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SUMS OF THE ADDITIVE DIVISOR PROBLEM TYPE AND THE INNER PRODUCT METHOD

Abstract. The inner product approach to the additive divisor problem with its cusp form analogues is surveyed, and a spectral summation formula for convolution sums involving Fourier coefficients of Maass forms is derived. An application to subconvexity estimates for Rankin–Selberg L-functions is announced.

Dedicated to the memory of Yu. V. Linnik

1. Introduction

Let $a(n)$ stand either for the divisor function $d(n)$, or for the Fourier coefficients of a cusp form (holomorphic or real-analytic) for the full modular group, and suppose for simplicity that $a(n)$ is real for all n. Sums of the additive divisor problem type, that is of the form

$$\sum_{n=1}^{\infty} a(n)a(n+f)W(n/f) \quad (f \geq 1) \tag{1.1}$$

with W a smooth function of compact support in the positive reals, are encountered in various problems of analytic number theory. The case $a(n) = d(n)$ occurred in the classical work of F. V. Atkinson [1] on the fourth moment of Riemann's zeta-function, and more recently in applications to Dirichlet L-functions [9] and exponential sums [5]. In [8], we treated mean value questions for Riemann's zeta-function and Hecke L-functions in a unified way on the basis of the spectral theory of the respective sums (1.1). As another application, we may mention our recent joint work with Y. Motohashi [10] on spectral averages and estimates for Hecke L-functions attached to Maass forms. The same argument applies even to certain Rankin–Selberg L-functions, but in this context the sums (1.1) related to Maass forms appear to be somewhat problematic. Namely, in that case, a spectral summation formula has not yet been established, whereas the other two cases mentioned above are perfectly understood thanks to the work of Y. Motohashi ([12], [13]). Also, we gave in [5] a summation formula of somewhat different structure for the sum (1.1) in the case $a(n) = d(n)$, and this can be extended to Maass forms as will be shown below. The underlying motivation for the last mentioned generalization is of course the need to find a substitute for Motohashi's formulae, with practical applications in mind.

We developed in [6–7] a unified approach to the sums (1.1), in all three cases, on the basis of the inner product method which is the topic of this paper. This method, the principles of which go back to A. Selberg [17], aims at a spectral decomposition and analytic continuation of the zeta-function

$$\sum_{n=1}^{\infty} a(n)a(n+f)(n+f)^{-s}.$$

Then, by Mellin transform techniques, one obtains a spectral formula for the sum (1.1). This argument is perfectly successful in the case of holomorphic cusp forms and leads to interesting concrete applications, as shown first by A. Good (see [3], [4]). Namely, if

$$\varphi(z) = \varphi(x+yi) = \sum_{n=1}^{\infty} a(n)e(nz) \tag{1.2}$$

is a holomorphic cusp form of weight $2k$ for the full modular group Γ, and

$$P_f(z,s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} (\Im\gamma(z))^s e(f\gamma(z))$$

is the non-holomorphic Poincaré series, with Γ_∞ denoting the subgroup of Γ generated by the shift $z \mapsto z+1$, then

$$\sum_{n=1}^{\infty} a(n)a(n+f)(n+f)^{-s-2k+1} = \frac{(4\pi)^{s+2k-1}}{\Gamma(s+2k-1)}\langle P_f(z,s), |\varphi(z)|^2 y^{2k}\rangle \tag{1.3}$$

for $\sigma = \Re s > 1$, where $\langle\cdot,\cdot\rangle$ denotes the Petersson inner product. The desired spectral decomposition now follows if this inner product is evaluated by Parseval's formula and unfolding.

A similar procedure in the case $a(n) = d(n)$ was carried out by L. A. Takhtajan and A. I. Vinogradov [18], with $d(n)$ appearing in the Fourier series of a non-holomorphic Eisenstein series. However, one does not get a precise analogue of (1.3) in this way, and there are certain error terms to be coped with. The same difficulty arises in the case of Maass forms. In a technical sense, a reason for this obstacle in the two last mentioned cases, compared with the holomorphic cusp forms, is the presence of K-Bessel functions in the respective Fourier series instead of the exponential function. For instance, the Fourier series of a Maass form $u(z)$ which is an eigenfunction of the hyperbolic Laplacian for the eigenvalue $1/4+\kappa^2$ is of the form (see [14], Lemma 1.4)

$$u(z) = y^{1/2} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \rho(n) K_{i\kappa}(2\pi|n|y)e(nx), \tag{1.4}$$

which should be compared with (1.2). Motohashi's formula for the additive divisor problem involves a contribution from the holomorphic cusp forms, and an analogous term is expected to appear in its tentative counterpart for the Maass forms as well. However, the inner product method does not produce anything like that, so it should be hidden in the error terms. A fully satisfactory version of this method would require working in the group $SL(2,\mathbb{R})$ rather than in the upper half-plane, as indicated by Y. Motohashi [15]. On the other hand, while the traditional line of argument is admittedly of somewhat limited scope, it has turned out to be a useful device in practice. An application alluded to above will be obtaining new subconvexity estimates for Rankin–Selberg zeta-functions, as a continuation of our joint work with Motohashi [10]; this topic will be commented in the end of the paper.

Let now $u(z)$ be a Maass form as in (1.4), and suppose, as usual, that it is a simultaneous eigenfunction for all Hecke operators T_n with the eigenvalues $t(n)$. Moreover, we suppose that $u(z)$ is even or odd as a function of x, and define ϵ (the parity symbol) to be 1 if u is even and -1 if it is odd. Then $\rho(n) = t(n)\rho(1)$ for $n \geq 1$, $\rho(-n) = \epsilon\rho(n)$, and $t(n)$ is a multiplicative arithmetic function. For basics of the spectral theory of automorphic functions and properties of Maass forms, we refer to the monograph of Y. Motohashi [14]. As an analogue of (1.3), we define

$$\zeta_f(s) = \frac{2(4\pi)^{s-1}}{|\rho(1)|^2\Gamma(s-1)}\langle P_f(z,s), |u(z)|^2\rangle \quad (\sigma > 1). \tag{1.5}$$

Here the factor in front of the inner product differs from that in [6], eq. (1.13), but asymptotically (that is for $|\Im s| \to \infty$) the definitions coincide by Stirling's formula.

We now outline an argument giving a spectral summation formula for the sum (1.1) in the case of Maass forms. In terms of the coefficients $t(n)$, this sum is of the form

$$A(f;W)=\sum_{n=1}^{\infty} t(n)t(n+f)W(n/f). \tag{1.6}$$

Let $M_f(s)$ be the Mellin transform of the function $\sqrt{(x-f)/x}W((x-f)/f)$, thus

$$\begin{aligned} M_f(s) &= \int_0^{\infty} \sqrt{\frac{x}{x+f}} W(x/f)(x+f)^{s-1}\,dx \\ &= f^s \int_0^{\infty} \sqrt{\frac{y}{y+1}} W(y)(y+1)^{s-1}\,dy. \end{aligned} \tag{1.7}$$

Then

$$\frac{1}{2\pi i}\int_{(a)} M_f(s)(n+f)^{-s}\,ds = \sqrt{\frac{n}{n+f}}W(n/f) \quad (a>1) \tag{1.8}$$

by the Mellin inversion formula; as usual, (a) denotes the line $\sigma = a$ with $s=\sigma+it$. Our goal is to show that

$$\zeta_f(s) \approx \sum_{n=1}^{\infty} t(n)t(n+f)\sqrt{\frac{n+f}{n}}(n+f)^{-s} \quad (\sigma>1), \tag{1.9}$$

so that

$$A(f;W) \approx \frac{1}{2\pi i}\int_{(a)} M_f(s)\zeta_f(s)\,ds. \tag{1.10}$$

Once (1.9) is made precise, then (1.10) becomes an identity as well, at the cost of a couple of additional correction terms of arithmetic nature. On the other hand, the inner product on the right of (1.5) can be calculated spectrally by Parseval's formula as in (1.3). Then, substituting the resulting spectral decomposition of $\zeta_f(s)$ into the integral on the right of (1.10), we end up with certain complex integrals which can be evaluated in terms of hypergeometric functions, and a summation formula for the sum $A(f;W)$ follows. This is in a nutshell the argument to be carried out below.

2. Spectral decomposition of $\zeta_f(s)$

To formulate Parseval's formula for automorphic functions, let $E(z,s) = P_0(z,s)$ be the non-holomorphic Eisenstein series and let $u_j(z)$ run over the orthonormal system of the Maass forms. Then, for two Γ-invariant functions f_1 and f_2 which are square integrable over a fundamental domain of Γ with respect to the hyperbolic measure $dx\,dy/y^2$, we have

$$\langle f_1, f_2\rangle = \frac{3}{\pi}\langle f_1, 1\rangle\overline{\langle f_2, 1\rangle} + \sum_{j=1}^{\infty}\langle f_1, u_j\rangle\overline{\langle f_2, u_j\rangle} \tag{2.1}$$

$$+\frac{1}{4\pi}\int_{-\infty}^{\infty}\langle f_1, E(\cdot, 1/2+ir)\rangle\overline{\langle f_2, E(\cdot, 1/2+ir)\rangle}\,dr.$$

We apply this to the inner product in (1.5). The first term on the right of (2.1) vanishes because $\langle P_f(z,s), 1\rangle = 0$. Further, the inner products of $P_f(z,s)$ against Eisenstein series and Maass forms were evaluated in [18], Lemma I.4. Let

$$c_j = \langle u_j, |u|^2\rangle,$$

$$c(r) = \langle E(\cdot, 1/2+ir), |u|^2\rangle$$

stand for the other inner products occurring in Parseval's formula. Then, denoting by κ_j the spectral parameter related to the Maass form u_j and writing $z_j = 1/2 + i\kappa_j$, we obtain the following spectral decomposition of $\zeta_f(s)$.

Lemma 1. *For $\sigma > 1$, we have*

$$\zeta_f(s) = \frac{f^{1/2-s}}{|\rho(1)|^2\Gamma(s)\Gamma(s-1)}\sum_{j=1}^{\infty} c_j\overline{\rho_j(1)}t_j(f)\Gamma(s-z_j)\Gamma(s-\overline{z_j})$$

$$+\frac{f^{1/2-s}}{2|\rho(1)|^2\sqrt{\pi}\Gamma(s)\Gamma(s-1)}\int_{-\infty}^{\infty}\frac{c(r)\sigma_{2ir}(f)\Gamma(s-1/2+ir)\Gamma(s-1/2-ir)}{(\pi f)^{ir}\Gamma(1/2-ir)\zeta(1-2ir)}\,dr.$$

In applications of the above formula, one needs mean value estimates for c_j and $c(r)$, which we quote from Lemmas 3 and 4 of [6].

Lemma 2. *For $K \geq 1$, we have*

$$\int_{-K}^{K} |c(r)|^2 \exp(\pi |r|)\, dr \ll K^{\varepsilon}, \tag{2.2}$$

$$\sum_{\kappa_j \leq K} |c_j|^2 \exp(\pi \kappa_j) \ll K^{\varepsilon}. \tag{2.3}$$

3. An arithmetic formula for $\zeta_f(s)$

An alternative representation for $\zeta_f(s)$ follows from the definition (1.5) by the unfolding method as in [5], Sec. 2.2. Substituting the Fourier series (1.4) and recalling the connection between $\rho(n)$ and $t(n)$, we obtain

$$\begin{aligned}\zeta_f(s) &= \frac{2(4\pi)^{s-1}}{|\rho(1)|^2 \Gamma(s-1)} \int_0^{\infty} e^{-2\pi f y} y^{s-2} \left(\int_{-1/2}^{1/2} |u(x+yi)|^2 e(fx)\, dx \right) dy \\ &= \sum_{n=1}^{\infty} t(n) t(n+f) \psi(s; n+f, n, f) + \frac{1}{2} \epsilon \sum_{n=1}^{f-1} t(n) t(f-n) \psi(s; n, f-n, f)\end{aligned} \tag{3.1}$$

with

$$\psi(s; m, n, f) = \frac{2^s}{\pi \Gamma(s-1)} \int_0^{\infty} K_{i\kappa}(my) K_{i\kappa}(ny) e^{-fy} y^{s-1}\, dy. \tag{3.2}$$

Next we express the integral (3.2) in terms of elementary functions by the formula (see [11], eq. (5.10.23))

$$K_{\nu}(x) = \int_0^{\infty} e^{-x \cosh t} \cosh(\nu t)\, dt \tag{3.3}$$

and derive the following analogue of Lemma 1 in [5].

Lemma 3. *For $\Re s > 0$, we have*

$$\psi(s; m, n, f) = \frac{m+n+f}{2\sqrt{mn}} \left(\frac{m+n+f}{2} \right)^{-s} + \rho(s; m, n, f), \tag{3.4}$$

where

$$\rho(s;m,n,f)=\frac{m+n+f}{\pi\sqrt{mn}}\int_0^\infty\int_0^{\pi/2}\left(\frac{1}{2}(m+n+f)+\sqrt{mn}r^2\right)^{-s}\times\left(1+\frac{2\sqrt{mn}r^2}{m+n+f}\right)Q_r\,dr\,d\theta$$

with

$$Q=Q(r,\theta;m,n,f)=\frac{\cos(\kappa v(r,\theta;n/m))\cos(\kappa w(r,\theta;m/n))}{\sqrt{1+r^2(\sqrt{n/m}\cos^2\theta+\sqrt{m/n}\sin^2\theta)+r^4\sin^2\theta\cos^2\theta}}$$

and

$$\begin{aligned}v(r,\theta;a)&=\cosh^{-1}(1+2\sqrt{a}r^2\cos^2\theta)\\&=\log\left(1+2\sqrt{a}r^2\cos^2\theta+2\sqrt[4]{a}r|\cos\theta|\sqrt{1+\sqrt{a}r^2\cos^2\theta}\right),\\ w(r,\theta;a)&=\cosh^{-1}(1+2\sqrt{a}r^2\sin^2\theta)\\&=\log\left(1+2\sqrt{a}r^2\sin^2\theta+2\sqrt[4]{a}r|\sin\theta|\sqrt{1+\sqrt{a}r^2\sin^2\theta}\right).\end{aligned}$$

Proof. First, by (3.2) and (3.3), we have

$$\psi(s;m,n,f)=\frac{2^s}{\pi\Gamma(s-1)}\int_0^\infty\int_0^\infty\left(\int_0^\infty e^{-(m\cosh v+n\cosh w+f)y}\cos(\kappa v)\cos(\kappa w)y^{s-1}\,dy\right)dvdw$$

$$=2^s\pi^{-1}(s-1)\int_0^\infty\int_0^\infty\frac{\cos(\kappa v)\cos(\kappa w)}{(m\cosh v+n\cosh w+f)^s}\,dv\,dw.$$

Reformulated next in the variables x and y such that $\cosh v=1+2\sqrt{n/m}x^2$ and $\cosh w=1+2\sqrt{m/n}y^2$, and then in the polar coordinates, this becomes

$$\frac{4(s-1)}{\pi((m+n+f)/2)^s}\int_0^\infty\int_0^{\pi/2}\left(1+\frac{2\sqrt{mn}r^2}{m+n+f}\right)^{-s}Q(r,\theta;m,n,f)r\,dr\,d\theta.$$

Finally, integration by parts with respect to r gives the formula (3.4).

4. The summation formula

Putting things together, we now deduce our summation formula on evaluating the integral on the right of (1.10) in two ways by use of Lemmas 1 and 3. We introduce the notation $(r \to -r)$ to denote an expression similar to the preceding one but with r replaced by $-r$.

Theorem. *Let $W(x)$ be a continuously differentiable function with compact support in the positive reals and let f be a natural number. Then*

$$A(f;W) = A_d + A_c + E, \tag{4.1}$$

where

$$A_d = \frac{f^{1/2}}{2\pi|\rho(1)|^2}\sum_{j=1}^{\infty} c_j(\frac{1}{2}+i\kappa_j)\overline{\rho_j(1)}t_j(f)\Psi(\kappa_j, W), \tag{4.2}$$

$$A_c = \frac{f^{1/2}}{4\pi^{3/2}|\rho(1)|^2}\int_{-\infty}^{\infty}\frac{c(r)(1/2+ir)\sigma_{2ir}(f)}{(\pi f)^{ir}\Gamma(1/2-ir)\zeta(1-2ir)}\Psi(r;W)\,dr \tag{4.3}$$

with

$$\Psi(r;W) = \frac{\Gamma^2(1/2+ir)}{\Gamma(1+2ir)}\coth(\pi r) \tag{4.4}$$

$$\times\int_0^{\infty} W(y)(y+1)^{-1/2}y^{-ir}F(-1/2+ir, 1/2+ir; 1+2ir; -1/y)\,dy$$

$$+(r \to -r),$$

and finally

$$E = -\frac{2}{\pi}\sum_{n=1}^{\infty} t(n)t(n+f)W_1(n,f) - \frac{\epsilon}{\pi}\sum_{n=1}^{f-1} t(n)t(f-n)W_2(n,f) \tag{4.5}$$

with

$$W_1(n,f) = \int_0^{\pi/2}\int_0^{\infty}\sqrt{1+r^4+(2n+f)r^2/\sqrt{n(n+f)}} \tag{4.6}$$

$$\times W((n+\sqrt{n(n+f)}r^2)/f)Q_r(r,\theta;n+f,n,f)\,dr\,d\theta$$

and

$$W_2(n,f) = \frac{1}{\sqrt[4]{n(f-n)}} \int\limits_0^{\pi/2} \int\limits_0^{\infty} \sqrt{f + \sqrt{n(f-n)}r^2} \tag{4.7}$$

$$\times r W(\sqrt{n(f-n)}r^2/f) Q_r(r,\theta;n,f-n,f)\,dr\,d\theta.$$

Proof. First we substitute the arithmetic formula for $\zeta_f(s)$, arising from (3.1) and Lemma 3, into the integral in (1.10). Then, by the Mellin inversion, this integral equals $A(f;W) - E$.

Next we calculate the same integral on substituting the definition of $M_f(s)$ from (1.7) and the spectral formula for $\zeta_f(s)$ from Lemma 1. Then we end up with integrals of the form

$$\frac{1}{2\pi i} \int\limits_{(a)} \frac{\Gamma(s-1/2+ir)\Gamma(s-1/2-ir)X^{s-1}}{\Gamma(s)\Gamma(s-1)}\,ds \quad (a>1)$$

with $X = y+1$. A closely analogous integral was evaluated in [9], Sec. 3.6, in terms of hypergeometric functions by the Mellin–Barnes formula, and the same argument shows that the preceding integral equals

$$X^{-1/2-ir}\cosh(\pi r)\frac{\Gamma(-2ir)\Gamma(1/2+ir)}{\pi\Gamma(-1/2-ir)} F(1/2+ir, 3/2+ir; 1+2ir; 1/X)$$

$$+(r \to -r)$$

(consider first the integral with $\Gamma(s-1+w)$ standing in place of $\Gamma(s-1)$ in the denominator, and put $w=0$ in the final expression). Here we apply the formula (see [11], Eq. (9.5.1))

$$F(\alpha,\beta;\gamma;z) = (1-z)^{-\alpha} F\left(\alpha, \gamma-\beta; \gamma; \frac{z}{z-1}\right),$$

valid for $|\arg(z)| < \pi$ and $\gamma \neq 0, -1, -2, \ldots.$, and after a rearrangement of the gamma-factors we get the terms A_d and A_c in (4.1).

Remark 1. It is interesting to compare the formula (4.1) with Motohashi's [12] identity for the sum

$$D(f;W) = \sum_{n=1}^{\infty} d(n)d(n+f)W(n/f).$$

The latter result is of the form

$$D(f;W) = D_r + D_d + D_c + D_h,$$

where D_r is a main term, D_h is a contribution of the holomorphic cusp forms,

$$D_d = f^{1/2}\sum_{j=1}^{\infty}\alpha_j t_j(f)H_j^2(1/2)\tilde{\Psi}(\kappa_j;W)$$

and

$$D_c = f^{1/2}\pi^{-1}\int_{-\infty}^{\infty}\frac{|\zeta(1/2+ir)|^4}{|\zeta(1+2ir)|^2}f^{-ir}\sigma_{2ir}(f)\tilde{\Psi}(r;W)\,dr.$$

Here $\alpha_j = |\rho_j(1)|^2/\cosh(\pi\kappa_j)$, $H_j(s)$ is the Hecke L-function attached to jth Maass form, and

$$\tilde{\Psi}(r;W) = \frac{1}{2}\int_0^{\infty}\Re\Big\{\Big(1-\frac{i}{\sinh(\pi r)}\Big)\frac{\Gamma^2(1/2+ir)}{\Gamma(1+2ir)}y^{-1/2-ir}$$
$$\times F(1/2+ir,1/2+ir;1+2ir;-1/y)\Big\}W(y)\,dy.$$

The analogy of A_d and A_c with D_d and D_c is apparent. In (4.2) and (4.3), we attached the factor $1/2+i\kappa_j$ or $1/2+ir$ to c_j or $c(r)$ in order to stress the analogy of (2.3) with the estimate (see [14], Theorem 3.4)

$$\sum_{\kappa_j\le K}\alpha_j H_j^4(1/2) \ll K^2\log^{15}K,$$

and likewise for (2.2).

In applications, it is convenient to transform the hypergeometric function in (4.4) by the formula (see [11], eq. (9.6.12))

$$F(\alpha,\beta;2\beta;z) = \left(\frac{1+\sqrt{1-z}}{2}\right)^{-2\alpha}F\left(\alpha,\alpha-\beta+\frac{1}{2};\beta+\frac{1}{2};\left(\frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}\right)^2\right),$$

for the new hypergeometric function can be expressed by its rapidly convergent series.

Remark 2. If W is an *oscillating* function, then one may expect cancellation in the integrals (4.6) and (4.7) defining $W_1(n,f)$ and $W_2(n,f)$. Also, it should be taken into account that the sum involving $W_1(n,f)$ in (4.5) is of the same structure as the original sum $A(f;W)$, so that the formula (4.1) can be iterated with the purpose of reducing the error term. In the course of the argument, we get additional but easier contributions of the type A_d and A_c.

As a typical example of an oscillating sum, let us consider a sum of the form

$$\sum_n t(n)t(n+f)g(n)e(\varphi(n)), \tag{4.8}$$

where $g(n)$ and $\varphi(n)$ are functions of support in $[M,2M]$ such that $g^{(j)}(x) \ll_j M^{-j}$ for $j \geq 0$, $|\varphi'(x)| \gg FM^{-1}$ and $\varphi^{(j)}(x) \ll_j FM^{-j}$ for $j \geq 2$, where M is a large parameter, and $F \gg M^{\eta_1}$, $f \ll M^{1-\eta_2}$ for some positive constants η_1 and η_2. The sum (4.8) is of the type (1.6) with $W(y) = g(fy)e(\varphi(fy))$. Repeated integration by parts with respect to r in (4.7) shows that the second sum in (4.5) is negligibly small, in fact it is $O(M^{-B})$. As to the first sum in (4.5), we may apply the iterative argument mentioned above. Note that the r-integration in (4.6) can be restricted to the range $r \ll F^{-1/2}M^{\varepsilon}$, for if the remaining r-integral is equipped with a smooth weight at the beginning of the range, then repeated integration by parts shows that the corresponding contribution to E is $O(M^{-B})$. For given r, we have a sum of the type (4.8), but since the relevant range for r is shorter than unity, we obtain a certain saving, and after finitely many iterations we end up with an expression which can be estimated trivially. Thus, all in all, we may virtually omit the term E since the most significant terms appeared already before the iteration got started.

Remark 3. The preceding remarks can be applied, in combination with the methods of [10], to the estimation of the Rankin–Selberg zeta-function

$$L(s, u \otimes u_j) = \zeta(2s) \sum_{n=1}^{\infty} t(n)t_j(n)n^{-s},$$

where u is a fixed Hecke invariant cusp form, either holomorphic or real analytic, with the Hecke eigenvalues $t(n)$. Letting κ_j stand for the spectral parameter of the Maass form u_j, we have for $t \geq 0$

$$L(1/2+it, u \otimes u_j) \ll \left((\kappa_j+t)^{2/3} + (\kappa_j+t)^{1/2}t^{1/4} + t\kappa_j^{-1/2}\right)(t+\kappa_j)^{\varepsilon}. \tag{4.13}$$

This improves the convexity bound $\ll t + \kappa_j$ except when t is very large compared with κ_j. Subconvexity estimates for Rankin–Selberg L-functions have been given previously by P. Sarnak [16], V. Blomer [2] and others, also for congruence subgroups. These results are weaker than (4.13) in the case of the full modular group, and the uniformity in t is not considered. A proof of (4.13) will be given in a forthcoming joint paper by Y. Motohashi and the author.

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