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V. A. Arzumanian

OPERATOR ALGEBRAS ASSOCIATED WITH POLYMORPHISMS

Abstract. A short survey of the problems and developments in the theory of operator algebras associated with semi-group dynamical systems is presented. The main part is an exposition of results concerning the algebras generated by multivalued transformations (polymorphisms).

0. Introduction. Operator algebras are among the most important objects of noncommutative ("quantum") mathematics. One of the stimuli is their applications in the algebraic approach to the foundations of quantum field theory and quantum statistical mechanics.

The standard way to obtain new algebras from simple components is the crossed product of an algebra and a group of its automorphisms. The replacement of the group action (one-to-one case) by a non-invertible one is a promising generalization at least by two reasons. Becoming more complicated, the theory, however, delivers remarkable examples in the simplest situations. Furthermore, real physical processes are usually non-invertible and must be described by the dynamical systems that reflect this feature.

In this note we represent a short survey of the problems and developments in the realization of the program of constructing the operator algebras associated with non-invertible mappings.

1. Endomorphisms (many-to-one case). The first step is the "semigroup measure" construction (by analogy with the group measure construction of W. Krieger [10]), which was presented by A. Vershik and the author in [1]. A von Neumann algebra $\mathcal{W}(X,\mu,T)$ was constructed starting with an endomorphism T with an invariant measure μ (i.e., the semigroup $\mathbb{Z}_+$, or, more precisely, the bicyclic semigroup – taking into account the adjoints – is considered). This algebra is a factor if and only if the endomorphism is ergodic. Thus it must be generated by an appropriately chosen group of automorphisms. The main point is the existence of a canonical conditional expectation to the diagonal (maximal commutative, or *Cartan*, in terminology of A.Vershik [13]) subalgebra through some auxiliary (noncommutative) subalgebra $\mathcal{W}_\infty(X,\mu,T)$ that plays the role of the diagonal subalgebra and is reduced to the latter if T is in fact invertible. Since the adjoints are not geometrical objects, the involvement of trajectory theory becomes a necessary tool: $\mathcal{W}(X,\mu,T)$ is isomorphic to the von Neumann algebra obtained by the Feldman-Moore construction [**9**] starting with a measurable equivalence relation corresponding to the trajectory partition. Moreover, the algebra $\mathcal{W}_\infty(X,\mu,T)$ is isomorphic to the von Neumann algebra obtained in the same way starting with an equivalence relation corresponding to the so-called *tail* partition ($x \sim y \Leftrightarrow T^n x = T^n y;\ \ x, y \in X,\ \ n \in \mathbb{Z}_+$). Hyperfiniteness is a consequence of Vershik's theorem: the trajectory partition of an endomorphism is *tame* (i.e., it can be approximated by a decreasing sequence of partitions with finite elements).

The situation with the type of the corresponding factors is the following. If an endomorphism with invariant (finite) measure is not in fact an automorphism (in which case the type is always II_1), the obtained factors are "wild" (i.e., of type III). All types III_λ, $\lambda \in]0,1]$, can be realized already in the case of a unilateral discrete Bernoulli shift for an appropriate choice of probability distribution on the generator. Let us call the *algebraic type* of an (ergodic) endomorphism a type of the corresponding factor, which is an invariant of the trajectory isomorphism. It can be shown that the type does not depend on entropy: there exists a pair of Bernoulli shifts with finite (but different) number of states (or a pair of Markov shifts with two states) such that they have the same entropy but different algebraic types (see [3], where the quasi-invariant case was also presented).

Another problem is to extend the constructions to topological dynamical systems. The main difference (and difficulty) here is that the adjoint operator does not represent a geometrical object. There are many ways to avoid this difficulty. An attempt of the construction was presented in the paper by A. Vershik and the author [2]. As appropriate tools, one can use Kumjian's localizations [11], Renault's groupoids [12] (i.e., see the paper by V.Deaconu [**6**]). Another point of view was developed by R. Exel [7]. In particular, in the work by R. Exel and A. Vershik [8], the algebras described above were investigated in detail (including such problems as faithfulness of covariant representation and simplicity).

Note that there is an evident relation between the above-mentioned constructions and the Cuntz algebras $\mathcal{O}_n$ [**5**]. Namely, the algebra $\mathcal{W}(X,\mu,T)$, where T is a unilateral Bernoulli shift with uniformly distributed n states, is the faithful representation of the algebra $\mathcal{O}_n$. Moreover, one can indicate in $\mathcal{O}_n$ another system of generators that adequately reflects the relations between the generators of the multiplier algebra and a single isometry.

3. Polymorphisms (many-to-many case). The following natural step is a generalization of these constructions to the case of multy-valued transformations. The question was first mentioned in [1] and directly posed to the author by Prof. A. Vershik.

Note firstly that the case "one to many" is the same as "many to one" since, as it will be seen later, the corresponding polymorphism is adjoint to an endomorphism (called *exomorphism*), and so it generates the same semigroup of isometries.

The following is Vershik's concept of the multi-valued mapping (polymorphism, see [14]) with insignificant simplifications sufficient for the main goal of the note. Let (X, μ) be a Lebesgue (standard measure) space with finite measure, and let π_1 and π_2 be the coordinate projections of $X \times X$ onto the first and the second factors. Polymorphism $\Pi = \Pi(\nu)$, where ν is a measure on $X \times X$, is the diagram $X \xleftarrow{\pi_1} (X \times X) \xrightarrow{\pi_2} X$ such that $\pi_i(\nu) = \mu$; $i = 1, 2$. Certainly, the polymorphism is completely determined by the corresponding bistochastic measure ν. The adducing definition gives to the notion a geometrical sense as the graph of the multi-valued mapping.

If X is finite, then the bistochastic measure is determined by a multiple of a so-called stochastic matrix. Another important example, which is recommended to keep in mind, is the following. Let $\mathbb{S}^1$ be the unit circle with the normalized Lebesgue measure. The polymorphism $\Pi_{n,m}$ is defined by the following relation: $x^n = y^m$; $x, y \in \mathbb{S}^1$; $n, m \in \mathbb{Z}_+$, with uniform measures on the preimages. The trajectory partition of this action can be treated as a winding of the unit torus. A composition (product) $\Pi_2\Pi_1$ of two polymorphisms $\Pi_1 = \Pi(\nu_1), \Pi_2 = \Pi(\nu_2)$ is determined by the unique measure ν on $X \times X$ such that the conditional measures ν^x are defined for a.a. $x \in X$ by the formula $\nu^x(E) = \int \nu_2^y(E) d\nu_1(y)$ for any measurable $E \subset X$. The adjoint $\Pi^* = \Pi(\nu^*)$ to the polymorphism $\Pi = \Pi(\nu)$ is determined by the measure $\nu^*(E \times F) = \nu(F \times E)$ for measurable E and F from X. Then, the set of all polymorphisms forms a $*$-semigroup with identity I (determined by the measure $\nu(E \times F) = \nu(E \cap F)$) and zero element Θ (determined by the measure $\nu = \mu \times \mu$)). An endomorphism T can be considered as a polymorphism Π_T with the measure $\nu(E \times F) = \mu(E \cap T^{-1}F)$. Thus, $\Pi_T \Pi_T^* = I$, and automorphisms stand out by the relation $\Pi_T^* = \Pi_{T^{-1}}$. Now, properly, the construction. Let $\Pi = \Pi(\nu)$ be a polymorphism and let $(U_\Pi f)(x) = \int f(y) d\nu^x(y)$ be the operator acting on $L_2(X, \mu)$. Then, clearly, $U_{\Pi_1 \Pi_2} = U_{\Pi_2} U_{\Pi_1}$, $U_{\Pi^*} = U_\Pi^*$. A polymorphism Π is called ergodic if $U_\Pi \chi^E = \chi^E$ implies $\mu(E) = 0$ or $\mu(E) = 1$, where χ^E is the indicator of $E \subset X$. Let $\mathcal{M}$ be the algebra of multipliers on $L^2(X, \mu)$ by functions from $L^\infty(X, \mu)$, and let $\mathcal{A} = \mathcal{C}(X, \mu, \Pi)$ be the uniformly closed $*$-algebra of operators on $L^2(X, \nu)$ generated by $\mathcal{M}$ and $U = U_\Pi$. This algebra can be graduated by $\mathbb{Z}$ (omitting the details: this corresponds to the relation of degrees of U and U^* in the generators). The subalgebra $\mathcal{C}_\infty(X, \mu, \Pi)$ that contains $\mathcal{M}$ and is generated by operators in which the sum of degrees of U and U^* are equal, plays the role of the "Cartan subalgebra" ("coefficients" in the graduation) and is reduced to $\mathcal{M}$ if Π is in fact an automorphism.

The important class of polymorphisms which permits to obtain advanced results is formed by the polymorphisms $\Pi = \Pi_S^* \Pi_T$ generated by a commuting pair of endomorphisms T and S, that is, the products of an exomorphism and an endomorphism. For simplicity, we suppose that all "preimages" are finite (conditional measures on the layers are discrete and finite). If $\Pi = \Pi_S^* \Pi_T$ is ergodic, then there exists on $\mathcal{A}$ a canonical conditional expectation $\mathcal{P}$ to the subalgebra $\mathcal{M}$. Note that the ergodicity of T and S imply the ergodicity of Π.

Thus, the regular representation $\pi = \pi_\varphi$ of the algebra $\mathcal{A}$ can be defined as the GNS-representation corresponding to the state $\varphi(a) = \int \mathcal{P}(a)(x) d\mu(x)$. Finally, the algebra $\mathcal{W}(X, \mu, \Pi)$ is the week closure of $\pi(\mathcal{A})$.

Theorem. *For an ergodic polymorphism $\Pi = \Pi_S^* \Pi_T$ the algebra $\mathcal{W}(X, \mu, \Pi)$ is a hyperfinite factor.*

Note that there exists a measurable equivalence relation R on $X \times X$ (generated by $x \sim y \Leftrightarrow Tx = Sy$) such that $\mathcal{W}(X, \mu, \Pi)$ is spatially isomorphic to the algebra $W(R)$ obtained by the Feldman–Moore construction. The relation R is ergodic if both T and S are ergodic. Since the polymorphism Π is a product of a commuting pair of endomorphism and exomorphism, the obtained algebra is hyperfinite.

Finally, about the type of the factor. It can be calculated using the modular function generated by the corresponding conditional measures. For example, if $\Pi = \Pi_{n,m}$, where n, m are mutually simple, $m > n$, then $\mathcal{W}(X, \nu, \Pi)$ is $\mathrm{III}_{n/m}$-factor. Thus the corresponding C^*-preimage is something like the fractal Cuntz algebra $\mathcal{O}_{n/m}$.

A variant of topological approach is presented in the paper by J. Renault and the author [4].

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Institute of Mathematics,
National Academy of Sciences
of Armenia

E-mail: vicar@instmath.sci.am;
victorar@mtu-net.ru

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