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PARAXIAL RAY THEORY FOR MAXWELL'S EQUATIONS

Abstract. Paraxial ray theory for Maxwell's equations in the case of an inhomogeneous isotropic medium with finite conductivity and smooth interfaces is developed. We show that the ray centered coordinates are suitable for describing amplitudes and polarization of waves in their propagation and reflection/refraction on a smooth interface. Expressions for the geometrical spreading and second order derivatives of the eikonal are obtained in terms of certain solutions of the equations in variations, i.e., equations which describe rays close to the central ray in linear approximation.

1. Introduction

In this paper, we develop the paraxial ray theory for Maxwell's equations in the case of an inhomogeneous isotropic medium with finite conductivity and smooth interfaces in order to apply it to modeling. The main idea of the paraxial approach is simple and can be explained as follows. If we consider a narrow ray tube surrounding a fixed central ray we may linearize Euler's equations for the rays from the ray tube because they are close to the central one. This results in significant simplification of Euler's equations, and therefore of the whole ray theory built on them. Note that these linearized equations for the rays close to the central one are known in classical mechanics as the equations in variations. Clearly, the paraxial ray theory is valid only in some vicinity of the central ray, however, keeping in mind our goal, it provides the following advantages.

Firstly, it gives rise to a natural way of solution of the dynamic ray tracing problem, and description of wave field polarization.

Then, this approach enables us to calculate second order derivatives of the eikonal on the central ray what is important for interpolation.

Lastly, the paraxial ray method provides, perhaps, the methodologically simplest way of deriving an individual Gaussian beam propagating

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along the central ray. It should be emphasized that only the leading term of the Gaussian beam can be constructed in this way. However, the leading term is sufficient for modeling with the Gaussian Beam method.

Note that the paraxial approach was actually developed and used by specialists in optical resonators for lasers in the 1960s. In particular, in the papers by Popov (1967, 1969) this method was developed for calculations of eigenfrequencies of open resonators in homogeneous and inhomogeneous media and the ray centered coordinates were first introduced.

Approximately at that period of time the paraxial method was extensively studied at Steklov Mathematical Institute in connection with Gaussian beams, at first by means of parabolic equation, and then by solving directly the eikonal and transport equations in a vicinity of the central ray. To this end see the review paper by Babich and Popov (1989).

In geophysics the paraxial approach was first used by Popov (1977) and Popov and Pšenčik (1978 a,b) for the dynamic ray tracing problem. Later this method was applied to different geophysical problems, see Červený and Hron (1980) and Červený (2001).

For recent discussion of this approach based on parabolic equation, see Kiselev (2000) and references therein.

The ray method for Maxwell's equations was suggested in pioneer works by Rytov (1938) and by Luneburg (1966).

Lecture Notes on ray optics by Luneburg were delivered in the mid 1940s but were published much later after the death of the author. They contain probably the first consistent description of the ray method as a mathematical tool for studying wave propagation problems in electrodynamics but, unfortunately, they remain actually inaccessible in scientific libraries.

Many aspects of the ray method can be found in Born and Wolf (1999), Felsen and Marcuvits (1973), see also references therein to Chapter 1, in particular, papers by R. Lewis, e.g., Lewis (1967), in Kravtsov and Orlov (1990). In the book by Babich, Buldyrev and Molotkov (1985), the reader will find detailed derivation of the basic equations of the space-time ray method in the case of an inhomogeneous medium with zero conductivity.

In the first auxiliary section of the paper we derive the basic equations of the ray method employing a bit nontraditional linear algebra technique. In following sections we integrate transport equations and solve the reflection/refraction problem on a smooth interface using, from the very beginning, the ray centered coordinates. It turns out that in the ray centered coordinates the systems of transport equations split into two independent equations and therefore the polarization of the wave field is naturally described by the basis vectors of the coordinate in use. The algorithm for construction the reflected/refracted wave in rather general case of an inhomogeneous medium with finite conductivity under consideration seems to be compact and relatively simple. Thus, it follows from our results, the ray centered coordinates are a suitable tool in the case of Maxwell's equations too.

In the present paper, we hold on, as much as possible, to a methodological approach for solving the eikonal and transport equations presented in Popov (2002) in the case of elastodynamics.

2. Basic equations of the ray theory

Consider an isotropic inhomogeneous medium with smooth interfaces in which the electric ϵ and magnetic μ permeabilities and the conductivity σ are smooth, i.e., differentiable as many times as we need, functions of the point between the interfaces. We assume also the density ρ of free charges is identically equal to zero.

Then, Maxwell's equations in a global Cartesian system of coordinates x_1, x_2, x_3 can be written down as follows (see, e.g., Stratton (1941)),

$$\operatorname{rot}\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0; \quad \operatorname{rot}\vec{H} - \frac{\partial \vec{D}}{\partial t} = \sigma\vec{E} \tag{1}$$

and

$$\operatorname{div}\vec{B} = 0, \quad \operatorname{div}\vec{D} = 0, \tag{2}$$

where

$$\vec{D} = \epsilon\vec{E}, \quad \vec{B} = \mu\vec{H} \tag{3}$$

and $\epsilon(x_1, x_2, x_3)$, $\mu(x_1, x_2, x_3)$, and $\sigma(x_1, x_2, x_3)$ are smooth functions of their arguments. We consider further a wave field harmonic in time and seek an approximate or asymptotic solution of the equations (1)–(3) in the form of a ray series

$$\vec{E} = e^{-i\omega(t-\tau)} \sum_{p=0}^{\infty} \frac{\vec{E}^{(p)}}{(i\omega)^p} \quad \text{and} \quad \vec{H} = e^{-i\omega(t-\tau)} \sum_{p=0}^{\infty} \frac{\vec{H}^{(p)}}{(i\omega)^p}, \tag{4}$$

where eikonal $\tau(x_1, x_2, x_3)$ and amplitudes $\vec{E}^{(p)}(x_1, x_2, x_3)$ and $\vec{H}^{(p)}(x_1, x_2, x_3)$, $p = 0, 1, 2, \ldots$, are so far unknown functions, and the circular frequency ω is supposed to be a large parameter, i.e., $\omega \to +\infty$.

By inserting the asymptotic series (4) into system of equations (1) and by collecting terms with equal power of ω and equating them to zero, we obtain the recurrent system of equations for the unknown amplitudes.

Pursuing our aim to construct only the leading terms in series (4), i.e., with p = 0, we have to study the first two systems of the recurrent equations.

For the leading terms $\vec{E}^{(0)}$ and $\vec{H}^{(0)}$ we obtain the following homogeneous system of linear algebraic equations

$$\begin{cases} \nabla\tau \times \ \vec{E}^{(0)} - \mu\vec{H}^{(0)} = 0, \\ \\ \nabla\tau \times \ \vec{H}^{(0)} + \epsilon\vec{E}^{(0)} = 0. \end{cases} \tag{5}$$

It follows from (5) that the three vectors $\nabla\tau$, $\vec{E}^{(0)}$ and $\vec{H}^{(0)}$ form the right-orientated set of mutually orthogonal vectors and therefore equations (2) are already satisfied for the leading term of the asymptotic series (4).

In order to simplify further investigations, let us introduce an auxiliary right-orientated basis of mutually orthogonal unit vectors $\vec{l}_1$, $\vec{l}_2$, $\vec{l}_3$:

$$\vec{l}_1 \times \vec{l}_2 = \vec{l}_3; \quad (\vec{l}_j, \vec{l}_k) = \delta_{jk}, \quad j,k = 1,2,3, \quad \vec{l}_3 = \frac{1}{|\nabla\tau|}\nabla\tau. \tag{6}$$

By projecting the field vectors onto basis (6)

$$\vec{E}^{(0)} = E_1^{(0)}\vec{l}_1 + E_2^{(0)}\vec{l}_2; \quad \vec{H}^{(0)} = H_1^{(0)}\vec{l}_1 + H_2^{(0)}\vec{l}_2, \tag{7}$$

we replace the vectorial system (5) by the following scalar one:

$$\begin{bmatrix} 0 & -|\nabla\tau| & -\mu & 0 \\ |\nabla\tau| & 0 & 0 & -\mu \\ \epsilon & 0 & 0 & -|\nabla\tau| \\ 0 & \epsilon & |\nabla\tau| & 0 \end{bmatrix} \begin{bmatrix} E_1^{(0)} \\ E_2^{(0)} \\ H_1^{(0)} \\ H_2^{(0)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{8}$$

Denote by N the main matrix of system (8), that is the matrix on the left-hand side in Eq. (8). The condition $\det N = 0$ gives rise to the eikonal equation in traditional form

$$(\nabla\tau, \nabla\tau) = \frac{1}{c^2}, \quad c = \frac{1}{\sqrt{\mu\,\epsilon}}, \tag{9}$$

and two linearly independent solutions of (8) have the form

$$ {}^{1}\vec{E}^{(0)} = \varphi_o^{(1)}\, \vec{l}_1; \quad {}^{1}\vec{H}^{(0)} = \varphi_o^{(1)} \sqrt{\frac{\epsilon}{\mu}}\, \vec{l}_2 \tag{10} $$

and

$$ {}^{2}\vec{E}^{(0)} = \varphi_o^{(2)}\, \vec{l}_2; \quad {}^{2}\vec{H}^{(0)} = -\varphi_o^{(2)} \sqrt{\frac{\epsilon}{\mu}}\, \vec{l}_1, \tag{11} $$

where $\varphi_o^{(1)}$ and $\varphi_o^{(2)}$ are so far unknown functions of the coordinates x_1, x_2, x_3.

On the next step we have to derive transport equations for functions $\varphi_o^{(1)}$ and $\varphi_o^{(2)}$. To this end, consider the equations for $\vec{E}^{(1)}$ and $\vec{H}^{(1)}$ in asymptotic expansion (4)

$$ \begin{cases} \nabla\tau \times \vec{E}^{(1)} - \mu\vec{H}^{(1)} = -\operatorname{rot}\vec{E}^{(0)}, \\ \\ \nabla\tau \times \vec{H}^{(1)} + \epsilon\vec{E}^{(1)} = -rot\vec{H}^{(0)} + \sigma\,\vec{E}^{(0)} \end{cases} . \tag{12} $$

By projecting (12) onto the basis (6), we represent it in the following scalar form

$$ \begin{bmatrix} 0 & -|\nabla\tau| & 0 & -\mu & 0 & 0 \\ |\nabla\tau| & 0 & 0 & 0 & -\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mu \\ \epsilon & 0 & 0 & 0 & -|\nabla\tau| & 0 \\ 0 & \epsilon & 0 & |\nabla\tau| & 0 & 0 \\ 0 & 0 & \epsilon & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} E_1^{(1)} \\ E_2^{(1)} \\ E_3^{(1)} \\ H_1^{(1)} \\ H_2^{(1)} \\ H_3^{(1)} \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}, \tag{13} $$

where the following additional notations are introduced

$$ a_j = (-\operatorname{rot}\vec{E}^{(0)},\, \vec{l}_j); \quad b_j = (-\operatorname{rot}\vec{H}^{(0)} + \sigma\vec{E}^{(0)},\, \vec{l}_j), \quad j = 1, 2, 3. \tag{14} $$

In order to explore system (13), we use the following well-known result in the theory of linear algebraic systems (Kronecker–Capelli theorem): inhomogeneous system (13) is solvable if and only if the rank of the main matrix is equal to the rank of the extended matrix of this system.

By employing the Gauss algorithm of successive excluding of variables for the rank calculation, we arrive at the conclusion: the inhomogeneous system (13) is solvable with respect to $\vec{E}^{(1)}$ and $\vec{H}^{(1)}$ if and only if the two following conditions on $\vec{E}^{(0)}$ and $\vec{H}^{(0)}$

$$ \epsilon\,(\operatorname{rot}\vec{E}^{(0)}, \vec{l}_2) + |\nabla\tau|\,(-\operatorname{rot}\vec{H}^{(0)} + \sigma\,\vec{E}^{(0)}, \vec{l}_1) = 0 \tag{15} $$

and

$$-\epsilon\,(\mathrm{rot}\,\vec{E}^{(0)},\vec{l}_1) + |\nabla\tau|\,(-rot\vec{H}^{(0)} + \sigma\,\vec{E}^{(0)},\vec{l}_2) = 0 \qquad (16)$$

hold true.

Precisely these conditions give rise to transport equations for $\varphi_o^{(1)}$ and $\varphi_o^{(2)}$ in formulas (10) and (11). Indeed, by substituting the field vectors $\vec{E}^{(0)}$ and $\vec{H}^{(0)}$ in (15) and (16) by linear combinations of solutions (10) and (11)

$$\vec{E}^{(0)} = {}^1\vec{E}^{(0)} + {}^2\vec{E}^{(0)}; \quad \vec{H}^{(0)} = {}^1\vec{H}^{(0)} + {}^2\vec{H}^{(0)}, \qquad (17)$$

we obtain, after some mathematics,

$$\begin{aligned} &2(\vec{l}_3,\nabla\varphi_o^{(1)}) + \\ &+\varphi_o^{(1)}\left[\sqrt{\frac{\mu}{\epsilon}}\Big(\vec{l}_3,\nabla\Big(\sqrt{\frac{\epsilon}{\mu}}4\Big)\Big) + (\mathrm{rot}\,\vec{l}_1,\vec{l}_2) - (\mathrm{rot}\,\vec{l}_2,\vec{l}_1) + \sqrt{\frac{\mu}{\epsilon}}\,\sigma\right] \\ &+\varphi_o^{(2)}\Big[(\mathrm{rot}\,\vec{l}_1,\vec{l}_1) + (\mathrm{rot}\,\vec{l}_2,\vec{l}_2)\Big] = 0 \end{aligned} \qquad (18)$$

and

$$\begin{aligned} &2(\vec{l}_3,\nabla\varphi_o^{(2)}) + \\ &+\varphi_o^{(2)}\left[\sqrt{\frac{\mu}{\epsilon}}\Big(\vec{l}_3,\nabla\Big(\sqrt{\frac{\epsilon}{\mu}}\Big)\Big) + (\mathrm{rot}\,\vec{l}_1,\vec{l}_2) - (\mathrm{rot}\,\vec{l}_2,\vec{l}_1) + \sqrt{\frac{\mu}{\epsilon}}\,\sigma\right] \\ &-\varphi_o^{(1)}\Big[(\mathrm{rot}\,\vec{l}_1,\vec{l}_1) + (\mathrm{rot}\,\vec{l}_2,\vec{l}_2)\Big] = 0. \end{aligned} \qquad (19)$$

Expressions (18) and (19) form a system of differential equations with respect to functions $\varphi_o^{(1)}$ and $\varphi_o^{(2)}$.

Now we have to integrate the basic equations (9), (18) and (19) in a vicinity of the central ray of a ray tube in the ray centered coordinates.

Since the eikonal equation (9) has the traditional form, we can just use the results developed, e.g., in the case of the acoustic wave equation.

As to equations (18) and (19), one can easily observe that they can be reduced to a system of ordinary differential equations due to

$$(\vec{l}_3\,,\nabla\varphi) = \left(\frac{\nabla\tau}{|\nabla\tau|},\nabla\varphi\right) = (\vec{t},\nabla\varphi) = \frac{d\,\varphi}{d\,s}, \qquad (20)$$

where $\vec{t}$ is a unit vector along a ray and s is its arc length.

3. Integration of transport equations

Further in the paper we extensively use the ray centered coordinates and therefore we should remind their definition together with some auxiliary but important results, see, e.g., Popov (2002).

Let us suppose that the central ray of a ray tube is given in the vector form $\vec{r} = \vec{r}_o(s)$, where $\vec{r}$ is the radius-vector in the global Cartesian coordinates x_1, x_2, x_3, and s is the arc length along this ray. The ray centered coordinates s, q_1, and q_2 are introduced by the formula

$$\vec{r}_M = \vec{r}_o(s) + q_1\,\vec{e}_1(s) + q_2\,\vec{e}_2(s) \tag{21}$$

in which $\vec{r}_M$ is the radius-vector of a point M in a vicinity of the ray $\vec{r}_o$ and $\vec{e}_j(s)$, $j = 1, 2$, are mutually orthogonal unit vectors located on the normal plane to the ray for each s and subjected to certain differential equations.

Assume that all material parameters ϵ, μ, σ, and, therefore, velocity $c = (\epsilon\,\mu)^{-(1/2)}$ are functions of s, q_1, q_2. Then the equations for $e_j(s)$ take the form

$$\frac{d\vec{e}_j}{ds} = \kappa_j(s)\,\vec{t}(s), \quad \kappa_j(s) = \frac{1}{c_o(s)}\,\frac{\partial c}{\partial q_j}\Big|_{q_1=q_2=0}, \quad c_o(s) = c(s,0,0), \tag{22}$$

where $\vec{t}(s)$ is a unit vector tangent to the central ray.

We need to use the equations in variations, i.e., equations which describe, in the first approximation, rays from a narrow ray tube surrounding the central one $\vec{r}_o(s)$. It should be emphasized that, similar to elastodynamics, the equations in variations underlie, in fact, the paraxial ray theory for Maxwell's equations. In Hamiltonian form, when along with coordinates q_1, q_2 we use slownesses p_1, p_2, they read

$$\begin{aligned} \frac{dq_j}{ds} &= c_o\,p_j; \\ \frac{dp_j}{ds} &= -\frac{1}{c_o^2}\sum_{k=1}^{2}\frac{\partial^2 c}{\partial q_j\,\partial q_k}\Big|_{q_1=q_2=0} q_k, \quad j = 1, 2. \end{aligned} \tag{23}$$

Then, we have to put into consideration two real solutions of the equations in variations

$$X^{(j)}(s) = \begin{bmatrix} q_1^{(j)}(s) \\ q_2^{(j)}(s) \\ p_1^{(j)}(s) \\ p_2^{(j)}(s) \end{bmatrix}, \quad j = 1, 2, \tag{24}$$

which satisfy the condition that their J-scalar product is equal to zero, i.e.,

$$(J\,X^{(1)}, X^{(2)}) \equiv p_1^{(1)}\,q_1^{(2)} + p_2^{(1)}\,q_2^{(2)} - q_1^{(1)}\,p_1^{(2)} - q_2^{(1)}\,p_2^{(2)} = 0. \tag{25}$$

Note that equation (25) does not fix solutions $X^{(j)}$ uniquely, we still have degrees of freedom for choosing them in accordance with a source of the wave field.

Let us dwell briefly on the eikonal equation (9).

In a vicinity of the central ray $\tau = \tau(s, q_1, q_2)$ can be constructed as a Taylor series on q_1, q_2

$$\tau(s, q_1, q_2) = \tau_o(s) + \frac{1}{2}\sum_{i,j=1}^{2} \Gamma_{ij}(s)\,q_i\,q_j + \cdots \equiv$$
$$\equiv \tau_o(s) + \frac{1}{2}\,(\Gamma\,\vec{q},\,\vec{q}) + \cdots, \tag{26}$$

where coefficients $\Gamma_{ij}(s)$ form a symmetric 2×2 matrix $\Gamma(s)$; vector $\vec{q}$ is formed by coordinates q_1, q_2,

$$\vec{q} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \quad \text{and} \quad \tau_o(s) = \int \frac{ds}{c_o(s)}. \tag{27}$$

Matrix Γ in (26) can be expressed in terms of solutions (24) as follows.

Introduce

$$Q = \begin{bmatrix} q_1^{(1)} & q_1^{(2)} \\ q_2^{(1)} & q_2^{(2)} \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} p_1^{(1)} & p_1^{(2)} \\ p_2^{(1)} & p_2^{(2)} \end{bmatrix} \tag{28}$$

then if $\det Q \neq 0$, put

$$\Gamma = P\,Q^{-1}. \tag{29}$$

We shall see later that in the case of Maxwell's equations $\det Q = 0$ on caustics.

It should be emphasized that condition (25) guarantees that the matrix Γ, defined by equations (28) and (29), is symmetrical.

Suppose now that two terms for the eikonal in equation (26) are constructed, i.e., $\tau_o(s)$ and matrix Γ are known, then we can integrate the transport equations (18) and (19) along the central ray $\vec{r}_o(s)$. To this end we have to calculate the scalar product $(\mathrm{rot}\ \vec{l}_i\,, \vec{l}_j)$ for $i, j = 1, 2$ precisely on the central ray $\vec{r}_o(s)$, i.e., when $q_1 = q_2 = 0$.

Let us build vectors $\vec{l}_1$ and $\vec{l}_2$ in the basis $\vec{t}$, $\vec{e}_1$, $\vec{e}_2$ of the ray centered coordinates. For that matter, consider a wave front $\tau(s, q_1, q_2) = \mathrm{const}$ and a set of curves on the wave front, defined by conditions $q_2 = q_{20} = \mathrm{const}$ and $q_1 = q_{10} = \mathrm{const}$, respectively. Let us determine $\vec{l}_1$ and $\vec{l}_2$ as a unit vector tangent to the curves $q_2 = \mathrm{const}$ and $q_1 = \mathrm{const}$, respectively.

Since on the wave front $\tau(s, q_1, q_2) = \mathrm{const}$

$$d\tau \equiv \frac{\partial\tau}{\partial s}\, ds + \frac{\partial\tau}{\partial q_1}\, dq_1 + \frac{\partial\tau}{\partial q_2}\, dq_2 = 0. \tag{30}$$

We derive that

$$\vec{l}_1 = \frac{1}{h_1}\Big(-\frac{\partial\tau}{\partial q_1}\,\vec{t} + \frac{\partial\tau}{\partial s}\,\vec{e}_1\Big); \quad h_1 = \Big[\Big(\frac{\partial\tau}{\partial q_1}\Big)^2 + \Big(\frac{\partial\tau}{\partial s}\Big)^2\Big]^{\frac{1}{2}} \tag{31}$$

and

$$\vec{l}_2 = \frac{1}{h_2}\Big(-\frac{\partial\tau}{\partial q_2}\,\vec{t} + \frac{\partial\tau}{\partial s}\,\vec{e}_2\Big); \quad h_2 = \Big[\Big(\frac{\partial\tau}{\partial q_2}\Big)^2 + \Big(\frac{\partial\tau}{\partial s}\Big)^2\Big]^{\frac{1}{2}}. \tag{32}$$

Obviously, both $\vec{l}_1$ and $\vec{l}_2$ are orthogonal to $\vec{l}_3 = \frac{\nabla\tau}{|\nabla\tau|}$ because they are placed on tangent planes to the wave fronts. Precisely on the central ray they coincide with the vectors $\vec{e}_1$ and $\vec{e}_2$, respectively. But, in general, they are not orthogonal to each other. However, in the narrow vicinity of the central ray they are orthogonal in linear approximation, see formula (26) for the eikonal, and it is sufficient for our goal due to rot is a first order differential operator.

Now based on formulas (31) and (32) for vectors $\vec{l}_1$ and $\vec{l}_2$, we are able to calculate $(\mathrm{rot}\ \vec{l}_k\,, \vec{l}_j)$, $k, j = 1, 2$, precisely on the central ray.

By employing expressions for rot $\vec{l}_j$ in the orthogonal ray centered coordinates, in which e.g.,

$$\mathrm{rot}\ \vec{l}_1 = \frac{1}{h}\begin{bmatrix} h\,\vec{t} & \vec{e}_1 & \vec{e}_2 \\ \frac{\partial}{\partial s} & \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} \\ -\frac{h}{h_1}\frac{\partial\tau}{\partial q_1} & \frac{1}{h_1}\frac{\partial\tau}{\partial s} & 0 \end{bmatrix} \tag{33}$$

and $h = 1 + \kappa_1 q_1 + \kappa_2 q_2$ is Lame's coefficient of s, q_1, q_2 coordinates, we obtain, after some mathematics,

$$(\mathrm{rot}\ \vec{l}_1, \vec{l}_1) = \frac{1}{h_1}\frac{\partial \tau}{\partial q_1}\frac{\partial}{\partial q_2}\Big(\frac{1}{h_1}\frac{\partial \tau}{\partial s}\Big) - \frac{1}{h_1}\frac{1}{h}\frac{\partial \tau}{\partial s}\frac{\partial}{\partial q_2}\Big(\frac{h}{h_1}\frac{\partial \tau}{\partial q_1}\Big). \tag{34}$$

Since on the central ray (cr) $q_1 \equiv q_2 \equiv 0$ and therefore $h = 1$ and $\frac{\partial \tau}{\partial q_1} = 0$, we ultimately deduce from the above formula

$$(\mathrm{rot}\ \vec{l}_1, \vec{l}_1)\Big|_{\mathrm{cr}} = -\Big(\frac{\partial \tau}{\partial s}\Big)^{-1}\Big|_{\mathrm{cr}} \frac{\partial^2 \tau}{\partial q_1\,\partial q_2}\Big|_{\mathrm{cr}}. \tag{35}$$

In the case of $(\mathrm{rot}\ \vec{l}_2, \vec{l}_2)$ a similar calculation leads to the result

$$(\mathrm{rot}\ \vec{l}_2, \vec{l}_2)\Big|_{\mathrm{cr}} = \Big(\frac{\partial \tau}{\partial s}\Big)^{-1}\Big|_{\mathrm{cr}} \frac{\partial^2 \tau}{\partial q_1\,\partial q_2}\Big|_{\mathrm{cr}}. \tag{36}$$

Thus, it follows from expressions (35) and (36) that the system of transport equations (18) and (19) is actually split in two independent equations of the same shape.

Corresponding calculations for $(\mathrm{rot}\ \vec{l}_1, \vec{l}_2) - (\mathrm{rot}\ \vec{l}_2, \vec{l}_1)$ on the central ray yield

$$\begin{aligned}(\mathrm{rot}\ \vec{l}_1, \vec{l}_2)\Big|_{\mathrm{cr}} - (\mathrm{rot}\ \vec{l}_2, \vec{l}_1)\Big|_{\mathrm{cr}} &= \Big(\frac{\partial \tau}{\partial s}\Big)^{-1}\Big|_{\mathrm{cr}} \Big[\frac{\partial^2 \tau}{\partial q_1^2} + \frac{\partial^2 \tau}{\partial q_2^2}\Big]\Big|_{\mathrm{cr}} = \\ &= c_0(s)\ \mathrm{tr}\,\Gamma, \end{aligned}\tag{37}$$

where, as usual, $\mathrm{tr}\,\Gamma = \Gamma_{11} + \Gamma_{22}$.

Thus, by collecting the results described above, we observe that both equations (18) and (19) take the following form

$$\frac{1}{\varphi_o^{(j)}}\frac{d\varphi_o^{(j)}}{ds} + \frac{1}{2}\sqrt{\frac{\mu}{\epsilon}}\frac{d}{ds}\Big(\sqrt{\frac{\epsilon}{\mu}}\Big) = -\frac{1}{2}\Big(c_o\ \mathrm{tr}\,\Gamma + \sigma\sqrt{\frac{\mu}{\epsilon}}\Big); \quad j = 1, 2. \tag{38}$$

By making use of the formula

$$\frac{d}{ds}\det Q = c_o\ \mathrm{tr}\,\Gamma\ \det Q \tag{39}$$

which can be verified by direct calculations, we obtain the general solution of the transport equation (38)

$$\begin{aligned}\varphi_o^{(j)}(s) &= \sqrt[4]{\frac{\mu}{\epsilon}}\ \frac{\psi_o^{(j)}}{\sqrt{|\det Q(s)|}}\eta(s), \quad j = 1, 2, \\ \eta(s) &= \exp\ \Big[-\frac{1}{2}\int_{s_0}^{s} \sigma\sqrt{\frac{\mu}{\epsilon}}\,ds\Big], \end{aligned}\tag{40}$$

where $\psi_o^{(j)}$ is a constant of integration.

Note that the conductivity σ of a medium gives rise to attenuation of the ray amplitude along the ray, which we denote from here on by $\eta(s)$, see equation(40).

Summarizing the results from the two previous sections, we obtain the following expressions for the field vectors $\vec{E}$ and $\vec{H}$ on the central ray in the zero-order ray approximation

$$\begin{cases} \vec{E} \simeq \varphi_o^{(1)}\, e^{-i\,\omega(t-\tau_o^{(1)})}\, \vec{e}_1 + \varphi_o^{(2)}\, e^{-i\,\omega(t-\tau_o^{(2)})}\, \vec{e}_2 \\ \\ \vec{H} \simeq \sqrt{\dfrac{\epsilon}{\mu}}\, \varphi_o^{(1)}\, e^{-i\,\omega(t-\tau_o^{(1)})}\, \vec{e}_2 - \sqrt{\dfrac{\epsilon}{\mu}}\, \varphi_o^{(2)}\, e^{-i\,\omega(t-\tau_o^{(2)})} \vec{e}_1 \end{cases} \tag{41}$$

where amplitudes $\varphi_o^{(j)}$ are defined by equations (40).

To calculate the geometrical spreading $|\det Q|$ along the central ray we must know two solutions (24) of the equations in variations (23). They should be specified by initial conditions at a source of the wave field and their J-scalar product must be equal to zero. Then, by means of formula (28), we obtain $|\det Q(s)|$ and therefore amplitude $\varphi_o(s)$ at each point on the ray $\vec{r}_o(s)$.

Observe that the field vectors in equations (41) are represented as a linear combination of two linearly independent asymptotic solutions of Maxwell's equations (1)–(3) and their polarization is described by the unit vectors $\vec{e}_1(s)$ and $\vec{e}_2(s)$ of the ray centered coordinates. Since these linearly independent solutions may be based on different families of rays (it depends on a source of the wave field) the eikonals in (41) may be different, i.e., $\tau^{(1)} \not\equiv \tau^{(2)}$. Finally, as the central ray $\vec{r}_o(s)$ of a ray tube is, in fact, an arbitrary one, formulas (41) hold for any ray.

4. Reduction to a 2.5D case

Consider a special case where all material parameters ϵ, μ, σ in Maxwell's equations do not depend on one coordinate, say, x_2. Suppose also that a source of the wave field and an observation point are placed on a plane $x_2 = 0$. Both these assumption provide simplifications of the final formulas, and the forward modeling problem can essentially be reduced to a 2D case.

Assume that at an initial point s_o on a central ray $\vec{r}_o(s)$ vector $\vec{e}_1(s_o)$ is on the plane $x_2 = 0$, while $\vec{e}_2(s_o)$ is orthogonal to it and directed along the x_2-axis. Then, as $\frac{\partial c}{\partial q_2} = \frac{\partial c}{\partial x_2} \equiv 0$ we obtain from equations (22) that

$\frac{d\vec{e}_2}{ds} \equiv 0$ and therefore $\vec{e}_2(s)$ remains orthogonal to the plane $x_2 = 0$. The equations in variations split in two independent system:

$$\frac{dq_1}{ds} = c_o\, p_1; \quad \frac{dp_1}{ds} = -\frac{1}{c_o^2} \frac{\partial^2 c}{\partial^2 q_1}\bigg|_{q_1=q_2=0} q_1 \tag{42}$$

and

$$\frac{dq_2}{ds} = c_o\, p_2; \quad \frac{dp_2}{ds} = 0. \tag{43}$$

The latter one can be integrated immediately and its general solution reads $q_2(s) = p_2(s_o) \int\limits_{s_o}^{s} c_o\, ds + q_2(s_o)$.

In order to construct the field vectors in zero-order approximation, i.e., formulas (26) for the eikonal and (40) for amplitudes, we may use $X^{(1)}(s)$ and $X^{(2)}(s)$ in the form

$$X^{(1)}(s) = \begin{bmatrix} q_1^{(1)}(s) \\ 0 \\ p_1^{(1)}(s) \\ 0 \end{bmatrix} \quad \text{and} \quad X^{(2)}(s) = \begin{bmatrix} 0 \\ q_2^{(2)}(s) \\ 0 \\ p_2^{(2)}(s) \end{bmatrix}. \tag{44}$$

Obviously, their J-scalar product is equal to zero.

By employing expressions (44) in equations (28) and (29), we easily obtain

$$\Gamma = \begin{bmatrix} \Gamma_{11} & 0 \\ 0 & \Gamma_{22} \end{bmatrix}, \quad \Gamma_{jj} = \frac{p_j^{(j)}(s)}{q_j^{(j)}(s)}, \quad j = 1, 2, \tag{45}$$

and

$$|\det Q(s)| = |q_1^{(1)}(s)|\, |q_2^{(2)}(s)|, \tag{46}$$

where $|q_2^{(2)}(s)|$ is usually called the orthogonal geometrical spreading.

By inserting the above results into equations (41), we obtain one asymptotic solution of Maxwell's equations having the electric vector $\vec{E}$ on the plane $x_2 = 0$, and a second one with the magnetic vector $\vec{H}$ being on that plane and that holds true all along the ray.

5. The reflection/refraction problem on a smooth interface

Suppose a smooth surface S separates two media with the material parameters ϵ_1, μ_1, σ_1 and ϵ_2, μ_2, σ_2 in the first and the second ones, respectively.

Denote by $\vec{n}$ a unit vector orthogonal to the surface, then classical boundary conditions on S in the electrodynamics read:

1. The normal components of $\vec{D}$ and $\vec{B}$ are continuous on S, i.e.,

$$(\vec{D}^{(1)}, \vec{n})\Big|_S = (\vec{D}^{(2)}, \vec{n})\Big|_S \quad \text{and} \quad (\vec{B}^{(1)}, \vec{n})\Big|_S = (\vec{B}^{(2)}, \vec{n})\Big|_S. \tag{47}$$

2. The tangential components of $\vec{E}$ and $\vec{H}$ are continuous on S, i.e.,

$$\vec{n} \times \vec{E}^{(1)}\Big|_S = \vec{n} \times \vec{E}^{(2)}\Big|_S \quad \text{and} \quad \vec{n} \times \vec{H}^{(1)}\Big|_S = \vec{n} \times \vec{H}^{(2)}\Big|_S, \tag{48}$$

where superscript indices 1 and 2 indicate the first and the second medium, respectively.

The aim of this section is to derive relationship among the asymptotic solutions (41) related to the incident, reflected and refracted central rays on the surface S.

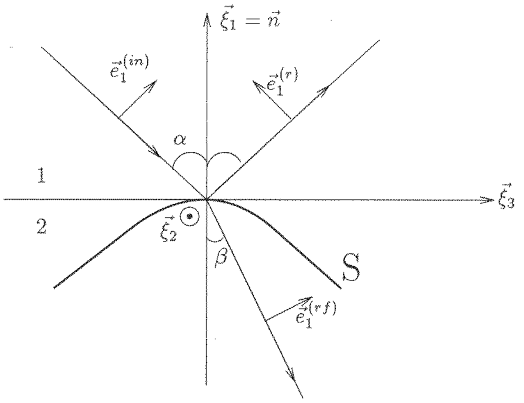

Fig. 1. The incident, reflected and refracted rays, and the local coordinates at the point of incidence.

First consider a 2.5D case where calculations turn out to be simpler. In this case we have to use additional assumption that the surface S is invariant with respect to translations along x_2-axis, i.e., S is a cylindrical surface orthogonal to the plane $x_2 = 0$.

In a vicinity of the point of incidence M we introduce local coordinates ξ_1, ξ_2, ξ_3 in such a way that the unit basis vectors $\vec{\xi_j}$, $j = 1, 2, 3$, form right-orientated system and $\vec{\xi_1} = \vec{n}$, $\vec{\xi_2}$ and $\vec{\xi_3}$ are tangent to S and $\vec{\xi_2}$ is orthogonal to the plane of incidence, see Fig. 1.

Orientation of the vectors $\vec{e}_1$, $\vec{e}_2$ of the ray centered coordinates, related to the incident (in), reflected (r) and refracted (rf) central rays at the point of incidence M, is supposed to be as depicted in Fig. 1.

In algebraic form it yields for the incident ray

$$\begin{cases} \vec{e}_1^{\,(\mathrm{in})}(s_*) = \sin\alpha\,\vec{\xi_1} + \cos\alpha\,\vec{\xi_3}, \\ \\ \vec{e}_2^{\,(\mathrm{in})}(s_*) = \vec{\xi_2} \end{cases} \tag{49}$$

for the reflected ray

$$\begin{cases} \vec{e}_1^{\,(\mathrm{r})}(s_*) = \sin\alpha\,\vec{\xi_1} - \cos\alpha\,\vec{\xi_3}, \\ \\ \vec{e}_2^{\,(\mathrm{r})}(s_*) = \vec{\xi_2}, \end{cases} \tag{50}$$

and for the refracted (or transmitted) ray

$$\begin{cases} \vec{e}_1^{\,(\mathrm{rf})}(s_*) = \sin\beta\,\vec{\xi_1} + \cos\beta\,\vec{\xi_3}. \\ \\ \vec{e}_2^{\,(\mathrm{rf})}(s_*) = \vec{\xi_2}, \end{cases} \tag{51}$$

where α is both the incident and the reflected angle, β is the refracted angle, and $s = s_*$ corresponds to the point of incidence M on the surface.

Consider further that the incident field in the medium 1 is described by the first type solution in (41) with the electric vector $\vec{E}$ being directed along $\vec{e}_1^{\,(\mathrm{in})}$ on the incident central ray. Omitting auxiliary indices of previous section, we write it as follows

$$\begin{aligned} \vec{E}^{(\mathrm{in})} &= \varphi_o^{(\mathrm{in})}\, e^{-i\omega(t-\tau_o^{(\mathrm{in})})}\,\vec{e}_1^{\,(\mathrm{in})}; \\ \\ \vec{H}^{(\mathrm{in})} &= \sqrt{\frac{\epsilon_1}{\mu_1}}\varphi_o^{(\mathrm{in})} e^{-i\,\omega(t-\tau_o^{(\mathrm{in})})}\,\vec{e}_2^{\,(\mathrm{in})}. \end{aligned} \tag{52}$$

In order to satisfy the boundary conditions (47) and (48), we introduce reflected $\vec{E}^{(\mathrm{r})}$, $\vec{H}^{(\mathrm{r})}$ and refracted $\vec{E}^{(\mathrm{rf})}$, $\vec{H}^{(\mathrm{rf})}$ field vectors and seek them in the following form

$$\begin{aligned} \vec{E}^{(\mathrm{r})} &= \varphi_o^{(\mathrm{r})}\, e^{-i\,\omega(t-\tau_o^{(\mathrm{r})})}\, \vec{e}_1^{\,(\mathrm{r})}; \\ \vec{H}^{(\mathrm{r})} &= \sqrt{\frac{\epsilon_1}{\mu_1}}\varphi_o^{(\mathrm{r})}\, e^{-i\,\omega(t-\tau_o^{(\mathrm{r})})}\, \vec{e}_2^{\,(\mathrm{r})} \end{aligned} \tag{53}$$

in first medium, and

$$\begin{aligned} \vec{E}^{(\mathrm{rf})} &= \varphi_o^{(\mathrm{rf})}\, e^{-i\,\omega(t-\tau_o^{(\mathrm{rf})})}\, \vec{e}_1^{\,(\mathrm{rf})}; \\ \vec{H}^{(\mathrm{rf})} &= \sqrt{\frac{\epsilon_2}{\mu_2}\varphi_o^{(\mathrm{rf})}}\, e^{-i\,\omega(t-\tau_o^{(\mathrm{rf})})}\, \vec{e}_2^{\,(\mathrm{rf})} \end{aligned} \tag{54}$$

in the second one.

In medium 1, we have sum of the incident and reflected fields $\vec{E}^{(1)} = \vec{E}^{(\mathrm{in})} + \vec{E}^{(\mathrm{r})}$, $\vec{H}^{(1)} = \vec{H}^{(\mathrm{in})} + \vec{H}^{(\mathrm{r})}$, while in medium 2 only the refracted fields.

By employing formulas (52)–(54) and (49)–(51) to the boundary conditions (47), and bearing in mind that the eikonals must be continuous at M, i.e., $\tau_o^{(\mathrm{in})}(s_*) = \tau_o^{(\mathrm{r})}(s_*) = \tau_o^{(\mathrm{rf})}(s_*)$, we obtain only one equation for $\varphi_o^{(\mathrm{r})}(s_*)$ and $\varphi_o^{(\mathrm{rf})}(s_*)$

$$\epsilon_1 \sin\alpha\, \varphi_o^{(\mathrm{in})}(s_*) + \epsilon_1 \sin\alpha\, \varphi_o^{(\mathrm{r})}(s_*) = \epsilon_2 \sin\beta\, \varphi_o^{(\mathrm{rf})}(s_*). \tag{55}$$

The second equation, coming from the condition on vectors $\vec{B}$, turns out to be already satisfied.

Analogous calculations in the case of boundary conditions (48) yield

$$\cos\alpha\, \varphi_o^{(\mathrm{in})}(s_*) - \cos\alpha\, \varphi_o^{(\mathrm{r})}(s_*) = \cos\beta\, \varphi_o^{(\mathrm{rf})}(s_*) \tag{56}$$

and

$$\sqrt{\frac{\epsilon_1}{\mu_1}}\, \varphi_o^{(\mathrm{in})}(s_*) + \sqrt{\frac{\epsilon_1}{\mu_1}}\, \varphi_o^{(\mathrm{r})}(s_*) = \sqrt{\frac{\epsilon_2}{\mu_2}}\, \varphi_o^{(\mathrm{rf})}(s_*). \tag{57}$$

Note that equations (55)–(57) form a system of three linear equations with respect to only two functions $\varphi_o^{(\mathrm{r})}$ and $\varphi_o^{(\mathrm{rf})}$. From two of them, i.e., (55) and (56), we get

$$\varphi_o^{(\mathrm{r})}(s_*) = R\, \varphi_o^{(\mathrm{in})}(s_*); \quad \varphi_o^{(\mathrm{rf})}(s_*) = T\, \varphi_o^{(\mathrm{in})}(s_*), \tag{58}$$

where

$$
\begin{aligned}
R &= \frac{\epsilon_2 \sin\beta\cos\alpha - \epsilon_1 \sin\alpha\cos\beta}{\epsilon_1 \sin\alpha\cos\beta + \epsilon_2 \sin\beta \cos\alpha}, \\
T &= \frac{\epsilon_1 \sin 2\alpha}{\epsilon_1 \sin\alpha\cos\beta + \epsilon_2 \sin\beta\cos\alpha}.
\end{aligned} \tag{59}
$$

Then, by inserting formulas (58) and (59) into equation (57) we obtain the following relation between the incident and refracted angles

$$
\sin\beta = \sqrt{\frac{\mu_1\,\epsilon_1}{\mu_2\,\epsilon_2}}\,\sin\alpha \tag{60}
$$

which holds true because of Snell's law. Thus, the boundary conditions on surface S give rise to a unique solution for $\varphi_o^{(\mathrm{r})}(s_*)$ and $\varphi_o^{(\mathrm{rf})}(s_*)$ at the incident point M.

Note that formulas (40) and (41) hold for the reflected and refracted rays too and in order to calculate the geometrical spreading $|\det Q^{(\mathrm{r})}|$ and $|\det Q^{(\mathrm{rf})}|$ on these rays we have to prolong the solutions (24) of the equations in variations along the reflected and refracted central rays. This problem was solved by Popov (1977) and Popov and Pšenčik (1978 a,b) for the dynamic ray tracing in elastodynamics. However, as the eikonal equation (9) and therefore Fermat's functional has the same form as in elastodynamics, the results of the above mentioned papers are valid in electrodynamics as well. So, we do not discuss this problem in the paper in detail.

To complete the problem of reflection/refraction on S in the case under consideration, we need to find formulas for the constant of integrations $\psi_o^{(\mathrm{r})}$ and $\psi_o^{(\mathrm{rf})}$ of the transport equation along the reflected and refracted rays, respectively.

For the incident ray the expression for amplitude $\varphi_o^{(\mathrm{in})}(s_*)$ at the point of incidence M comes out from (40)

$$
\varphi_o^{(\mathrm{in})}(s_*) = \sqrt[4]{\frac{\mu_1}{\epsilon_1}}\,\frac{\psi_o^{(\mathrm{in})}}{\sqrt{|\det Q^{\mathrm{in}}(s_*)|}}\,\eta^{(\mathrm{in})}(s_*) \tag{61}
$$

in which all terms are supposed to be known including the constant $\psi_o^{(\mathrm{in})}$.

As for $\varphi_o^{(\mathrm{r})}$ and $\varphi_o^{(\mathrm{rf})}$, we take them in the form

$$\varphi_o^{(\mathrm{r})}(s_*) = \sqrt[4]{\frac{\mu_1}{\epsilon_1}}\,\frac{\psi_o^{(\mathrm{r})}}{\sqrt{|\det Q^{(\mathrm{r})}(s_*)|}};$$
$$\varphi_o^{(\mathrm{rf})}(s_*) = \sqrt[4]{\frac{\mu_2}{\epsilon_2}}\,\frac{\psi_o^{(\mathrm{rf})}}{\sqrt{|\det Q^{(\mathrm{rf})}(s_*)|}} \tag{62}$$

because their attenuation factors $\eta^{(\mathrm{r})} = \eta^{(\mathrm{rf})} = 1$ at s_*. Indeed, M is the initial point for the reflected and refracted rays.

Finally, by inserting formulas (61) and (62) into (58), we obtain desirable relationship among the $\psi_o^{(\mathrm{in})}$, $\psi_o^{(\mathrm{r})}$ and $\psi_o^{(\mathrm{rf})}$:

$$\psi_o^{(\mathrm{r})} = R\,\psi_o^{(\mathrm{in})}\,\eta^{(\mathrm{in})}(s_*);$$
$$\psi_o^{(\mathrm{rf})} = T\,\sqrt[4]{\frac{\mu_1\,\epsilon_2}{\mu_2\,\epsilon_1}}\,\sqrt{\frac{\cos\beta}{\cos\alpha}}\,\psi_o^{(\mathrm{in})}\,\eta^{(\mathrm{in})}(s_*). \tag{63}$$

Note that in equations (63) we used relations between the geometrical spreadings

$$|\det Q^{(\mathrm{in})}(s_*)| = |\det Q^{(\mathrm{r})}(s_*)|$$

and

$$|\det Q^{(\mathrm{rf})}(s_*)| = \frac{\cos\beta}{\cos\alpha}\,|\det Q^{(\mathrm{r})}(s_*)|.$$

In the case of another polarization of the incident wave field (see Eqs. (41)) corresponding calculations are similar and we do not describe them in the paper.

Consider now the reflection/refraction problem in the general case of a 3D medium.

The main difference between 2.5D and the general case consists in the following. The polarization of the field vectors related to the incident ray is fixed at the initial point, say, at the source, and, when arrived at the interface S, it will not coincide, in principle, with the plane of incidence. It means that unit vectors of the ray centered coordinates, related to the ray $\vec{r}_o^{\,(\mathrm{in})}(s)$, at the point M will be rotated on some angle $\Omega^{(\mathrm{in})}$ with respect to their position depicted in Fig. 1. Apparently, the polarization of the reflected and refracted waves will form angles with vectors $\vec{e}_j^{\,(\mathrm{r})}(s_*)$

and $\vec{e}_j^{\,(\mathrm{rf})}(s_*)$, $j = 1, 2$, too. We denote these angles by $\Omega^{(\mathrm{r})}$ and $\Omega^{(\mathrm{rf})}$, respectively. So, instead of equations (49)–(51), we have to develop new ones.

Denote by ${}'\vec{e}_1^{\,(\mathrm{in})}(s)$, ${}'\vec{e}_2^{\,(\mathrm{in})}(s)$ the unit vectors describing the initial polarization of the field vectors along the incident ray. Then at the point $s = s_*$ on S we get

$$\begin{bmatrix} {}'\vec{e}_1^{\,(\mathrm{in})}(s_*) \\ {}'\vec{e}_2^{\,(\mathrm{in})}(s_*) \end{bmatrix} = \begin{bmatrix} \cos\Omega^{(\mathrm{in})} & \sin\Omega^{(\mathrm{in})} \\ -\sin\Omega^{(\mathrm{in})} & \cos\Omega^{(\mathrm{in})} \end{bmatrix} \begin{bmatrix} \vec{e}_1^{\,(\mathrm{in})}(s_*) \\ \vec{e}_2^{\,(\mathrm{in})}(s_*) \end{bmatrix}, \tag{64}$$

where orientation of $\vec{e}_1^{\,(\mathrm{in})}(s_*)$ and $\vec{e}_2^{\,(\mathrm{in})}(s_*)$ is depicted in Fig. 1. So now, instead of equations (49), we obtain the following connecting formulas between ${}'\vec{e}_j^{\,(\mathrm{in})}(s_*)$, $j = 1, 2$, and local coordinates ξ_1, ξ_2, ξ_3, which will be used in the boundary conditions,

$$\begin{cases} {}'\vec{e}_1^{\,(\mathrm{in})}(s_*) = \cos\Omega^{(\mathrm{in})}\,\sin\alpha\,\vec{\xi}_1 + \\ +\sin\Omega^{(\mathrm{in})}\,\vec{\xi}_2 + \cos\Omega^{(\mathrm{in})}\,\cos\alpha\,\vec{\xi}_3 \\ {}'\vec{e}_2^{\,(\mathrm{in})}(s_*) = -\sin\Omega^{(\mathrm{in})}\,\sin\alpha\,\vec{\xi}_1 + \\ +\cos\Omega^{(\mathrm{in})}\,\vec{\xi}_2 - \sin\Omega^{(\mathrm{in})}\,\cos\alpha\,\vec{\xi}_3. \end{cases} \tag{65}$$

Analogously, for the reflected and refracted waves we use the following formulas

$$\begin{cases} {}'\vec{e}_1^{\,(\mathrm{r})}(s_*) = \cos\Omega^{(\mathrm{r})}\,\sin\alpha\,\vec{\xi}_1 + \\ +\sin\Omega^{(\mathrm{r})}\,\vec{\xi}_2 - \cos\Omega^{(\mathrm{r})}\,\cos\alpha\,\vec{\xi}_3 \\ {}'\vec{e}_2^{\,(\mathrm{r})}(s_*) = -\sin\Omega^{(\mathrm{r})}\,\sin\alpha\,\vec{\xi}_1 + \\ +\cos\Omega^{(\mathrm{r})}\,\vec{\xi}_2 + \sin\Omega^{(\mathrm{r})}\,\cos\alpha\,\vec{\xi}_3 \end{cases} \tag{66}$$

and

$$\begin{cases} {}'\vec{e}_1^{\,(\mathrm{rf})}(s_*) = \cos\Omega^{(\mathrm{rf})}\,\sin\beta\,\vec{\xi}_1 + \\ +\sin\Omega^{(\mathrm{rf})}\,\vec{\xi}_2 + \cos\Omega^{(\mathrm{rf})}\,\cos\beta\,\vec{\xi}_3 \\ {}'\vec{e}_2^{\,(\mathrm{rf})}(s_*) = -\sin\Omega^{(\mathrm{rf})}\,\sin\beta\,\vec{\xi}_1 + \\ +\cos\Omega^{(\mathrm{rf})}\,\vec{\xi}_2 - \sin\Omega^{(\mathrm{rf})}\,\cos\beta\,\vec{\xi}_3, \end{cases} \tag{67}$$

where the vectors ${}'\vec{e}_j^{\,(\mathrm{r})}(s_*)$ and ${}'\vec{e}_j^{\,(\mathrm{rf})}(s_*)$, $j = 1, 2$, describe the polarization of the reflected and refracted field vectors at the point $s = s_*$ on S, respectively.

Consider further a special case when the electric vectors $\vec{E}^{\,(\mathrm{in})}$, $\vec{E}^{\,(\mathrm{r})}$ and $\vec{E}^{\,(\mathrm{rf})}$ are oriented along ${}'\vec{e}_1^{\,(\mathrm{in})}$, ${}'\vec{e}_1^{\,(\mathrm{r})}$ and ${}'\vec{e}_1^{\,(\mathrm{rf})}$ (see (41)). The second type of polarization can be studied similarly and we do not discuss it in this paper.

By employing equations (65)–(67) in the boundary conditions (47) and (48) we obtain, after some mathematics, the following system of equations

$$\left\{\begin{array}{l} -\epsilon_1 \cos\Omega^{(\mathrm{r})} \sin\alpha\, {\varphi_o}^{(\mathrm{r})} + \epsilon_2 \cos\Omega^{(\mathrm{rf})} \sin\beta\, {\varphi_o}^{(\mathrm{rf})} = \\ \epsilon_1 \cos\Omega^{(\mathrm{in})} \sin\alpha\, {\varphi_o}^{(\mathrm{in})} \\ \cos\Omega^{(\mathrm{r})} \cos\alpha\, {\varphi_o}^{(\mathrm{r})} + \cos\Omega^{(\mathrm{rf})} \cos\beta\, {\varphi_o}^{(\mathrm{rf})} = \\ \cos\Omega^{(\mathrm{in})} \cos\alpha\, \varphi_o^{(\mathrm{in})} \\ \sqrt{\epsilon_1\mu_1} \sin\Omega^{(\mathrm{r})} \sin\alpha\, {\varphi_o}^{(\mathrm{r})} - \sqrt{\epsilon_2\mu_2} \sin\Omega^{(\mathrm{rf})} \sin\beta\, {\varphi_o}^{(\mathrm{rf})} = \\ -\sqrt{\epsilon_1\mu_1} \sin\Omega^{(\mathrm{in})} \sin\alpha\, \varphi_o^{(\mathrm{in})} \\ \sqrt{\frac{\epsilon_1}{\mu_1}} \sin\Omega^{(\mathrm{r})} \cos\alpha\, {\varphi_o}^{(\mathrm{r})} + \sqrt{\frac{\epsilon_2}{\mu_2}} \sin\Omega^{(\mathrm{rf})} \cos\beta\, {\varphi_o}^{(\mathrm{rf})} = \\ \sqrt{\frac{\epsilon_1}{\mu_1}} \sin\Omega^{(\mathrm{in})} \cos\alpha\, \varphi_o^{(\mathrm{in})} \\ -\sin\Omega^{(\mathrm{r})}\, {\varphi_o}^{(\mathrm{r})} + \sin\Omega^{(\mathrm{rf})}\, {\varphi_o}^{(\mathrm{rf})} = \sin\Omega^{(\mathrm{in})}\, \varphi_o^{(\mathrm{in})} \\ -\sqrt{\frac{\epsilon_1}{\mu_1}} \cos\Omega^{(\mathrm{r})}\, {\varphi_o}^{(\mathrm{r})} + \sqrt{\frac{\epsilon_2}{\mu_2}} \cos\Omega^{(\mathrm{rf})}\, {\varphi_o}^{(\mathrm{rf})} = \\ \sqrt{\frac{\epsilon_1}{\mu_1}} \cos\Omega^{(\mathrm{in})}\, \varphi_o^{(\mathrm{in})}. \end{array}\right. \tag{68}$$

This time we have to solve the system (68) with respect to ${\varphi_o}^{(\mathrm{r})}$, ${\varphi_o}^{(\mathrm{rf})}$ and angles $\Omega^{(\mathrm{r})}$, $\Omega^{(\mathrm{rf})}$, which describe the polarization of the reflected and refracted field vectors $\vec{E_o}^{(\mathrm{r})}$, $\vec{E_o}^{(\mathrm{rf})}$ and $\vec{H_o}^{(\mathrm{r})}$, $\vec{H_o}^{(\mathrm{rf})}$ in accordance with the first terms in (41). To this matter, we treat this system in the following way.

Firstly, we solve some equations from (68) with respect to sine and cosine of the angles $\Omega^{(\mathrm{r})}$, $\Omega^{(\mathrm{rf})}$:

$$\cos\Omega^{(\mathrm{r})} = \frac{\varphi_o^{(\mathrm{in})}}{{\varphi_o}^{(\mathrm{r})}}\, a_\epsilon \cos\Omega^{(\mathrm{in})}; \quad \sin\Omega^{(\mathrm{r})} = \frac{\varphi_o^{(\mathrm{in})}}{{\varphi_o}^{(\mathrm{r})}}\, a_\mu \sin\Omega^{(\mathrm{in})} \tag{69}$$

and

$$\cos\Omega^{(\mathrm{rf})} = \frac{\varphi_o^{(\mathrm{in})}}{{\varphi_o}^{(\mathrm{rf})}}\, b_\epsilon \cos\Omega^{(\mathrm{in})}; \quad \sin\Omega^{(\mathrm{rf})} = \sqrt{\frac{\epsilon_1\,\mu_2}{\epsilon_2\,\mu_1}}\, \frac{\varphi_o^{(\mathrm{in})}}{{\varphi_o}^{(\mathrm{rf})}}\, b_\mu \sin\Omega^{(\mathrm{in})}, \tag{70}$$

where the following additional notations are introduced

$$a_\epsilon = \frac{\epsilon_2 \cos\alpha \sin\beta - \epsilon_1 \sin\alpha \cos\beta}{\epsilon_1 \sin\alpha \cos\beta + \epsilon_2 \cos\alpha \sin\beta};$$
$$a_\mu = \frac{\mu_2 \cos\alpha \sin\beta - \mu_1 \sin\alpha \cos\beta}{\mu_1 \sin\alpha \cos\beta + \mu_2 \cos\alpha \sin\beta} \tag{71}$$

and

$$b_\epsilon = \frac{\epsilon_1 \sin 2\alpha}{\epsilon_1 \sin\alpha \cos\beta + \epsilon_2 \cos\alpha \sin\beta},$$
$$b_\mu = \frac{\mu_1 \sin 2\alpha}{\mu_1 \sin\alpha \cos\beta + \mu_2 \cos\alpha \sin\beta}. \tag{72}$$

Then, by using the trigonometric identity $\sin^2 \alpha + \cos^2 \alpha = 1$, we exclude $\Omega^{(\mathrm{r})}$ and $\Omega^{(\mathrm{rf})}$ in equations (69) and (70)

$$\varphi_o{}^{(\mathrm{r})} = \pm\sqrt{a_\epsilon^2 \cos^2 \Omega^{(\mathrm{in})} + a_\mu^2 \sin^2 \Omega^{(\mathrm{in})}}\ \varphi_o^{(\mathrm{in})} \tag{73}$$

and

$$\varphi_o{}^{(\mathrm{rf})} = \pm\sqrt{b_\epsilon^2 \cos^2 \Omega^{(\mathrm{in})} + \frac{\epsilon_1 \mu_2}{\epsilon_2 \mu_1}\, b_\mu^2 \sin^2 \Omega^{(\mathrm{in})}}\ \varphi_o^{(\mathrm{in})}, \tag{74}$$

where the proper sign, i.e., $+$ or $-$, should be chosen. For that matter, let us come back with formulas (73) and (74) to 2.5D case, where $\Omega^{(\mathrm{in})} = 0$ and therefore $\varphi_o{}^{(\mathrm{r})} = \pm a_\epsilon\, \varphi_o^{(\mathrm{in})}$ and $\varphi_o{}^{(\mathrm{rf})} = \pm b_\epsilon\, \varphi_o^{(\mathrm{in})}$. By inserting the latter formulas in equation (57), we observe that (57) is satisfied only for the sign $+$. Thus, based on the fact that the reflection/refraction phenomenon must be continuous with respect to angle $\Omega^{(\mathrm{in})}$, we find the correct sign "+" in both equations (73) and (74). In their turn, formulas (73) and (74) give rise to the desired relationship among the constants of integration $\psi_o^{(\mathrm{in})}$, $\psi_o{}^{(\mathrm{r})}$ and $\psi_o{}^{(\mathrm{rf})}$. Indeed, by using expression (61) for $\varphi_o^{(\mathrm{in})}$ and by taking into account the relationship among the geometrical spreadings on S, we obtain

$$\psi_o{}^{(\mathrm{r})} = +\sqrt{a_\epsilon^2 \cos^2 \Omega^{(\mathrm{in})} + a_\mu^2 \sin^2 \Omega^{(\mathrm{in})}}\, \eta^{(\mathrm{in})}(s_*)\, \psi_o^{(\mathrm{in})} \tag{75}$$

and

$$\psi_o{}^{(\mathrm{rf})} = \tag{76}$$

$$+\sqrt{\frac{\cos\beta}{\cos\alpha}}\,\sqrt[4]{\frac{\mu_1\,\epsilon_2}{\mu_2\,\epsilon_1}}\,\sqrt{b_\epsilon^2\,\cos^2\,\Omega^{(\mathrm{in})}+\frac{\epsilon_1\,\mu_2}{\epsilon_2\,\mu_1}\,b_\mu^2\,\sin^2\,\Omega^{(\mathrm{in})}}\;\eta^{\,(\mathrm{in})}(s_*)\,\psi_o^{(\mathrm{in})}.$$

Summarizing the calculations above described, consider in more detail the algorithm for extension of the field vectors along the reflected/refracted ray when the incident field vectors are known at the incident point M.

Clearly, formulas (40) and (41) hold for an arbitrary ray, i.e., the incident, reflected or refracted one, if the proper values of the material parameters ϵ, μ and σ of a medium are employed.

In order to use them for the reflected/refracted ray, we need to know: i) the initial orientation of the polarization vectors $\vec{e}_1(s_*)$ and $\vec{e}_2(s_*)$ at the point of incidence $s = s_*$ on surface S, ii) the value of the integration constant ψ_o in the transport equation (40).

By inserting equations (73) and (74), taken with upper sign $+$, into (69) and (70) and by solving the latter ones on angles $\Omega^{(\mathrm{r})}$ and $\Omega^{(\mathrm{rf})}$, we obtain information about the polarization plane for the reflected and refracted rays, respectively.

The initial data for the polarization vectors $'\vec{e}_j^{\,(\mathrm{r})}(s_*)$ and $'\vec{e}_j^{\,(\mathrm{rf})}(s_*)$, $j = 1, 2$, then come out from formulas (66) and (67).

As to the integration constants $\psi_o{}^{(\mathrm{r})}$ and $\psi_o{}^{(\mathrm{rf})}$, they are given by expressions (75) and (76).

Concluding this section, let us consider the case of normal incidence where the concept of the incident plane has no sense. In this case, an uncertainty appears in formulas for a_ϵ, a_μ, b_ϵ and b_μ (see equations (71), (72)), and in formulas for R and T in (59). To resolve the uncertainty, let us assume the incident α and refracted β angles are small but not equal to zero, i.e., we have nearly vertical incidence.

Under this assumption we have $\sin\alpha \simeq \alpha$, $\sin\beta \simeq \beta$ and $\cos\alpha \simeq \cos\beta \simeq 1$, what results in the following formulas

$$\begin{aligned} a_\epsilon &= \frac{\sqrt{\mu_1\,\epsilon_1\,\epsilon_2}-\epsilon_1\,\sqrt{\mu_2}}{\sqrt{\mu_1\,\epsilon_1\,\epsilon_2}+\epsilon_1\,\sqrt{\mu_2}} = R, \\ a_\mu &= \frac{\sqrt{\mu_1\,\mu_2\,\epsilon_1}-\mu_1\,\sqrt{\epsilon_2}}{\sqrt{\mu_1\,\mu_2\,\epsilon_1}+\mu_1\,\sqrt{\epsilon_2}} \end{aligned} \tag{77}$$

and

$$
\begin{aligned}
b_\epsilon &= \frac{2\,\epsilon_1\,\sqrt{\mu_2}}{\sqrt{\mu_1\,\epsilon_1\,\epsilon_2} + \epsilon_1\,\sqrt{\mu_2}} = T, \\
b_\mu &= \frac{2\,\mu_1\,\sqrt{\epsilon_2}}{\sqrt{\mu_1\,\mu_2\,\epsilon_1} + \mu_1\,\sqrt{\epsilon_2}}.
\end{aligned} \tag{78}
$$

The latter formulas can be used for vertical and nearly vertical incidence. In this case $\Omega^{(\mathrm{in})} = 0$ and since both a_ϵ and a_μ in (77) have no uncertainty we get from the second equation (69) that $\sin\Omega^{(\mathrm{r})} = 0$. At the same time on the right-hand side of the first equation (69) we obtain that $\cos\Omega^{(\mathrm{r})} = 1$. This yields $\Omega^{(\mathrm{r})} = 0$. Similarly, one can verify that $\Omega^{(\mathrm{rf})} = 0$ too.

Conclusions.

Based on the above results we conclude that the dynamic ray tracing for Maxwell's equations, when directly performed for the field vectors $\vec{E}$ and $\vec{H}$ or $\vec{D}$ and $\vec{B}$ in the ray centered coordinates, insignificantly differs from one in the case of a scalar wave field. It seems that the only additional trouble comes out from the requirement for the unit vectors $\vec{e}_1$ and $\vec{e}_2$ to be adjusted to the polarization of the field vectors.

We would like also to note that the formulas derived in this paper make a technical basis for the use of the Gaussian Beam method for modeling in the case of Maxwell's equations. We plan to discuss this problem in detail in further publications.

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