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DIVISOR WEIGHTED SUMS

Abstract. Estimates for sums of divisor functions are required in many arguments of analytic number theory. Linnik was among the first to show how to handle such questions. We provide bounds which are particularly suitable for sparse sequences.

1. Introduction

Given a sequence $\mathcal{A} = \big(a_n\big)$ of nonnegative real numbers perhaps the first question to ask is how large is the sum

$$A(x) = \sum_{n \leqslant x} a_n. \tag{1.1}$$

Very often, arguments from analytic number theory lead us next to study the weighted sums

$$D_r(x) = \sum_{n \leqslant x} \tau_r(n) a_n, \tag{1.2}$$

where $\tau_r(n)$ is the divisor function whose generating function is $\zeta(s)^r$, that is

$$\tau_r(n) = \sum_{n_1 \ldots n_r = n} 1. \tag{1.3}$$

Trivially we get

$$D_r(x) \ll A(x) x^{\varepsilon}, \tag{1.4}$$

because $\tau_r(n) \ll n^{\varepsilon}$ for any $\varepsilon > 0$, with the implied constant depending on ε and r. Estimate (1.4) is sometimes sufficient but frequently not.

One should expect the bound

$$D_r(x) \ll A(x)(\log x)^{r-1} \tag{1.5}$$

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because the average value of $\tau_r(n)$ is of order $(\log n)^{r-1}$. More precisely, we have

$$\sum_{n\leqslant x} \tau_r(n) \sim \frac{1}{(r-1)!}\, x(\log x)^{r-1}. \tag{1.6}$$

This best possible bound can be generalized to $D_r(x)$ provided we have sufficiently good bounds for the congruence sums

$$A_d(x) = \sum_{\substack{n\leqslant x\\ n\equiv 0 (\bmod d)}} a_n. \tag{1.7}$$

Using the inequality

$$\tau_r(n) \leqslant r \sum_{\substack{d\mid n\\ d\leqslant n^{1-\frac{1}{r}}}} \tau_{r-1}(d) \tag{1.8}$$

(for $r = 2$ this is the familiar Dirichlet hyperbola trick) we obtain

$$D_r(x) \leqslant r \sum_{d\leqslant x^{1-\frac{1}{r}}} \tau_{r-1}(d) A_d(x). \tag{1.9}$$

Suppose we have

$$A_d(x) \ll g(d)A(x) \tag{1.10}$$

for all $d \leqslant x^{1-\frac{1}{r}}$, where $g(d)$ is a multiplicative function satisfying

$$g(p^\ell) \ll p^{-\ell} \tag{1.11}$$

for all p and all $\ell \geqslant 1$, and

$$\prod_{p\leqslant x} \big(1+g(p)\big) \ll \log x. \tag{1.12}$$

Then

$$\sum_{d\leqslant x} g(d)\tau_{r-1}(d) \ll (\log x)^{r-1}$$

giving (1.5).

Although for most sequences one would consider the bound (1.10) holds in some ranges, it is unlikely to hold for such large d. Definitely one should not expect it for $d > A(x)$.

If one is willing to give up on getting the optimal bound then an application of Cauchy's inequality is an option (or Hölder's inequality). This would give

$$D_r(x) \ll \left(x(\log x)^{r^2-1}\right)^{\frac{1}{2}} \Big(\sum_{n \leqslant x} |a_n|^2\Big)^{\frac{1}{2}} \tag{1.13}$$

since

$$\sum_{n \leqslant x} \tau_r^2(n) \ll x(\log x)^{r^2-1}. \tag{1.14}$$

This would be better than losing x^ε, as in (1.4), and also satisfactory provided that $A(x) \gg x(\log x)^{-A}$ for some A. Unfortunately, many of the most interesting sequences are sparse in the sense that

$$A(x) \ll x^{1-\delta} \tag{1.15}$$

for some $\delta > 0$ and in such a case this bound is useless.

If $\delta > \frac{1}{r}$ in (1.15) then condition (1.10) is also unrealistic and we cannot claim (1.5) nor anything stronger than (1.4).

Our goal is to obtain reasonable bounds for sparse sequences, for any δ with $0 < \delta < 1$. The idea is to replace (1.8) by a similar inequality which runs over much smaller divisors, an inequality of the shape

$$\tau_r(n) \leqslant c \sum_{\substack{d|n \\ d \leqslant n^{1/k}}} f(d). \tag{1.16}$$

Here k is any integer $k \geqslant 2$, c is a constant which is allowed to depend on r and k, and $f(d)$ is a suitable function. We require $f(d)$ to be multiplicative. We want f to be reasonably small and are willing to sacrifice the size of c.

Assuming inequality (1.16) one derives along the above lines that

$$D_r(x) \ll A(x)L(x), \tag{1.17}$$

where

$$L(x) \leqslant \sum_{d \leqslant x} f(d)\, g(d) \tag{1.18}$$

provided that (1.10) holds for all $d \leqslant x^{1/k}$. Since we can choose k arbitrarily large, condition (1.10) may be expected to hold in this range even for very sparse sequences.

In practice we shall have

$$L(x) \ll (\log x)^B \tag{1.19}$$

for some B depending on r and k, which is good enough although not the optimal bound.

2. Main results

Although in applications one normally does not care about the implied constants we would like to deliver rather neat bounds of type (1.16). First we establish a quite strong inequality, but only for squarefree n.

Theorem 2.1. *Let $k \geqslant 2$, $r \geqslant 2$ and n squarefree. Then*

$$\tau_r(n) \leqslant k^{(k\log r)^2} \sum_{\substack{d|n \\ d \leqslant n^{\frac{1}{k}}}} f(d), \tag{2.1}$$

where $f(d)$ is the multiplicative function given by

$$f(p) = \frac{r^k}{k}. \tag{2.2}$$

Remark. This result is general and quite strong. However, for small values of k and r we have already established still stronger results in Lemmas 6.2 and 6.3 of [3].

Proof. We begin with the inequality

$$\tau_k(n) = \sum_{n_1 \ldots n_k = n} 1 \leqslant k \sum_{\substack{d|n \\ d \leqslant n^{\frac{1}{k}}}} \tau_{k-1}\Big(\frac{n}{d}\Big).$$

Hence

$$\frac{\tau_k(n)}{\tau_{k-1}(n)} \leqslant k \sum_{\substack{d|n \\ d \leqslant n^{\frac{1}{k}}}} \frac{1}{\tau_{k-1}(d)}.$$

Let $f(d) > 1$ and $0 < \alpha < 1$. By Hölder's inequality, this is bounded by

$$\frac{\tau_k(n)}{\tau_{k-1}(n)} \leqslant k \Bigg(\sum_{\substack{d|n \\ d \leqslant n^{\frac{1}{k}}}} f(d) \Bigg)^{\alpha} \Bigg(\sum_{d|n} \frac{1}{f(d)^{\frac{\alpha}{1-\alpha}} \tau_{k-1}(d)^{\frac{1}{1-\alpha}}} \Bigg)^{1-\alpha}.$$

Hence

$$\Big(\frac{\tau_k(n)}{\tau_{k-1}(n)}\Big)^{\frac{1}{\alpha}}\Bigg(\sum_{d|n} h(d)^{-1}\Bigg)^{\frac{\alpha-1}{\alpha}} \leqslant k^{\frac{1}{\alpha}} \sum_{\substack{d|n\\ d\leqslant n^{\frac{1}{k}}}} f(d),$$

where we have put $h(d) = f(d)^{\frac{\alpha}{1-\alpha}}\tau_{k-1}(d)^{\frac{1}{1-\alpha}}$. To get (1.16) from this we require

$$\tau_r(n) \leqslant \Big(\frac{\tau_k(n)}{\tau_{k-1}(n)}\Big)^{\frac{1}{\alpha}}\Bigg(\sum_{d|n} h(d)^{-1}\Bigg)^{\frac{\alpha-1}{\alpha}}.$$

Note that we have expanded the summation to all divisors $d \mid n$, so the above inequality contains multiplicative functions (supported on square-free integers). Thus it suffices to check this for n prime. In this case, we require

$$r^{\alpha}\Big(1+\frac{1}{h}\Big)^{1-\alpha} \leqslant \frac{k}{k-1}$$

where

$$h = h(p) = f(p)^{\frac{\alpha}{1-\alpha}}(k-1)^{\frac{1}{1-\alpha}} = f^{\frac{\alpha}{1-\alpha}}(k-1)^{\frac{1}{1-\alpha}}$$

for $f(p) = f$ constant. Therefore we require

$$r^{\alpha}\left(1+f^{\frac{-\alpha}{1-\alpha}}(k-1)^{\frac{-1}{1-\alpha}}\right)^{1-\alpha} \leqslant \frac{k}{k-1}. \tag{2.3}$$

□

To simplify this condition we are going to apply some elementary inequalities.

Lemma 2.2. *For $0<\beta<1$ and $0 \leqslant x \leqslant \frac{1}{k-1}$, we have*

$$(1+x)^{\beta} \leqslant 1 + \left[k^{\beta} - (k-1)^{\beta}\right] x^{\beta}. \tag{2.4}$$

Proof. Consider the function

$$A(x) = 1 + \left[k^{\beta} - (k-1)^{\beta}\right] x^{\beta} - (1+x)^{\beta}.$$

We have $A(0) = A\big(\frac{1}{k-1}\big) = 0$ and

$$A'(x) = \left[k^{\beta} - (k-1)^{\beta}\right] \beta x^{\beta-1} - \beta(1+x)^{\beta-1}.$$

Hence $A'(0) = +\infty$ and $A'(x)$ has one zero, so $A(x)$ is concave. Therefore $A(x) \geqslant 0$ for $0 \leqslant x \leqslant \frac{1}{k-1}$, completing the proof. □

Writing the well-known inequality

$$1+\frac{1-\beta}{k-1}\leqslant\left(\frac{k}{k-1}\right)^{1-\beta}$$

in the form

$$k^{\beta}-(k-1)^{\beta}\leqslant\beta k^{\beta-1}$$

we get by (2.4):

Corollary 2.3. *For* $0\leqslant\beta\leqslant 1$, $0\leqslant x\leqslant\frac{1}{k-1}$, *we have*

$$(1+x)^{\beta}\leqslant 1+\frac{\beta}{k}(kx)^{\beta}. \tag{2.5}$$

By Corollary 2.3, our goal (2.3) reduces to showing that

$$r^{\alpha}\left(1+\frac{1-\alpha}{k-1}(kf)^{-\alpha}\right)\leqslant\frac{k}{k-1}. \tag{2.6}$$

We choose $kf=r^k$ and transform (2.6) to

$$(k-1)r^{\alpha k}-kr^{\alpha(k-1)}+1\leqslant\alpha. \tag{2.7}$$

Putting

$$H(x)=(k-1)e^{kx}-ke^{(k-1)x}+1$$

we write (2.7) as

$$H(x)\leqslant\alpha\quad\text{for}\quad x=\alpha\log r.$$

We have

$$H'(x)=k(k-1)\left[e^{kx}-e^{(k-1)x}\right]$$

and

$$\begin{aligned}H''(x)&=k(k-1)\left[ke^{kx}-(k-1)e^{(k-1)x}\right]\\&\leqslant k(k-1)e^{kx}\leqslant 2k^2\quad\text{if}\quad kx\leqslant\log\frac{2k}{k-1}.\end{aligned}$$

Note that $H(0)=H'(0)=0$, hence

$$H(x)\leqslant(kx)^2\quad\text{if}\quad kx\leqslant\log\frac{2k}{k-1}. \tag{2.8}$$

Now we want $(kx)^2\leqslant\alpha$ which for $x=\alpha\log r$ means

$$\alpha\leqslant(k\log r)^{-2}. \tag{2.9}$$

Note that $kx = \alpha k \log r = (k \log r)^{-1} \leqslant \log \frac{2k}{k-1}$ for any $k \geqslant 2$, $r \geqslant 2$. This completes the proof of Theorem 2.1.

If n is not assumed to be squarefree the arguments of Theorem 2.1 can be modified but the proof and the result would not be so elegant. Hence, for the general case we apply a different reasoning which is quite short but leads to a slightly weaker result.

Theorem 2.4. *Let $k \geqslant 2$ and $r \geqslant 2$. Any $n \geqslant 1$ has a divisor $d \mid n$, $d \leqslant n^{1/k}$ such that*

$$\tau_r(n) \leqslant \big(2\tau(d)\big)^{(r-1)k\frac{\log k}{\log 2}}. \tag{2.10}$$

Proof. We have

$$\tau_r(p^\alpha) = \binom{\alpha + r - 1}{r - 1} \leqslant (\alpha + 1)^{r-1}.$$

Hence $\tau_r(n) \leqslant \tau(n)^{r-1}$ and the result follows from Lemma 2 of [1]. □

Corollary 2.5. *Let $k \geqslant 2$, $r \geqslant 2$, $n \geqslant 1$. We have*

$$\tau_r(n) \leqslant \sum_{\substack{d \mid n \\ d \leqslant n^{1/k}}} \big(2\tau(d)\big)^{(r-1)k\frac{\log k}{\log 2}}. \tag{2.11}$$

3. Remarks

From Theorem 2.4, we obtain

$$L(x) \ll (\log x)^{k^{(r-1)k}} \tag{3.1}$$

and so, by (1.16),

$$D_r(x) \ll A(x)(\log x)^{k^{(r-1)k}}. \tag{3.2}$$

To apply Theorem 2.1, we assume the sequence $\mathcal{A}$ is supported on squarefree numbers and obtain stronger results, namely

$$L(x) \ll (\log x)^{r^k/k} \tag{3.3}$$

and

$$D_r(x) \ll A(x)(\log x)^{r^k/k}. \tag{3.4}$$

It is useful to have some idea how close our theorems are to best possible. Consider the case of n being the product of very many (say ℓ

of them) distinct but nearly equal primes with $\ell \equiv 0 (\mathrm{mod}\, k)$. Any $d \mid n$, $d \leqslant n^{\frac{1}{k}}$ must contain no more than $\frac{\ell}{k}$ primes. Hence,

$$\sum_{\substack{d \mid n \\ d \leqslant n^{1/k}}} f(d) \leqslant \sum_{s \leqslant \frac{\ell}{k}} \binom{\ell}{s} f^s \leqslant \ell \binom{\ell}{\ell/k} f^{\frac{\ell}{k}}. \tag{3.5}$$

Since $\tau_r(n) = r^\ell$ our inequality (1.15) implies

$$r^\ell \leqslant c\ell \binom{\ell}{\ell/k} f^{\frac{\ell}{k}}. \tag{3.6}$$

Taking ℓth roots and letting $\ell \to \infty$ we obtain, by Stirling's formula,

$$r \leqslant k(k-1)^{\frac{1}{k}-1} f^{\frac{1}{k}} \tag{3.7}$$

that is

$$f \geqslant \frac{r^k}{k}\Big(1 - \frac{1}{k}\Big)^{k-1}. \tag{3.8}$$

Thus Theorem 2.1 cannot be improved beyond this value which we expect is best possible. Note that Theorem 2.1 is close to this; only the factor $\big(1 - \frac{1}{k}\big)^{k-1}$ is lost (about $\frac{1}{e}$).

Ideas as described in this paper were in one form or another used by a number of people. Y. V. Linnik used them daily, see, for example, [4]. Moreover, inequalities of type (1.15) occur, for example, in the works [5, 1, 2, 3].

References

1. J. B. Friedlander, H. Iwaniec, *Asymptotic sieve for primes.* — Ann. Math. **148** (1998) 1041–1065.
2. J. B. Friedlander, H. Iwaniec, *A polynomial divisor problem.* — to appear.
3. J. B. Friedlander, H. Iwaniec, *The illusory sieve.* — to appear.
4. Y. V. Linnik, *The dispersion method in binary additive problems.* — Amer. Math. Soc. (Providence) (1963).
5. D. Wolke, *A new proof of a theorem of van der Corput.* — JLMS **5** (1972) 609–612.

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