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# AMENABLE ACTIONS OF NONAMENABLE GROUPS

Abstract. We give two ways of constructing amenable (in the sense of Greenleaf) actions of nonamenable groups. In the first part of the paper we construct a class of faithful transitive amenable actions of the free group using Schreier graphs. In the second part we show that every finitely generated residually finite group can be embedded into a bigger residually finite group, which acts level-transitively on a locally finite rooted tree, so that the induced action on the boundary of the tree is amenable on every orbit.

## 1. Introduction

Since 1929, when von Neumann [25] introduced the notion of an invariant mean on a group (and, more generally, on a $G$-set) there is a permanent interest in the study of the phenomenon known as amenability. Amenable objects like groups, semigroups, algebras, graphs, metric spaces, operator algebras, etc. play an important role in various areas of mathematics. A big progress in understanding of the structure of the class of amenable groups and in the study of asymptotic characteristics of them like growth of Følner sets (the notion introduced by A. M. Vershik in [24], drift, entropy, etc. was reached in the past two decades [24, 12, 17, 13, 4, 3, 6, 5, 2].

An important role in the propaganda of the idea of amenability is played by, perhaps the best introductory source to the subject of amenable groups, the book by Greenleaf [10] where the following question is formulated.

**Q1.** Let $X$ be a $G$-set and there is an invariant mean for the pair $(G, X)$. Does this imply that the group $G$ is amenable?

Here one must add some extra conditions in order to avoid immediate negative answer to the question. Namely, one has to assume that the group $G$ acts faithfully (otherwise the pair $(F_m, F_m/N)$ would be a trivial counterexample where $F_m$ is a free group of rank $m \geqslant 2$, $N \triangleleft F_m$ is a

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normal subgroup such that the quotient $F_m/N$ is amenable and $F_m$ acts on $F_m/N$ in the standard way). The second reasonable assumption is the transitivity of the action of $G$ on $X$. Otherwise one can take $X$ equal to a union of $G$-orbits and then the existence of an invariant mean for $(G, X)$ would follow from the existence of an invariant mean for any pair $(G, Gx)$, where $x \in X$. Of course, the action of $G$ on orbits can be nonfaithful even in the case where it is faithful on $X$, but certainly a transitive amenable pair $(G, X)$ with nonamenable $G$ can be viewed as a more interesting example giving the negative solution of the above question. So we reformulate the question of Greenleaf as follows.

**Q2.** Let a group $G$ act transitively and faithfully on a set $X$. If the pair $(G, X)$ is amenable (i.e., there is a $G$-invariant mean on $X$), does this imply the amenability of $G$?

Surprisingly, the question of Greenleaf did not attract attention of a large community of mathematicians, although it was solved (in the negative) in [23]. But recently the interest to this question came back and a number of new constructions are on the way to print. This is stimulated, in particular, by the observation made in [19] that coamenability of subgroups (a subgroup $H < G$ is coamenable if a pair $(G, G/H)$ is amenable) behaves differently from coamenability of normal subgroups.

We are going to present in this note two constructions of amenable actions of nonamenable groups. In the first construction, $G$ is a free noncommutative group and the action $(G, X)$ is viewed as a Schreier graph of $G$, so that amenability of the action is converted to amenability of the graph.

In the second example we use the methods of the theory of groups acting on rooted trees developed in [22, 14, 1, 9]. We start with an arbitrary nonamenable residually finite group $G$, realize it as a group acting on a spherically homogeneous rooted tree $T$ and then extend it to a group $\widetilde{G}$ (also acting on a rooted tree) in a way which guarantees the amenability of the pair $(\widetilde{G}, \widetilde{G}/P)$ where $P$ is a parabolic subgroup (i.e., the stabilizer of a point of the boundary $\partial T$).

The next question naturally arises as a part of our investigation.

For which nonamenable groups $G$ there is a faithful transitive and amenable action $(G, X)$? (I.e., there is a coamenable subgroup $H < G$ whose core

$$\bigcap_{g \in G} g^{-1} H g$$

is trivial). Let us call such groups NAA groups. Observe that groups with Kazhdan $T$-property are not NAA groups.

As far as we know, Y. Glasner and N. Monod [8] have other constructions of amenable pairs $(G, X)$ with nonamenable $G$. It would be interesting to get more on such constructions.

## 2. The first construction

In our first construction the group $G$ is the free group $F_m$ of rank $m \geqslant 2$, the set $X$ is the set $F_m/H$ of cosets $gH$, $g \in F_m$, where $H < F_m$ is a coamenable subgroup with trivial core (so that the left action $(F_m, F_m/H)$ is amenable, faithful and transitive).

The group $H$ will be constructed in a combinatorial-geometric way via the construction of a $2m$-regular amenable graph (with some extra properties) which will be converted to a Schreier graph $\Gamma = \Gamma(F_m, H, S)$ where $S = \{a_1, \dots, a_m\}$ is a free set of generators of $F_m$.

Recall that the set of vertices of the graph $\Gamma$ is identified with the set of left cosets $gH$, $g \in F_m$, and two "vertices" $gH$ and $hH$ are connected by an oriented edge labelled by $s$ if $gH = shH$, where $s \in S \cup S^{-1}$. Obviously, the degree of each vertex of this graph is $2m$. The amenability of the pair $(F_m, H)$ is equivalent to the amenability of the graph $\Gamma$, which can be defined as the existence of a sequence $\{F_n\}$ of finite subsets of $\Gamma$ with the property that $|\partial F_n|/|F_n| \to 0$ as $n \to \infty$, where $\partial F_n$ is the boundary of $F_n$ (for amenability of graphs, see [4]). One of the properties that insure amenability of a graph is subexponentiality of the growth [4] (which means that the number of vertices in $\Gamma$ of combinatorial distance $\leqslant n$ from a distinguished vertex $v_0$ grows slower than exponential functions). Another criterion of co-amenability of subgroups is given in [11]. It says that a subgroup $H$ of the free group $F_m$ of rank $m$ is coamenable if and only if the growth exponent of $H$ in $F_m$ (with respect to the standard generating set of $F_m$) is equal to $2m - 1$.

It is known (see Section IV.15 in [15] and [18]) that every $2m$-regular (nonoriented and without labelling of edges) graph $\Delta$ can be converted to a Schreier graph of a free group $F_m$ by putting an orientation on the edges and labelling of the edges by the elements of the set $S \cup S^{-1}$. Therefore, any example of a $2m$-regular graph of subexponential growth leads to a construction of an amenable pair $(F_m, F_m/H)$. A free generating set of the subgroup $H < F_m$ can be found in the following way.

Construct a spanning subtree $T$ in $\Delta$ and take the set $E_0$ of the edges of $\Delta$ that do not belong to $T$. For each $e \in E_0$, let $t_e$ be the path $peq$

where $p$ is the geodesic path in $T$ joining the initial vertex $v_0$ of $\Delta$ with the beginning of the edge $e$ and $q$ is the geodesic path in $T$ joining the endpoint of $e$ with $v_0$. Let $w_e$ be the word read along the path $t_e$. Then $\{w_e, e \in E_0\}$ is a free set of generators of $H$.

There are plenty of $2m$-regular graphs of subexponential (even polynomial) growth. The problem is only in getting the faithfulness of the action of $F_m$ on $F_m/H$, i.e., in showing that the core

$$\bigcap_{g \in F_m} g^{-1}Hg \tag{1}$$

is trivial. The last step in our construction is to show how to construct the graph $\Delta$ which guarantees the triviality of the core (1).

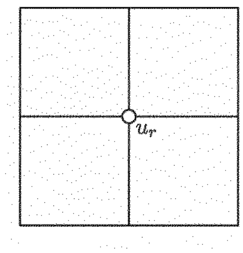


Fig. 1.

A word $w$ over the alphabet $S \cup S^{-1}$ represents an element of $H$ if and only if the path $l_w$ in $\Delta$ starting at the vertex $v_0$ and determined by the word $w$ is closed. If we change the reference vertex $v_0$ by a vertex $u_0$, then we must replace the group $H$ by its conjugate $g^{-1}Hg$, where $g$ is an element given by a word that can be read on any path joining $v_0$ with $u_0$. Thus, if we construct a graph $\Delta$ with the property that for any nonempty freely reduced word $w$ there is a vertex $u$ of the graph such that the path beginning at $u$ and determined by the word $w$ is not closed, then the core of $H$ will be trivial. This property is satisfied if for any positive integer $r$ there is a vertex $u_r$ of the graph such that the length of any back-trackless loop in $\Delta$ beginning at $u_r$ is greater than $r$ (i.e., the neighborhood of $u_r$ in $\Delta$ of radius $r$ is a tree).

Construction of a $2m$-regular graph that satisfies all the listed properties is easy. Start with an $m$-dimensional grid $\mathbb{Z}^m = \Delta_0$, where $m > 1$,

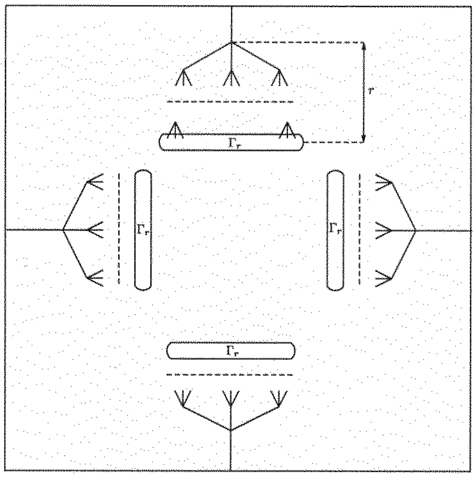


Fig. 2.

and make a sequence of local surgeries in it by replacement at the $r$-th step the 1-neighborhood of a vertex $u_r$ of $\Delta_0$ (see Figure 2) by the graph shown next in Figure 2, where $\Gamma_r$ is any $(2m-1)$-regular graph with $2m(2m-1)^{r-1}$ vertices.

The graph $\Delta_0$ has polynomial growth of degree $2m$. It is clear that if we choose the sequence $\{u_r\}_{r=1}^{\infty}$ of vertices in $\Delta_0$ such that the distance $d_r$ of $u_r$ from the origin 0 of $\Delta_0 = \mathbb{Z}^m$ is growing very fast, then the graph $\Delta$ obtained from $\Delta_0$ by such local reconstructions will have polynomial growth (and hence will be amenable) and the core of the corresponding group $H$ will be trivial.

## 3. Automorphism groups of rooted trees

Let $\mathsf{X} = (X_1, X_2, \dots)$ be a sequence of finite sets and let $\mathsf{X}^*$ denote the set of words $x_1 x_2 \dots x_n$, where $x_i \in X_i$, together with the empty word $\varnothing$. Set

$$\mathsf{X}^n = \{x_1 \dots x_n \ : \ x_i \in X_i\} = X_1 \times \cdots \times X_n$$

and $\mathsf{X}^0 = \{\varnothing\}$. Then $\mathsf{X}^* = \bigsqcup_{n \geqslant 0} \mathsf{X}^n$.

We can transform $\mathsf{X}^*$ into a rooted tree in a natural way: connect every vertex $v \in \mathsf{X}^n$ to the vertices of the form $vx$ for $x \in X_{n+1}$. The empty word $\varnothing$ is the root of the tree. The tree $\mathsf{X}^*$ is called the *spherically-homogeneous tree* of the spherical index $(|X_1|, |X_2|, \dots)$. The spherical index determines the tree uniquely up to an isomorphism of rooted trees.

Set $\mathsf{X}_n = (X_{n+1}, X_{n+2}, \dots)$. The spherically homogeneous tree $\mathsf{X}^*$ is called *regular* if its spherical index is constant. In this case we may assume that the sequence $\mathsf{X} = (X_1, X_2, \dots)$ is also constant. In this case $\mathsf{X}_n = \mathsf{X} = (X, X, \dots)$.

The *boundary* $\mathsf{X}^\omega$ of the tree $\mathsf{X}^*$ is identified with the set of the infinite words of the form $x_1x_2\dots$, where $x_i \in X_i$. The disjoint union $\mathsf{X}^\omega \sqcup \mathsf{X}^*$ has a natural topology defined by the base consisting of the *cylindrical sets*

$$v\mathsf{X}^\omega_{|v|} \sqcup v\mathsf{X}^*_{|v|}$$

of words starting with a given finite word $v$. Here $|v|$ denotes the length of the word $v$, i.e., $v \in \mathsf{X}^{|v|}$. The topological space $\mathsf{X}^\omega \sqcup \mathsf{X}^*$ is compact and totally disconnected. The subspace $\mathsf{X}^\omega$ is homeomorphic to the Cantor set and the topology on it is the direct product topology of the finite discrete sets $X_i$. The subset $\mathsf{X}^*$ is discrete and dense in $\mathsf{X}^\omega \sqcup \mathsf{X}^*$.

The boundary $\mathsf{X}^\omega$ also has a natural measure (that we call *Bernoulli measure*) equal to the direct product of the uniform probability measures on the sets $X_i$. It is the unique measure invariant under the action of the full automorphism group of $\mathsf{X}^*$.

We are interested in groups acting faithfully on the tree $\mathsf{X}^*$ by automorphisms. Every such an action extends in a unique way to an action by homeomorphisms on $\mathsf{X}^\omega \sqcup \mathsf{X}^*$. The obtained action on $\mathsf{X}^\omega$ is measure-preserving.

An action of $G$ on $\mathsf{X}^*$ is said to be *level-transitive* if it is transitive on every level $\mathsf{X}^n$ of the tree $\mathsf{X}^*$. The action is level-transitive if and only if the induced action on $\mathsf{X}^\omega$ is minimal, i.e., has dense orbits (see [9]). In particular, if an action of $G$ on $\mathsf{X}^*$ is faithful and level-transitive, then the restriction of the action to any $G$-orbit of $\mathsf{X}^\omega$ is also faithful.

An action is level-transitive if and only if it is ergodic with respect to the Bernoulli measure on $\mathsf{X}^\omega$.

If $g$ is an automorphism of $\mathsf{X}^*$ and $v \in \mathsf{X}^n$ is a word, then the *restriction* $g|_v$ is the automorphism of the tree $\mathsf{X}^*_n$ defined by the condition that

$$(vw)^h = v^h w^{g|_v}$$

for all $w \in \mathsf{X}^*_n$. It is easy to see that $g|_v$ is uniquely defined and is an automorphism of $\mathsf{X}^*_n$.

If the sequence $\mathsf{X}$ is constant (and hence $\mathsf{X}_n = \mathsf{X}$ for all $n$), then an automorphism $g$ of $\mathsf{X}^*$ is said to be *finite-state* if the set $\{g|_v : v \in \mathsf{X}^*\}$ is finite. The set of all finite-state automorphisms is a countable subgroup of the automorphism group of $\mathsf{X}^*$, is called the *group of finite automata* and is denoted $\mathcal{F}(\mathsf{X})$.

Given an automorphism $g$ of the tree $\mathsf{X}^*$, we say that an infinite word $x_1x_2\ldots \in \mathsf{X}^\omega$ is *$g$-rigid* if there exists $n$ such that $g|_{x_1\ldots x_n}$ is trivial. The set of $g$-rigid points is obviously open.

We say that an automorphism $g$ of $\mathsf{X}^*$ is *almost finitary* if the set of $g$-rigid points of $\mathsf{X}^\omega$ has full measure.

**Proposition.** *The set of almost finitary automorphisms of $\mathsf{X}^*$ is a group.*

**Proof.** We have the following obvious properties of restrictions:

$$g|_{v_1v_2} = g|_{v_1}|_{v_2}, \quad (g_1g_2)|_v = (g_1|_v)(g_2|_{v^{g_1}}), \quad (g^{-1})|_v = \left(g|_{v^{g^{-1}}}\right)^{-1}.$$

Suppose that $g$ is almost finitary. Then for almost every sequence $x_1x_2\ldots \in \mathsf{X}^\omega$ there exists $n$ such that $g|_{(x_1\ldots x_n)^{g^{-1}}} = 1$, since $g^{-1}$ is measure-preserving. But then

$$g^{-1}|_{x_1\ldots x_n} = \left(g|_{(x_1\ldots x_n)^{g^{-1}}}\right)^{-1} = 1,$$

which proves that $g^{-1}$ is almost finitary.

Suppose now that $g_1, g_2$ are almost finitary. Then for almost every $x_1x_2\ldots \in \mathsf{X}^\omega$ there exists $n$ such that $g_1|_{x_1\ldots x_n} = 1$ and $g_2|_{(x_1\ldots x_n)^{g_1}} = 1$, since $g_1$ is measure-preserving. Then we have

$$(g_1g_2)|_{x_1\ldots x_n} = g_1|_{x_1\ldots x_n} \cdot g_2|_{(x_1\ldots x_n)_1^g} = 1,$$

i.e., $g_1g_2$ is also almost finitary. □

Denote by $\mathcal{A}(\mathsf{X})$ the group of almost finitary automorphisms of $\mathsf{X}^*$. We have the following examples of almost finitary automorphisms.

**Proposition.** *Let $\mathsf{X}$ be constant. An element $g \in \mathcal{F}(\mathsf{X})$ is almost finitary if for every $v \in \mathsf{X}^*$ there exists $u \in \mathsf{X}^*$ such that $g|_{vu} = 1$.*

**Proof.** Let $\{g_1, g_2, \ldots, g_n\} = \{g|_v : v \in X^*\}$ be the set of the states of $g$. There exists $v_1$ such that $g_1|_{v_1} = 1$. There exists $v_2$ such that $g_2|_{v_1v_2} = 1$, and further, by induction, there exists a sequence $v_1, v_2, \ldots, v_n$ of words such that $g_i|_{v_1v_2\ldots v_i} = 1$ for $i = 1, \ldots, n$. Then for every $i = 1, \ldots, n$ we have

$$g_i|_{v_1v_2\ldots v_n} = g_i|_{v_1v_2\ldots v_i}|_{v_{i+1}\ldots v_n} = 1.$$

We conclude that for every word containing the word $w = v_1 v_2 \ldots v_n$, i.e., for every word of the form $u_1 w u_2$, we have

$$g|_{u_1 w u_2} = g|_{u_1}|_w|_{u_2} = g_i|_w|_{u_2} = 1|_{u_2} = 1$$

for some $i$.

Thus, if an infinite word $x_1 x_2 \ldots \in \mathsf{X}^\omega$ contains $w = v_1 v_2 \ldots v_n$ as a subword, then $g|_{x_1 \ldots x_n} = 1$ for some $n$, i.e., $x_1 x_2 \ldots$ is $g$-rigid. But it is obvious that the set of infinite words containing a given finite word $w$ has full measure. $\square$

Let $g$ be a homeomorphism of a compact topological space $\mathcal{X}$. A point $\xi \in \mathcal{X}$ is said to be *$g$-regular* if either $\xi^g \neq \xi$, or $g$ fixes pointwise a neighborhood of $\xi$. If $G$ is a homeomorphism group of $\mathcal{X}$, then a point $\xi \in \mathcal{X}$ is said to be $G$-regular if it is $g$-regular for every $g \in G$.

Suppose that $G$ is a countable homeomorphism group of a compact space $\mathcal{X}$. One can prove the following properties of $G$-regular points (see [9] and [21].

1. The set of $G$-regular points is co-meager, i.e., is an intersection of a countable set of open dense sets.
2. Suppose that $G$ is generated by a finite generating set $S$ and that the action is minimal on $\mathcal{X}$, i.e., that every $G$-orbit is dense. Then for every $G$-regular point $\xi$ the Schreier graph $\Gamma(G, G_\xi, S)$ is locally contained in the Schreier graph $\Gamma(G, G_\zeta, S)$ for every $\zeta \in \mathcal{X}$.

Here $G_\zeta$ denotes the stabilzier of $\zeta$ in $G$. A graph $\Gamma_1$ is *locally contained* in a graph $\Gamma_2$ if for every vertex $v_1$ of $\Gamma_1$ and every $R \in \mathbb{N}$ there exists a vertex $v_2$ of $\Gamma_2$ such that the ball in $\Gamma_1$ of radius $R$ with center in $v_1$ is isomorphic as a labelled graph with the ball in $\Gamma_2$ of radius $R$ with center in $v_2$. The balls are viewed as subgraphs of $\Gamma_i$ with the induced graph structure. It is easy to see that if a graph $\Gamma$ locally contains an infinite amenable graph, then $\Gamma$ is amenable.

Note that if $g|_{x_1 x_2 \ldots x_n}$ is trivial, then all the points of the cylindrical set $x_1 x_2 \ldots x_n \mathsf{X}_n^\omega$ are $g$-regular. Consequently, every $g$-rigid point of $\mathsf{X}^\omega$ is $g$-regular. In particular, if $G$ is a countable subgroup of $\mathcal{A}(\mathsf{X})$, then almost every point of $\mathsf{X}^\omega$ is $G$-regular.

**Theorem.** *If a finitely-generated group $G \leqslant \mathcal{A}(\mathsf{X})$ is level-transitive, then for every point $\xi \in X^\omega$ the $G$-space $G/G_\xi$ is amenable.*

**Proof.** Almost every $G$-orbit on $\mathsf{X}^\omega$ consists of $G$-rigid sequences. Hence, for almost every $w \in \mathsf{X}^\omega$ and for every $g \in G$ the sequences $w$ and $w^g$ are *co-final*, i.e., are of the form $w = v_1 w'$ and $w^g = v_2 w'$, where $v_1, v_2 \in \mathsf{X}^n$ for some $n$ and $w' \in \mathsf{X}_n^*$.

The co-finality equivalence relation is *hyperfinite*, i.e., is a union of an increasing sequence of measurable equivalence relations with finite equivalence classes. Namely, the co-finality relation is equal to $\bigcup_{n\geqslant 1} E_n$, where $E_n$ is the equivalence relation consisting of the pairs $(w_1, w_2) \in \mathsf{X}^\omega \times \mathsf{X}^\omega$ such that $w_1 = v_1 w'$ and $w_2 = v_2 w'$ for some $w' \in \mathsf{X}_n^\omega$ and $v_1, v_2 \in \mathsf{X}^n$.

We see, therefore, that the $G$-orbit equivalence relation (the equivalence relation whose equivalence classes are $G$-orbits) is a sub-relation of the hyperfinite co-finality relation, up to sets of measure zero. The group $G$ acts by measure-preserving transformations on $\mathsf{X}^\omega$, hence, by Theorem 1 of [16], the Schreier graphs $\Gamma(G, G_w, S)$ are amenable for almost every $w \in \mathsf{X}^\omega$. Almost every point of $\mathsf{X}^\omega$ is $G$-regular. Hence, there exists a $G$-regular point $w$ such that the Schreier graph $\Gamma(G, G_w, S)$ is amenable. But the graph $\Gamma(G, G_w, S)$ is locally contained in the graph $\Gamma(G, G_\xi, S)$ for any $\xi \in \mathsf{X}^\omega$. Consequently, the space $G/G_\xi$ is amenable for any $\xi$.

## 4. Second construction

Let us show how the last theorem can be used to construct amenable actions of non-amenable groups. This construction was inspired by a "tree-wreathing" construction of S. Sidki.

Let $G$ be any finitely-generated residually finite non-amenable group. It is known that it acts faithfully on a spherically homogeneous rooted tree $\mathsf{X}^*$. Take an additional letter $\$ \notin X_i$ and consider a new sequence $\mathsf{Y} = (X_1 \cup \{\$\}, X_2 \cup \{\$\}, \dots)$. The tree $\mathsf{X}^*$ is in a natural way a sub-tree of the tree $\mathsf{Y}^*$. We can extend the action of $G$ on $\mathsf{X}^*$ to an action on $\mathsf{Y}^*$ in the following way. Suppose that $w \in \mathsf{Y}^*$ is arbitrary. If $w \in \mathsf{Y}^*$ does not belong to $\mathsf{X}^*$, then it can be uniquely written in the form $w = v\$u$ for $v \in \mathsf{X}^*$ and $u \in \mathsf{Y}_{|v|+1}^*$. Then we set

$$w^g = v^g \$ u$$

for all $g \in G$. It is easy to see that we thus get an action of $G$ on $\mathsf{Y}^*$, which extends the original action on $\mathsf{X}^*$, and thus is also faithful. Note also that (in the case of a constant sequence $\mathsf{X}$) the obtained action is finite-state if and only if the original action on $\mathsf{X}^*$ is finite-state.

The obtained action of $G$ on $\mathsf{Y}^*$ is an action by almost finitary automorphisms since every sequence $y_1 y_2 \ldots \in \mathsf{Y}^\omega$ containing the letter $\$$ is

$G$-rigid. Hence, we get an embedding of $G$ into $\mathcal{A}(\mathsf{Y})$. However, the action of $G$ is not level-transitive on $\mathsf{Y}^*$.

But it is easy to embed $G < \mathcal{A}(\mathsf{Y})$ into a level-transitive subgroup of $\mathcal{A}(\mathsf{Y})$. It is sufficient to take, for instance, any level-transitive finitely-generated subgroup $H < \mathcal{A}(\mathsf{Y})$ and consider $F = \langle G, H\rangle$. Then $F$ is a non-amenable subgroup of $\mathcal{A}(\mathsf{Y})$, and thus, by the above theorem, the $F$-space $F/F_w$ is amenable for every $w \in \mathsf{Y}^\omega$.

Probably, the simplest example of the group $H$ is the infinite cyclic group generated by the *adding machine*. We identify the alphabets $Y_i$ with the sets $\{0, 1, \dots d_i - 1\}$, where $(d_1, d_2, \dots)$ is the spherical index of $\mathsf{Y}^*$, and define the *adding machines* $a_n$ acting on $\mathsf{Y}_n^*$ by the recurrent rule

$$(iw)^{a_n} = \begin{cases} (i+1)w & \text{for} \quad i = 0, 1, \dots, d_{n+1} - 2 \\ 0w^{a_{n+1}} & \text{for} \quad i = d_{n+1} - 1. \end{cases}$$

Then $a_0$ is an automorphism of $\mathsf{Y}^*$ generating a level-transitive cyclic group.

An explicit construction can be done in the following way. Consider the constant sequence $\mathsf{X} = (X, X, \dots)$, where $X = \{0, 1\}$, and define three automorphisms $a, b, c$ of $\mathsf{X}^*$ by the inductive rules

$$\begin{aligned} (0w)^a &= 1\left(w^b\right) \qquad & (1w)^a &= 0\left(w^b\right) \\ (0w)^b &= 0\left(w^a\right) \qquad & (1w)^b &= 1\left(w^c\right) \\ (0w)^c &= 0\left(w^c\right) \qquad & (1w)^c &= 1\left(w^a\right). \end{aligned}$$

It is known that the group generated by the automorphisms $a, b, c$ is isomorphic to the free product $C_2 * C_2 * C_2$ of three groups of order 2, and thus is non-amenable. The definition of the transformations $a, b$, and $c$ implies that they are finite-state. This example of a three-state automaton was found by our students E. Muntyan and D. Savchuk. A proof that it generates the free product $C_2 * C_2 * C_2$ can be found in [20].

The above construction gives the following non-amenable group with amenable Schreier graphs. It is the group $G$ generated by four transformations $a, b, c, d$ acting on the tree defined over the alphabet $Y = \{0, 1, 2\}$ and satisfying the recursions

$$\begin{aligned} (0w)^a &= 1\left(w^b\right) \qquad & (1w)^a &= 0\left(w^b\right) \qquad & (2w)^a &= 2w \\ (0w)^b &= 0\left(w^a\right) \qquad & (1w)^b &= 1\left(w^c\right) \qquad & (2w)^b &= 2w \\ (0w)^c &= 0\left(w^c\right) \qquad & (1w)^c &= 1\left(w^a\right) \qquad & (2w)^c &= 2w \\ (0w)^d &= 1w \qquad & (1w)^d &= 2w \qquad & (2w)^d &= 0\left(w^d\right). \end{aligned}$$

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