \leqslant_V b$, and $\varphi(\varphi^{-1}(a)) = a$, $\psi(\psi^{-1}(b)) = b$, we have $\varphi(\varphi^{-1}(a)) \leqslant_V \psi(\psi^{-1}(b))$. Thus $(\varphi^{-1}(a), \psi^{-1}(b)) \in \bar{r}$. Then there exists $(a', b') \in r$ such that $\varphi^{-1}(a) \leqslant_P a'$ and $b' \leqslant_Q \psi^{-1}(b)$. Since $(a', b') \in r$, by hypothesis, we have $a' \leqslant_P \eta(b')$. Since $b' \leqslant_Q \psi^{-1}(b)$ and η is isotone, we have $\eta(b') \leqslant_P \eta(\psi^{-1}(b))$. Then we have $\varphi^{-1}(a) \leqslant_P \eta(\psi^{-1}(b))$, and $g(a) \leqslant_P (b)$.

Let $a \in V \setminus P'$. Then $g(a) := \eta(\psi^{-1}(a))$. On the other hand, $\eta(\psi^{-1}(a)) \leqslant_P \eta(\psi^{-1}(b))$. Indeed: We have

$$(a, b) \in \leqslant_V \cap ((V \setminus P') \times (V \setminus P')) = \leqslant_{V \setminus P}.$$

Then, since ψ^{-1} is isotone we have $\psi^{-1}(a) \leqslant_Q \psi^{-1}(b)$. Since η is isotone, we have $\eta(\psi^{-1}(a)) \leqslant_P \eta(\psi^{-1}(b))$. Thus we have $g(a) \leqslant_P g(b)$.

2) Let $x \in P'$. Then $g(x) = \varphi^{-1}(x)$. (This is by the definition of g).

$\Leftarrow$ Let g: $V \to P$ be an isotone mapping such that $g(x) = \varphi^{-1}(x)$ $\forall x \in P'$. We consider the mapping

$$\eta : Q \to P, \quad a \mapsto g(\psi(a)).$$

1) The mapping η is isotone. Indeed, let $a, b \in Q$, $a \leqslant_Q b$. Since

$$(\psi(a), \psi(b)) \in \leqslant_{V \setminus P'} = \leqslant_V \cap ((V \setminus P') \times (V \setminus P')) \subseteq \leqslant_V,$$

we have $\psi(a) \leqslant_V \psi(b)$. Since g is isotone, we have $g(\psi(a)) \leqslant_P g(\psi(b))$.

2) If $(a,b) \in r$, then $a \leqslant_P \eta(b)$. Indeed,

$$\begin{aligned}(a,b) \in r &\Rightarrow (a,b) \in r = \{(a,b) \in P \times Q | \varphi(a) \leqslant_V \psi(b)\}\\ &\Rightarrow (a,b) \in P \times Q, \psi(a) \leqslant_V \psi(b)\\ &\Rightarrow \varphi(a), \psi(b) \in V, \varphi(a) \leqslant_V \psi(b)\\ &\Rightarrow g(\varphi(a)) \leqslant_P g(\psi(b)) \text{ (since } g \text{ is isotone).}\end{aligned}$$

Besides, since $a \in P$, we have $\varphi(a) \in P'$. By hypothesis, we have $g(\varphi(a)) = \varphi^{-1}(\varphi(a))$. Since φ is an isomorphism, we have $\varphi^{-1}(\varphi(a)) = a$. Since $b \in Q$, we have $\eta(b) := g(\psi(b))$. Thus $a \leqslant_P \eta(b)$.

Example of extensions which are retract

We consider the sets

$$P := \{1,3,5,7,9\} \quad \text{and} \quad Q := \{2,4,6,8,10\}$$

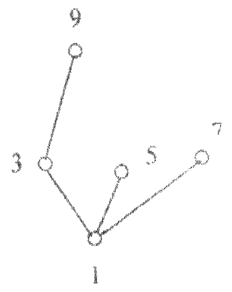
with the orders "$\leqslant_P$", "$\leqslant_Q$" on P, Q, respectively, defined by:

$$\leqslant_{P}:= \{(a,b) \in P \times P \mid a|b\}, \ \leqslant_{Q}:= \{(a,b) \in Q \times Q \mid a|b\}$$
$$(a|b \quad \text{means } a \text{ divides} \quad b).$$

We give the covering relations and the figures of P and Q.

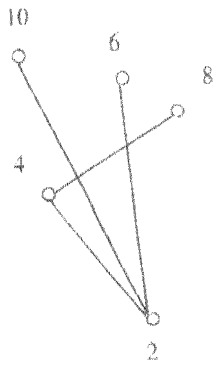
$\prec_P = \{(1,3),(1,5),(1,7),(3,9)\}$

$\prec_Q = \{(2,4),(2,6),(2,10),(4,8)\}$



$(P, \leqslant_P)$

Fig. 1



$(Q, \leqslant_Q)$

Fig. 2.

Let $r = \{(3,6),(5,10)\}$. According to the Theorem 1 the set $V := P \cup Q$ with the relation "$\leqslant_V$" on V defined by $\leqslant_V := \leqslant_P \cup \leqslant_Q \cup \bar{r}$, is an extension of P by Q.

The determination of $\bar{r}$.

Let $(a,b) \in \bar{r}$. Then there exists $(a',b') \in r$ such that $a \leqslant_P a'$, $b' \leqslant_Q b$.

Since $(a',b') \in r$, we have $(a',b') = (3,6)$ or $(a',b') = (5,10)$.

Let $(a',b') = (3,6)$. Then $a' = 3$, $b' = 6$. Since $a \leqslant_P a' = 3$, we have $a = 1$ or $a = 3$. Since $6 = b' \leqslant_Q b$, we have $b = 6$. Then

$$(a,b) = (1,6) \quad \text{or} \quad (a,b) = (3,6).$$

Let $(a',b') = (5,10)$. Then $a' = 5$, $b' = 10$. Since $a \leqslant_P a' = 5$, we have $a = 1$ or $a = 5$. Since $10 = b' \leqslant_Q b$, we have $b = 10$.

Then $(a,b) = (1,10)$ or $(a,b) = (5,10)$.

Since $r \subseteq \bar{r}$, we have $(3,6) \in \bar{r}$ and $(5,10) \in \bar{r}$.

On the other hand,

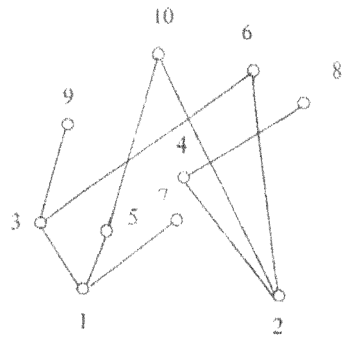
$(1,6) \in \bar{r}$ since $(1,6) \in P \times Q$, $(3,6) \in r$, $1 \leqslant_P 3$, $6 \leqslant_Q 6$.

$(1,10) \in \bar{r}$ since $(1,10) \in P \times Q$, $(5,10) \in r$, $1 \leqslant_P 5$, $10 \leqslant_Q 10$.

Hence we have $\bar{r} = \{(1,6),(1,10),(3,6),(5.10)\}$.

We give the covering relation and the figure of V.

$\prec_V = \{(1,3),(1,5),(1,7),(2,4),(2,6),(2,10),(3,6),(3,9),(4,8),(5,10)\}$



$(V, \leqslant_V)$

Fig. 3.

The extension V is retract. In fact, we remark first that

$$\bar{r} = \{(a,b) \in P \times Q | i_P(a) \leqslant_V i_P(b)\} = \{(a,b) \in P \times Q | a \leqslant_V b\}.$$

Let η be the mapping defined by:

$$\eta : Q \to P, \backslash : q \mapsto \begin{cases} 1 & \text{if} \quad q = 2 \\ 3 & \text{if} \quad q = 4, 6, 8 \\ 5 & \text{if} \quad q = 10. \end{cases}$$

The mapping η is isotone. Moreover, for each $(a, b) \in r$, we have $a \leqslant_P \eta(b)$. Indeed,

$$\begin{aligned} (3,6) \in r \quad &\text{and} \quad 3 \leqslant_P \eta(6) = 3, \\ (5,10) \in r \quad &\text{and} \quad 5 \leqslant_P \eta(10) = 5. \end{aligned}$$

Example of extensions which are not retract

We consider the ordered sets $(P, \leqslant_P)$ and $(Q, \leqslant_Q)$ defined by:

$P := \{p | p \in N, p \text{ prime}, 1 < p \leqslant 60 \text{ or } p = 1\}$, $\leqslant_P = \{(a, b) \in P \times P \mid a|b\}$,

$Q := \{2^n \cdot 23 | n \in N\}$, $\leqslant_Q := \{(a, b) \in Q \times Q | \, a|b\}$

($a|b$ means a divides b).

Let $r := \{(2, 2 \cdot 23), \, (23, 2 \cdot 23)\}$

We have

$$\bar{r} = \{(1, 2^n \cdot 23), (2, 2^n \cdot 23), (23, 2^n \cdot 23); n \in N\}.$$

From the Theorem above, the set $V := P \cup Q$ with the relation $\leqslant_V := \leqslant_P \cup \leqslant_Q \cup \bar{r}$ is an ordered set, P is an ideal of V, and $V(P, Q, i_p : P \to P, i_Q : Q \to Q, \leqslant, r)$ is an extension of P by Q.

We remark that:

$$\bar{r} = \{(a, b) \in P \times Q | \, i_P(a) \leqslant_V \, i_P(b)\} = \{(a, b) \in P \times Q | \, a \leqslant_V \, b\}.$$

This is the extension considered in the Example 1 in [10], its figure is given in [10], and this extension is not retract. Indeed:

Suppose it is. Then, there is an isotone mapping

$$\eta : Q \to P \quad \text{such that} \quad (a, b) \in r \Rightarrow a \leqslant_P \eta(b).$$

Since $(2, 2 \cdot 23) \in r$, we have $2 \leqslant_P \eta(2 \cdot 23)$. Then, by the definition of "$\leqslant_P$", we have $\eta(2 \cdot 23) = 2$. Since $(23, 2 \cdot 23) \in r$, we have $23 \leqslant_P \eta(2 \cdot 23)$. Then, by the definition of "$\leqslant_P$", we have $\eta(2 \cdot 23) = 23$. Contradiction.

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