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ON THE DEGENERATIONS OF FINITE DIMENSIONAL NILPOTENT COMPLEX LEIBNIZ ALGEBRAS

Abstract. In the present paper, a theorem on closed families of Leibniz algebras with respect to the Zarisky topology is proved. Statements concerning degenerations of finite dimensional nilpotent Leibniz algebras to the special classes of nilpotent Leibniz algebras are provided.

1. Introduction

Let V be a vector space of dimension n over the field of complex numbers $\mathbb{C}$. An n-dimensional Leibniz algebra L may be considered as an element λ of the affine variety $\mathrm{Hom}(V \otimes V, V)$ via the bilinear mapping $\lambda : L \otimes L \to L$ defining the Leibniz bracket on L. The set of Leibniz algebra structures is an algebraic subset $\mathrm{Leib}_n(\mathbb{C})$ of the variety $\mathrm{Hom}(V \otimes V, V)$ and the linear reductive group $\mathrm{GL}_n(\mathbb{C})$ acts on $\mathrm{Leib}_n(\mathbb{C})$ by $(g * \lambda)(x, y) = g(\lambda(g^{-1}(x), g^{-1}(y)))$. The orbits under this action are the isomorphic classes of algebras. For the given two algebras λ and μ we will say that λ degenerates to μ, if μ lies in the Zariski closure of the orbit λ. We denote this by $\lambda \to \mu$. It is easy to see that any nilpotent Leibniz algebra degenerates to the Abelian algebra.

There are algebras the orbits of which are open in $\mathrm{Leib}_n(\mathbb{C})$. These algebras are called rigid. The orbits of the rigid algebras give irreducible components of the variety $\mathrm{Leib}_n(\mathbb{C})$. Hence to describe the variety $\mathrm{Leib}_n(\mathbb{C})$ it suffices to describe all rigid Leibniz algebras and rigid families of Leibniz algebras. By Noetherian argument there are finite number of those.

Definition 1. *An algebra L over a field F is called a Leibniz algebra if it satisfies the following Leibniz identity:*

$$[x, [y, z]] = [[x, y], z] - [[x, z], y],$$

where $[\cdot, \cdot]$ denotes the multiplication in L.

Let L be a Leibniz algebra. We put:

$$L^1 = L, \quad L^{k+1} = [L^k, L], \;\; k \in N.$$

Definition 2. *A Leibniz algebra L is called nilpotent if there exists an integer $s \in N$ such that $L^1 \supset L^2 \supset \ldots \supset L^s = \{0\}$. The smallest integer s for which $L^s = 0$ is called the nilindex of L.*

The set $\mathrm{LN}_n(\mathbb{C})$ of n-dimensional nilpotent Leibniz algebras can be included into the above mentioned n^3-dimensional affine space by the following way: let $\{e_1, e_2, \ldots, e_n\}$ be a basis of the Leibniz algebra λ. Then the multiplication table of λ is represented by a point (γ_{ij}^k) of this affine space as follows:

$$\lambda(e_i, e_j) = \sum_{k=1}^{n} \gamma_{ij}^k e_k.$$

Thus, the algebra λ correspond to the point (γ_{ij}^k). γ_{ij}^k are called *structure constants* of λ. The Leibniz identity and the condition of nilpotency give polynomial relations among γ_{ij}^k. Therefore, $\mathrm{LN}_n(\mathbb{C})$ is a closed subset of $\mathbb{C}^{n^3}$.

Definition 3. *A Leibniz algebra λ is said to degenerate to another Leibniz algebra μ, if μ is represented by a structure which lies in the Zariski closure of the $\mathrm{GL}_n(\mathbb{C})$-orbit of a structure which represents λ. In this case, the entire orbit $\mathrm{Orb}(\mu)$ lies in the closure of $\mathrm{Orb}(\lambda)$. We denote this by $\lambda \to \mu$.*

Remark 1. Degeneration is transitive, that is if $\lambda \to \mu$ and $\mu \to \nu$ then $\lambda \to \nu$.

We will provide a few useful facts from the algebraic groups theory, concerning the degeneration theory. The first of them on constructive subsets of algebraic varieties, the closures of which with respect to Euclidean and Zariski topologies coincide. It is not hard to see that $\mathrm{GL}_n(\mathbb{C})$-orbits are constructive sets. Therefore the usual Euclidean topology on $\mathbb{C}^{n^3}$ leads to the same degenerations as does the Zariski topology, that is the following condition will imply that $\lambda \to \mu$:

$$\exists g_t \in \mathrm{GL}_n(\mathbb{C}(t)) \;\text{ such that }\; \lim_{t \to 0} g_t * \lambda = \mu,$$

where $\mathbb{C}(t)$ is the field of fractions of the polynomial ring $\mathbb{C}[t]$.

The second fact is on closure of $\mathrm{GL}_n(\mathbb{C})$-orbits stating that the boundary of each orbit consists of union of orbits whose dimension is strictly less than dimension of the given orbit. It follows that each irreducible component of the variety, on which algebraic group acts, does not contain two orbits of maximal dimensions, that is the orbit of maximal dimension in this sense is unique. It is obvious that their representatives are rigid.

Remark 2. It is easy to note that the rigid algebras can not be degenerated by any others.

Description of any class of algebras variety is a very difficult problem. Results in [1–4], concerning applications of algebraic group theory to the description of the variety of unitary associative and Lie algebras gave impulse to investigation of the same problem for other classes of algebras. Some investigations deal with the applications to this problem of the nonstandard analysis methods [5]. In this paper we are going to pay our attention to some properties of the degenerations of finite dimensional complex Leibniz algebras.

1. Main results

In this section, we are going to present some statements which are useful for investigation of a variety of a given class of algebras.

For a given Leibniz algebra λ we put:

- $R(\lambda) = \{x \in \lambda | [\lambda, x] = 0\}$ – the right annihilator of λ;
- $L(\lambda) = \{x \in \lambda | [x, \lambda] = 0\}$ – the left annihilator of λ;
- $Z(\lambda) = \{x \in \lambda | [x, \lambda] = [\lambda, x] = 0\}$ – the center of λ;
- $\mathrm{Aut}(\lambda)$ – the group of automorphisms of λ;
- $\lambda^k = [\lambda^{k-1}, \lambda]$ – the kth degree of λ;
- $SA(\lambda)$ – the maximal Abelian subalgebra of λ;
- $\mathrm{Com}(\lambda)$ – the maximal commutative subalgebra of λ;
- $\mathrm{Lie}(\lambda)$ – the maximal Lie subalgebra of λ.

Theorem 1. *For any m, $r \in N$ the following subsets of $\mathrm{LN}_n(\mathbb{C})$ are closed relatively to the Zariski topology:*

1. $\{\lambda \in \mathrm{LN}_n(\mathbb{C}) \mid \dim \lambda^m \leqslant r\}$;
2. $\{\lambda \in \mathrm{LN}_n(\mathbb{C}) \mid \dim R(\lambda) \geqslant m\}$;
3. $\{\lambda \in \mathrm{LN}_n(\mathbb{C}) \mid \dim L(\lambda) \geqslant m\}$;
4. $\{\lambda \in \mathrm{LN}_n(\mathbb{C}) \mid \dim Z(\lambda) \geqslant m\}$;

5. $\{\lambda \in \mathrm{LN}_n(\mathbb{C}) \mid \dim \mathrm{Aut}(\lambda) > m\}$
6. $\{\lambda \in \mathrm{LN}_n(\mathbb{C}) \mid \dim SA(\lambda) \geqslant m\}$
7, $\{\lambda \in \mathrm{LN}_n(\mathbb{C}) \mid \dim \mathrm{Com}(\lambda) \geqslant m\}$
8. $\{\lambda \in \mathrm{LN}_n(\mathbb{C}) \mid \dim \mathrm{Lie}(\lambda) \geqslant m\}$

Below for the sake of convenience brackets $[,]$ are omitted and we assume that the undefined products are zero.

Proof. 1. Indeed, let $\{e_1, e_2, \dots, e_n\}$ be a basis of λ. Consider a set of polynomials of degree m at n indeterminates. The cardinal number of this set will be denoted by $P_{m,n}$, that is

$$P_{m,n} = \mathrm{card}\{U(x_1, x_2, \dots, x_n)\colon \deg U(x_1, x_2, \dots, x_n) = m\}.$$

One considers

$$U_\ell(e_1, e_2, \dots, e_n) = f_1^\ell(\gamma_{i,j}^k)e_1 + f_2^\ell(\gamma_{i,j}^k)e_2 + \dots + f_n^\ell(\gamma_{i,j}^k)e_n,$$

where $\gamma_{i,j}^k$ are the structural constants of λ and $\ell = \overline{1, P_{m,n}}$. We will consider the following $(P_{m,n} \times n)$-order matrix

$$\begin{pmatrix} f_1^1(\gamma_{i,j}^k) & \dots & f_n^1(\gamma_{i,j}^k) \\ \dots & \dots & \dots \\ f_1^{P_{m,n}}(\gamma_{i,j}^k) & \dots & f_n^{P_{m,n}}(\gamma_{i,j}^k)) \end{pmatrix}.$$

The condition $\dim \lambda^k \leq m$ is equivalent to the condition that all $(m+1)$-order minors of this matrix are equal to zero, that gives polynomial conditions on the set $\{\gamma_{i,j}^k\}$.

2. Let $\{e_1, e_2, \dots, e_n\}$ be a basis of λ and $a \in \lambda$. By $[a]$ we denote the line, consisting of the coordinates of a in this basis. Let us consider the following matrix consisting of coordinate lines of the elements $e_i e_j$, that is $[\lambda] = |[e_i e_j]|$, where $i, j = \overline{1, n}$. Then $\dim R(\lambda) \leq m$ gives us the condition $\mathrm{rank}[\lambda] < n - m + 1$. But the last condition means that all $(n-m+1)$-order minors of $[\lambda]$ are equal to zero, that in turn means the existence of the polynomial conditions among structure constants of λ.

It should be noted that similar considerations can be applied to prove items 3 and 4.

Item 5 follows from the above mentioned fact on closure of orbits if we mean the following relation between dimensions of $\mathrm{GL}_n(\mathbb{C})$-orbits and automorphism's group:

$$\dim O(\lambda) = n^2 - \dim \mathrm{Aut}(\lambda).$$

Items 6–8 are special cases of the following more general fact: let B be a Borel subgroup of $\mathrm{GL}_n(\mathbb{C})$ and λ, μ in $\mathrm{LN}_n(\mathbb{C})$. If $\lambda \to \mu$ and λ lies in B-stable closed subset $R \subset \mathrm{LN}_n(\mathbb{C})$ then μ must also be in R. It is not hard to check that subsets 6–8 are stable with respect to Borel subgroup consisting of lower triangular matrices. The proof is complete. □

Corollary. *An algebra λ does not degenerate to μ if one of the following conditions is valid:*

1. $\dim \lambda^m < \dim \mu^m$ *for some* m,
2. $\dim R(\lambda) > \dim R(\mu)$,
3. $\dim L(\lambda) > \dim L(\mu)$,
4. $\dim Z(\lambda) > \dim Z(\mu)$,
5. $\dim \mathrm{Aut}(\lambda) \geqslant \dim \mathrm{Aut}(\mu)$,
6. $\dim SA(\lambda) > \dim SA(\mu)$,
7. $\dim \mathrm{Com}(\lambda) > \dim \mathrm{Com}(\mu)$,
8. $\dim \mathrm{Lie}(\lambda) > \dim \mathrm{Lie}(\mu)$.

This proposition can be used to establish existence or nonexistence of degenerations of one algebra to another in a given class of algebras.

Proposition 2. *Let λ be a non-Lie algebra in $\mathrm{LN}_n(\mathbb{C})$. Then $\lambda \to \mu \oplus C^{n-2}$, where μ is two-dimensional non-Abelian nilpotent Leibniz algebra.*

Proof. Since λ is non-Lie Leibniz algebra there exists an x such that $xx = y$, where $y \neq 0$. These two elements are linearly independent. Indeed, if we have $\alpha_1 x + \alpha_2 y = 0$ for these elements then multiplying both sides of this equality by x on the left we obtain that $\alpha_1 xx = \alpha_1 y = 0$, it follows $\alpha_1 = 0$ and $\alpha_2 = 0$.

Thus, x and y can be included to a basis of λ: $e_1 = x$, $e_2 = y$, $e_3, \ldots, e_n$. Then taking the following family g_t in $\mathrm{GL}_n(C(t))$: $g_t(e_1) = t^{-1}e_1$, $g_t(e_i) = t^{-2}e_i$ $(2 \leqslant i \leqslant n)$, we obtain that $\lambda \to \mu \oplus C^{n-2}$, where μ is defined by the following multiplications table: $e_1 e_1 = e_2$. But by J.-L. Loday the algebra μ is the unique non-Abelian two-dimensional nilpotent Leibniz algebra [6]. The proof is complete. □

Definition 4. *An n-dimensional Leibniz algebra L is said to be nulfiliform if $\dim L^i = n - i + 1$, where $1 \leqslant i \leqslant n + 1$.*

It should be noted that the nulfiliform Leibniz algebra is unique and it is rigid in $\mathrm{LN}_n(\mathbb{C})$.

Definition 5. *An n-dimensional Leibniz algebra L is said to be filiform if $\dim L^i = n - i$, where $2 \leqslant i \leqslant n$.*

Theorem 3. *Any n-dimensional nulfiliform and non-Lie filiform $(n \geq 4)$ Leibniz algebra degenerates to the algebra $\nu \oplus C^{n-3}$, where ν is three-dimensional non-Abelian nilpotent Leibniz algebra with the following multiplications table:*

$$e_2e_2 = e_1, \quad e_2e_3 = e_1.$$

Proof. First we consider of nulfiliform algebra case. By Lemma 1 in [7], a complex n-dimensional nulfiliform Leibniz algebra can be represented on a basis $\{e_1, e_2, \dots, e_n\}$ by the following multiplications table:

$$e_ie_1 = e_{i+1} \quad (1 \leqslant i \leqslant n-1).$$

It is easy to check that the following family g_t in $\mathrm{GL}_n(C(t))$:

$$g_t(e_1) = t^{-2}e_2, \ g_t(e_2) = t^{-4}e_1, \ g_t(e_3) = t^{-3}e_2 - t^{-3}e_3,$$
$$g_t(e_4) = t^{-5}e_1 + t^{-4}e_4, \ g_t(e_i) = t^{-1}e_i \ (5 \leqslant i \leqslant n)$$

gives us corresponding transformations.

Now, let us consider the filiform Leibniz algebras case. By Theorem 2 in [7], any complex n-dimensional filiform non-Lie Leibniz algebra is isomorphic to one of the following algebras:

$$e_0e_0 = e_2, \ e_ie_0 = e_{i+1} \ (1 \leqslant i \leqslant n-2),$$
$$e_0e_1 = \alpha_3e_3 + \alpha_4e_4 + \dots + \alpha_{n-2}e_{n-2} + \theta e_{n-1},$$
$$e_ie_1 = \alpha_3e_{i+2} + \alpha_4e_{i+3} + \dots + \alpha_{n-i}e_{n-1} \quad (1 \leqslant i \leqslant n-3),$$

and

$$e_0e_0 = e_2, \ e_ie_0 = e_{i+1} \ (2 \leqslant i \leqslant n-2),$$
$$e_0e_1 = \beta_3e_3 + \beta_4e_4 + \dots + \beta_{n-1}e_{n-1},$$
$$e_1e_1 = \gamma e_{n-1}, \ e_ie_1 = \beta_3e_{i+2} + \beta_4e_{i+3} + \dots + \beta_{n-i}e_{n-1} \quad (2 \leqslant i \leqslant n-3),$$

where $\{e_0, e_1, \dots, e_{n-1}\}$ is a basis of L.

Checking up the following family g_t of transformations from $\mathrm{GL}_n(C(t))$:

$$g_t(e_0) = t^{-1}e_2, \ g_t(e_1) = t^{-1}e_2 - t^{-1}e_3, \ g_t(e_2) = t^{-2}e_1,$$
$$g_t(e_3) = t^{-1}e_0, \ g_t(e_i) = t^{-1}e_i \ (4 \leqslant i \leqslant n-1)$$

it is easy to conclude that the first class degenerates to the $\nu \oplus C^{n-3}$.

As for the second class then it can be degenerated to the $\nu \oplus C^{n-3}$ by the following family of transformations g_t in $\mathrm{GL}_n(C(t))$:

$$g_t(e_0) = t^{-2}e_2, \quad g_t(e_1) = t^{-3}e_0, \quad g_t(e_2) = t^{-4}e_1 + t^{-2}e_2 - t^{-2}e_3,$$
$$g_t(e_3) = t^{-4}e_1, \quad g_t(e_i) = te_i \quad (4 \leqslant i \leqslant n-1).$$

The proof is complete. □

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