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A LIMIT THEOREM FOR THE HURWITZ ZETA-FUNCTION WITH ALGEBRAIC IRRATIONAL PARAMETER

Abstract. A limit theorem in the sense of weak convergence of probability measures in the space of analytic functions for the Hurwitz zeta-function with algebraic irrational parameter is obtained.

To the memory of Professor J. V. Linnik

1. Introduction

The Hurwitz zeta-function $\zeta(s,\alpha)$, $s=\sigma+it$, $0<\alpha\leqslant 1$, is defined, for $\sigma>1$, by

$$\zeta(s,\alpha)=\sum_{m=0}^{\infty}\frac{1}{(m+\alpha)^s}.$$

Moreover, it has analytic continuation to the whole complex plane, except for a point $s=1$, which is a simple pole with residue 1.

Limit theorems in the sense of the weak convergence of probability measures in various spaces for the function $\zeta(s,\alpha)$ with transcendental or rational parameter α can be found in [4]. A limit theorem on the complex plane $\mathbb{C}$ for $\zeta(s,\alpha)$ with algebraic irrational α was obtained in [5]. Denote by $\text{meas}\{A\}$ the Lebesgue measure of a measurable set $A\subset\mathbb{R}$, and let

$$\nu_T^t(\dots)=\frac{1}{T}\text{meas}\{t\in[0,T]:\dots\},$$

where in place of dots a condition satisfied by t is to be written. Let $\mathcal{B}(S)$ stand for the class of Borel sets of the space S. For $\sigma>\frac{1}{2}$ and algebraic irrational α, in [5] it was proved that the probability measure

$$\nu_T^t\left(\zeta(\sigma+it,\alpha)\in A\right),\quad A\in\mathcal{B}(\mathbb{C}),$$

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converges weakly to some explicitly given probability measure as $T \to \infty$.

The function $\zeta(s,\alpha)$ is holomorphic in the half-plane $D = \{s \in \mathbb{C} : \sigma > 1\}$. The aim of this paper is to obtain for $\zeta(s,\alpha)$ a limit theorem in the space $H(D)$ of holomorphic on D functions equipped with the topology of uniform convergence on compacta. To state the theorem we need one topological structure.

It is known [2] that at least 51% elements of the system $L(\alpha) = \{\log(m+\alpha) : m = 0,1,2,\dots\}$ are linearly independent over the field of rational numbers $\mathbb{Q}$ if α is algebraic irrational. Denote the system of linearly independent numbers $\log(m+\alpha)$ by $I(\alpha)$, and let $D(\alpha) = L(\alpha) \setminus I(\alpha)$. Moreover, let $\mathfrak{M}(\alpha) = \{m : \log(m+\alpha) \in I(\alpha)\}$ and $\mathfrak{N}(\alpha) = \{m : \log(m+\alpha) \in D(\alpha)\}$. If $m \in D(\alpha)$, then the set $\{\log(m+\alpha)\} \bigcup I(\alpha)$ is linearly dependent our $\mathbb{Q}$, therefore there exists a finite set $\log(m_1+\alpha), \dots, \log(m_n+\alpha) \in I(\alpha)$ and integer numbers $k_0, k_1, \dots, k_n$, $k_j \neq 0$, $j = 0, \dots, n$, such that

$$k_0 \log(m+\alpha) = k_1 \log(m_1+\alpha) + \ldots + k_n \log(m_n+\alpha),$$

the last expression being unique.

Hence

$$m+\alpha = (m_1+\alpha)^{\frac{k_1}{k_0}} \dots (m_n+\alpha)^{\frac{k_n}{k_0}}. \tag{1}$$

Let $\gamma = \{s \in \mathbb{C} : |s| = 1\}$ be the unit circle on $\mathbb{C}$. Define

$$\widehat{\Omega} = \prod_{m \in \mathfrak{M}(\alpha)} \gamma_m,$$

where $\gamma_m = \gamma$ for all $m \in \mathfrak{M}(\alpha)$. With the product topology and pointwise multiplication the infinite-dimensional torus $\widehat{\Omega}$ is a compact topological Abelian group, therefore on $(\widehat{\Omega}, \mathcal{B}(\widehat{\Omega}))$ the probability Haar measure m_H can be defined. This leads to a probability space $(\widehat{\Omega}, \mathcal{B}(\widehat{\Omega}), m_H)$. Denote by $\widehat{\omega}(m)$ the projection of $\widehat{\omega} \in \widehat{\Omega}$ to the coordinate space γ_m, $m \in \mathfrak{M}(\alpha)$. If $m \in \mathfrak{N}(\alpha)$ and relation (1) holds, we take

$$\omega(m) = \widehat{\omega}^{\frac{k_1}{k_0}}(m_1) \dots \widehat{\omega}^{\frac{k_n}{k_0}}(m_n), \tag{2}$$

where the principal values of roots are taken. So, we can define the function $\omega(m)$ for all $m = 0,1,2,\dots$ by

$$\omega(m) = \begin{cases} \widehat{\omega}(m) & \text{if } m \in \mathfrak{M}(\alpha), \\ \text{the right-hand side of (2)} & \text{if } m \in \mathfrak{N}(\alpha). \end{cases} \tag{3}$$

Now we define an $H(D)$-valued random element $\zeta(s,\alpha,\omega)$ on the probability space $(\widehat{\Omega}, \mathcal{B}(\widehat{\Omega}), m_H)$ by the formula

$$\zeta(s,\alpha,\omega) = \sum_{m=0}^{\infty} \frac{\omega(m)}{(m+\alpha)^s}, \quad s \in D.$$

Really, $\zeta(s,\alpha,\omega)$ is an $H(D)$-valued random element because the series defining it converges uniformly on compact subsets of D for all $\omega(m)$.

Our aim is to obtain the weak convergence of the probability measure

$$P_T(A) = \nu_T^{\tau}\left(\zeta(s+i\tau,\alpha) \in A\right), \quad A \in \mathcal{B}(H(D)).$$

Theorem 1. *Assume that α is an algebraic irrational number. Then the probability measure P_T converges weakly to the distribution of the random element $\zeta(s,\alpha,\omega)$, i.e., to the measure*

$$m_H\left(\widehat{\omega} \in \widehat{\Omega} : \zeta(s,\alpha,\omega) \in A\right), \quad A \in \mathcal{B}(H(D)),$$

as $T \to \infty$.

2. Limit theorems for Dirichlet polynomials

In this section, we will consider the Dirichlet polynomials

$$\zeta_n(s,\alpha) = \sum_{m=0}^{n} \frac{1}{(m+\alpha)^s}, \quad \zeta_n(s,\alpha,\omega_0) = \sum_{m=0}^{n} \frac{\omega_0(m)}{(m+\alpha)^s},$$

where ω_0 is a fixed function given by (3). For this we will use the following statement proved in [5].

Lemma 2. *Let α be an algebraic irrational number. Then the probability measure*

$$Q_T(A) = \nu_T^{\tau}\left(\left((m+\alpha)^{-i\tau} : m \in \mathfrak{M}(\alpha)\right) \in A\right), \quad A \in \mathcal{B}(\widehat{\Omega}),$$

converges weakly to the Haar measure m_H as $T \to \infty$.

Now we define two probability measures

$$P_{T,n}(A) = \nu_T^{\tau}\left(\zeta_n(s+i\tau,\alpha) \in A\right), \quad A \in \mathcal{B}(H(D))$$

and

$$\widehat{P}_{T,n}(A) = \nu_T^{\tau}\left(\zeta_n(s+i\tau,\alpha,\omega_0) \in A\right), \quad A \in \mathcal{B}(H(D)).$$

Theorem 3. *Let α be an algebraic irrational number. Then the probability measures $P_{T,n}$ and $\widehat{P}_{T,n}$ both converge weakly to the same probability measure on $(H(D), \mathcal{B}(H(D)))$ as $T \to \infty$.*

Proof. Define a function $h : \widehat{\Omega} \to H(D)$ by the formula

$$h(\widehat{\omega}) = \sum_{m=0}^{n} \frac{1}{(m+\alpha)^s \omega(m)},$$

where $\omega(m)$ is defined by (3). The function h is continuous and

$$h\left((m+\alpha)^{-i\tau},\, m \in \mathfrak{M}(\alpha)\right) = \zeta_n(s+i\tau, \alpha).$$

Therefore, by Theorem 5.1 from [1] and Lemma 2, we obtain that the probability measure $P_{T,n} = Q_T h^{-1}$ converges weakly to the measure $m_H h^{-1}$ as $T \to \infty$.

Now let $h_1 : \widehat{\Omega} \to \widehat{\Omega}$ be given by $h_1(\widehat{\omega}) = \widehat{\omega} \cdot \widehat{\omega}_0^{-1}$, $\widehat{\omega} \in \widehat{\Omega}$, and $\widehat{\omega}_0$ is a fixed element of $\widehat{\Omega}$ defining ω_0. Then we have that

$$\zeta_n(s+i\tau, \alpha, \omega_0) = h(h_1(\widehat{\omega})). \tag{4}$$

Similarly as above, we have that $\widehat{P}_{T,n}$ converges weakly to the measure $m_H \widehat{h}^{-1}$ as $T \to \infty$, where by (4)

$$\widehat{h}(\widehat{\omega}) = \sum_{m=0}^{n} \frac{\omega_0(m)}{(m+\alpha)^s \omega(m)} = h(h_1(\widehat{\omega})).$$

Since the Haar measure m_H is invariant with respect to translations by points in $\widehat{\Omega}$, we find that

$$m_H \widehat{h}^{-1} = m_H (h(h_1))^{-1} = (m_H h_1^{-1}) h^{-1} = m_H h^{-1},$$

and the theorem is proved.

3. Limit theorems for absolutely convergent Dirichlet series

Define one more probability measure $\widehat{P}_T$ on $(H(D), \mathcal{B}(H(D)))$ by

$$\widehat{P}_T(A) = \nu_T^{\tau}\left(\zeta(s+i\tau, \alpha, \omega) \in A\right).$$

Theorem 4. *Let α be an algebraic irrational number. Then the probability measures P_T and $\widehat{P}_T$ both converge weakly to the same probability measure on $(H(D), \mathcal{B}(H(D)))$ as $T \to \infty$.*

Proof. By Theorem 3, the probability measures $P_{T,n}$ and $\widehat{P}_{T,n}$ both converges weakly to the measure $P_n = m_H h_n^{-1}$ as $T \to \infty$. First we will prove that the family of probability measures $\{P_n\}$ is tight.

Let θ_T be a random variable defined on a certain probability space $(\Omega, \mathcal{B}(\Omega), \mathbb{P})$ such that

$$\mathbb{P}(\theta_T \in A) = \frac{1}{T} \int_0^T I_A(t) \mathrm{d}t, \quad A \in \mathcal{B}(\mathbb{R}),$$

where I_A is the indicator function of the set A. Define

$$X_{T,n}(s) = \zeta_n(s + i\theta_T, \alpha).$$

Then, by Theorem 3, we have that

$$X_{T,n} \xrightarrow[T \to \infty]{\mathcal{D}} X_n, \tag{5}$$

where X_n is an $H(D)$-valued random element with the distribution P_n, and $\xrightarrow{\mathcal{D}}$ means the convergence in distribution. Define on $H(D)$ a metric which induces its topology. It is well known, see, for example, [3], that there exists a sequence $\{K_l\}$ of compacts subsets of D such that

$$D = \bigcup_{l=1}^{\infty} K_l.$$

Moreover, the sets K_l can be chosen to satisfy $K_l \subset K_{l+1}$, and if K is a compact of D, then $K \subseteq K_l$ for some l. For $f, g \in H(D)$, let

$$\rho(f, g) = \sum_{l=1}^{\infty} 2^{-l} \frac{\rho_l(f, g)}{1 + \rho_l(f, g)},$$

where

$$\rho_l(f, g) = \sup_{s \in K_l} |f(s) - g(s)|.$$

Then it is easily seen that ρ is a metric on $H(D)$ which induces its topology.

By the Chebyshev inequality, for every positive M_l,

$$\mathbb{P}\left(\sup_{s\in K_l} |X_{T,n}(s)| > M_l\right) \leqslant \frac{1}{TM_l} \int_0^T \sup_{s\in K_l} |\zeta_n(s+i\tau,\alpha)| \mathrm{d}\,\tau,$$

and thus

$$\limsup_{T\to\infty} \mathbb{P}\left(\sup_{s\in K_l} |X_{T,n}(s)| > M_l\right) \leqslant$$
$$\leqslant \frac{1}{M_l} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \sup_{s\in K_l} |\zeta_n(s+i\tau,\alpha)| \mathrm{d}\,\tau.$$

The series for $\zeta(s,\alpha)$ converges absolutely on D and uniformly on compact subsets of D. Therefore

$$\sup_{n\geqslant 0} \limsup_{T\to\infty} \frac{1}{T} \int_0^T \sup_{s\in K_l} |\zeta_n(s+i\tau,\alpha)| \mathrm{d}\,\tau \leqslant R_l < \infty.$$

Let $\varepsilon > 0$ be arbitrary, and $M_l = \frac{R_l 2^l}{\varepsilon}$. Then two last inequalities show that

$$\limsup_{T\to\infty} \mathbb{P}\left(\sup_{s\in K_l} |X_{T,n}(s)| > M_l\right) \leqslant \frac{\varepsilon}{2^l}. \tag{6}$$

Define the function $h : H(D) \to \mathbb{R}$ by

$$h(f) = \sup_{s\in K_l} |f(s)|, \quad f \in H(D).$$

Then the continuity of h, (5), and Theorem 5.1 of [1], yield

$$\sup_{s\in K_l} |X_{T,n}(s)| \xrightarrow[T\to\infty]{\mathcal{D}} \sup_{s\in K_l} |X_n(s)|.$$

Hence, in view of (6),

$$\mathbb{P}\left(\sup_{s\in K_l} |X_n(s)| > M_l\right) \leqslant \frac{\varepsilon}{2^l}$$

and

$$\mathbb{P}(X_n(s) \in H_\varepsilon) \geqslant 1 - \varepsilon \tag{7}$$

for all $n \geqslant 0$, where

$$H_\varepsilon = \{f \in H(D) : \sup_{s \in K_l} |f(s)| \leqslant M_l, \quad l \geqslant 1\}.$$

Since the set H_ε is compact and by (7),

$$P_n(H_\varepsilon) \geqslant 1 - \varepsilon$$

for all $n \geqslant 0$, we have that the family of probability measures $\{P_n\}$ is tight. Hence by the Prokhorov theorem [1] it is relatively compact.

By the definition of $\zeta_n(s, \alpha)$, for $s \in D$,

$$\lim_{n \to \infty} \zeta_n(s, \alpha) = \zeta(s, \alpha),$$

the convergence being uniform on compact subsets of D. Therefore, for every $\varepsilon > 0$,

$$\lim_{n \to \infty} \limsup_{T \to \infty} \nu_T^\tau \left(\rho(\zeta_n(s + i\tau, \alpha), \zeta(s + i\tau, \alpha)) \geqslant \varepsilon\right) \leqslant$$

$$\leqslant \lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{\varepsilon T} \int_0^T \rho(\zeta_n(s + i\tau, \alpha), \zeta(s + i\tau, \alpha)) \mathrm{d}\,\tau = 0. \tag{8}$$

Define

$$X_T(s) = \zeta(s + i\theta_T, \alpha).$$

Then (8) shows that, for every $\varepsilon > 0$,

$$\lim_{n \to \infty} \limsup_{T \to \infty} \mathbb{P}\left(\rho(X_{T,n}(s), X_T(s)) \geqslant \varepsilon\right) = 0. \tag{9}$$

Now let $\{P_{n_1}\} \subset \{P_n\}$ be such that P_{n_1} converges weakly to some measure P as $n_1 \to \infty$. Then

$$X_{n_1} \xrightarrow[n_1 \to \infty]{\mathcal{D}} P.$$

This, (5) and (9) show that the hypotheses of Theorem 4.2 from [1], are satisfied. Therefore,

$$X_T \xrightarrow[T \to \infty]{\mathcal{D}} P, \tag{10}$$

or the probability measure P_T converges weakly to P as $T \to \infty$.

The relation (10) shows that the measure P is independent of the choice of the sequence n_1. This, the relative compactness of $\{P_n\}$ and Theorem 2.3 of [1] now imply the relation

$$X_n \xrightarrow[T\to\infty]{\mathcal{D}} P. \tag{11}$$

The series for $\zeta(s,\alpha,\omega)$ also converges absolutely on D and thus uniformly on compact subsets of D. Therefore, repeating the above arguments for the random elements

$$\widehat{X}_{T,n}(s) = \zeta_n(s+i\theta_T,\alpha,\omega)$$

and

$$\widehat{X}_T(s) = \zeta(s+i\theta_T,\alpha,\omega),$$

and applying (11), we obtain that the probability measure $\widehat{P}_T$ also converges weakly to the measure P as $T\to\infty$.

4. Proof of Theorem 1

To prove Theorem 1 it remains to identify the limit measure in Theorem 4. For this we will apply some facts from the ergodic theory.

Let $a_\tau = \{(m+\alpha)^{-i\tau},\ m\in\mathfrak{M}(\alpha)\}$, $\tau\in\mathbb{R}$. Then $\{a_\tau:\tau\in\mathbb{R}\}$ is a one-parameter group. Define $\{\varphi_\tau:\tau\in\mathbb{R}\}$ taking $\varphi_\tau(\widehat{\omega}) = a_\tau\widehat{\omega}$, $\widehat{\omega}\in\widehat{\Omega}$. Then $\{\varphi_\tau:\tau\in\mathbb{R}\}$ is an one-parameter group of measurable transformations on $\widehat{\Omega}$.

Lemma 5. *Let α be an algebraic irrational number. Then the one-parameter group $\{\varphi_\tau:\tau\in\mathbb{R}\}$ is ergodic.*

Proof. The lemma is proved in [5].

Proof of Theorem 1. Let $A\in\mathcal{B}(H(D))$ be a continuity set of the probability measure P defined in the proof of Theorem 4. Then we have that

$$\lim_{T\to\infty}\nu_T^\tau\left(\zeta(s+i\tau,\alpha,\omega)\in A\right) = P(A). \tag{12}$$

Now we fix the set A, and on $(\widehat{\Omega},\mathcal{B}(\widehat{\Omega}),m_H)$ define the random variable θ by

$$\theta(\widehat{\omega}) = \begin{cases} 1 & \text{if } \zeta(s,\alpha,\omega)\in A,\\ 0 & \text{if } \zeta(s,\alpha,\omega)\notin A.\end{cases}$$

Denote by $E(\theta)$ the expectation of θ. Then we have that

$$E(\theta) = \int\limits_{\widehat{\Omega}} \theta \mathrm{d}\, m_H = m_H(\widehat{\omega} \in \widehat{\Omega} : \zeta(s,\alpha,\omega) \in A) \tag{13}$$

is the distribution P_ζ of the random element $\zeta(s,\alpha,\omega)$. Lemma 5 implies the ergodicity of the random process $\theta(\varphi_\tau(\widehat{\omega}))$. Consequently, by the Birkhoff–Khintchine theorem (see, for example, [6]),

$$\lim_{T\to\infty} \frac{1}{T} \int\limits_0^T \theta(\varphi_\tau(\widehat{\omega}))\mathrm{d}\,\tau = E(\theta) \tag{14}$$

for almost all $\widehat{\omega} \in \widehat{\Omega}$. On the other hand,

$$\frac{1}{T} \int\limits_0^T \theta(\varphi_\tau(\widehat{\omega}))\mathrm{d}\,\tau = \nu_T^\tau\left(\zeta(s+i\tau,\alpha,\omega) \in A\right).$$

Therefore, by (13) and (14),

$$\lim_{T\to\infty} \nu_T^\tau\left(\zeta(s+i\tau,\alpha,\omega) \in A\right) = P_\zeta(A)$$

for almost all $\widehat{\omega} \in \widehat{\Omega}$. This and (12) yield that

$$P(A) = P_\zeta(A)$$

for continuity sets A of P. Hence $P(A) = P_\zeta(A)$ for all $A \in \mathcal{B}(H(D))$, since the continuity sets constitute the determining class.

The theorem is proved.

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