Записки научных
семинаров ПОМИ
Том 326, 2005 г.

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PSEUDO–SELF–AFFINE TILINGS IN $\mathbb{R}^d$

Abstract. It is proved that every pseudo-self-affine tiling in $\mathbb{R}^d$ is mutually locally derivable with a self-affine tiling. A characterization of pseudo-self-similar tilings in terms of derived Voronoï tessellations is a corollary. Previously, these results were obtained in the planar case, jointly with Priebe Frank. The new approach is based on the theory of graph-directed iterated function systems and substitution Delone sets developed by Lagarias and Wang.

1. Introduction

Self-affine tilings have been much studied, see [17, 14] and the references therein. They arise, in particular, in connection with Markov partitions of toral automorphisms and as models for quasicrystals. A *self-affine tiling* has the property that, if we expand it by a certain expanding linear map, then the original tiling may be obtained by subdividing the expanded tiles according to a prescribed rule.

A *pseudo-self-affine tiling* has a less rigid hierarchical structure: if the entire tiling is expanded by a certain linear map, then one can recover the original tiling at any point by looking in a finite "window" around that point in the expanded tiling. If the expanding linear map is a similitude, we say that the tiling is *pseudo-self-similar*. A famous example of a pseudo-self-similar tiling is the Penrose tiling with rhombi tiles.

Pseudo-self-affine tilings were introduced by N. Priebe Frank [9, 10]. E. A. Robinson, Jr. conjectured that such tilings are *mutually locally derivable* from self-affine tilings (precise definitions are given in the next section). In paper [11], joint with N. Priebe Frank, we settled the conjecture for pseudo-self-similar tilings in $\mathbb{R}^2$ and used it to complete the characterization of pseudo-self-similar tilings in $\mathbb{R}^2$ started in [10].

Here we establish the conjecture in full generality. The only caveat is that the tiles of the resulting self-affine tiling need not be connected (in [11] the tiles were topological disks); however, having such a tiling

Supported in part by NSF grant #DMS-0355187.

is sufficient for many purposes; in particular, it yields a characterization of pseudo-self-similar tilings in $\mathbb{R}^d$. The approach is different: whereas in [11] we used the method of "redrawing the boundary," here we obtain the prototiles as an attractor of a graph-directed iterated function system, applying the results of Lagarias and Wang [5].

The work of A. M. Vershik and co-authors on arithmetic constructions of Markov and sofic partitions [18, 4, 16] has many points of contact with the theory of self-affine tilings; it played an important role in the development of the subject. I am happy to dedicate this paper to Anatoly Moiseevich Vershik, with gratitude and affection.

2. Preliminaries

We begin with tiling preliminaries, following [11] and [6]. See also [14] for a recent survey.

We fix a set of types (or colors) labeled by $\{1, \dots, m\}$. A *tile* in $\mathbb{R}^d$ is defined as a pair $T = (A, i)$ where $A = \mathrm{supp}(T)$ (the support of T) is a compact set in $\mathbb{R}^d$ which is the closure of its interior, and $i = \ell(T) \in \{1, \dots, m\}$ is the type of T. We emphasize that the tiles are not assumed to be homeomorphic to the ball or even connected. A *tiling* of $\mathbb{R}^d$ is a set $\mathcal{T}$ of tiles such that $\mathbb{R}^d = \bigcup\{\mathrm{supp}(T) : T \in \mathcal{T}\}$ and distinct tiles have disjoint interiors.

A $\mathcal{T}$*-patch* P is a finite subset of the tiling $\mathcal{T}$. Denote by $\mathcal{T}^*$ the set of all $\mathcal{T}$-patches. The *support of a patch* P is defined by $\mathrm{supp}(P) = \bigcup\{\mathrm{supp}(T) : T \in P\}$. The *diameter of a patch* P is $\mathrm{diam}(P) = \mathrm{diam}(\mathrm{supp}(P))$. The *translate* of a tile $T = (A, i)$ by a vector $g \in \mathbb{R}^d$ is $T + g = (A + g, i)$. The translate of a patch P is $P + g = \{T + g : T \in P\}$. We say that two patches P_1, P_2 are *translationally equivalent* if $P_2 = P_1 + g$ for some $g \in \mathbb{R}^d$.

A *patch with a marked tile* is a pair (P, T) where P is a patch and $T \in P$. Two patches with marked tiles (P_1, T_1) and (P_2, T_2) are said to be translationally equivalent if $P_2 = P_1 + g$ and $T_2 = T_1 + g$ for some $g \in \mathbb{R}^d$.

Given an invertible linear map ψ in $\mathbb{R}^d$ and a tiling $\mathcal{T}$, we can consider a new tiling $\psi\mathcal{T} = \{(\psi(\mathrm{supp}(T)), \ell(T)) : T \in \mathcal{T}\}$.

For a tile T and $\Lambda \subset \mathbb{R}^d$ we denote

$$T + \Lambda := \{T + g : g \in \Lambda\}.$$

We always assume that:

- any two $\mathcal{T}$-tiles with the same type (color) are translationally equivalent. (Hence there are finitely many $\mathcal{T}$-tiles up to translation.)

- the tiling $\mathcal{T}$ has *finite local complexity* (FLC), that is, for any $R > 0$ there are finitely many $\mathcal{T}$-patches of diameter less than R up to translation equivalence.

Given a tiling $\mathcal{T}$, we can choose one tile for each translation equivalence class, thereby obtaining a *prototile set* $\{T_1, \dots, T_m\}$ (note that $T_i \in \mathcal{T}$ by assumption). Then we can write

$$\mathcal{T} = \bigcup_{i=1}^{m} (T_i + \Lambda_i) \tag{2.1}$$

for some uniformly discrete sets Λ_i.

Notation. We fix a metric in $\mathbb{R}^d$ (equivalent to the Euclidean metric), and write, for $F \subset \mathbb{R}^d$:

$$N_R(F) = \{x \in \mathbb{R}^d : \ \mathrm{dist}(x, F) \leqslant R\}.$$

Thus $N_R(x) = N_R(\{x\})$ is the closed ball of radius R centered at x.

Definition 2.1. *A tiling $\mathcal{T}$ is called repetitive if for any patch $P \subset \mathcal{T}$ there is a real number $R > 0$ such that for any $x \in \mathbb{R}^d$ there is a $\mathcal{T}$-patch P' such that* $\mathrm{supp}(P') \subset N_R(x)$ *and P' is a translate of P. The minimal such R, denoted $R(P)$, is called the repetitivity radius of P.*

If the tiling $\mathcal{T}$ is repetitive, then the sets Λ_i in (2.1) are both uniformly discrete and relatively dense. Such sets are called *Delone sets*.

Tiling Dynamical Systems. Although in this paper we do not deal with dynamical systems directly, they provide a useful framework for the concepts and results. We recall the set-up briefly.

The *tiling space* is $X_{\mathcal{T}} = \overline{\{-g + \mathcal{T} : g \in \mathbb{R}^d\}}$, where $X_{\mathcal{T}}$ carries a well-known tiling topology. It is based on the idea that two tilings are close if after a small translation they agree on a large ball around the origin. The reader is referred to [14] for details. The space $X_{\mathcal{T}}$ is compact [15] (see also [14]), and the group $\mathbb{R}^d$ acts on it continuously by translations, so that we get a *tiling dynamical system* associated with $\mathcal{T}$. It is well-known (see [14]) that $\mathcal{T}$ is repetitive if and only if the tiling dynamical system $(X_{\mathcal{T}}, \mathbb{R}^d)$ is minimal, that is, its every orbit is dense.

Next we define local derivability, which is the key concept in this paper. First, some more notation: to a subset $F \subset \mathbb{R}^d$ and a tiling $\mathcal{T}$ we associate a $\mathcal{T}$-patch as follows:

$$[F]^{\mathcal{T}} = \{T \in \mathcal{T} : \operatorname{supp}(T) \cap F \neq \emptyset\}.$$

We write $[x]^{\mathcal{T}} = [\{x\}]^{\mathcal{T}}$ for notational convenience.

Definition 2.2 (See [2].**).** *Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two tilings. We say that $\mathcal{T}_2$ is locally derivable (LD) from $\mathcal{T}_1$ with radius $R > 0$ if for all $x, y \in \mathbb{R}^d$,*

$$[N_R(x)]^{\mathcal{T}_1} = [N_R(y)]^{\mathcal{T}_1} + (x - y) \;\Rightarrow\; [x]^{\mathcal{T}_2} = [y]^{\mathcal{T}_2} + (x - y). \tag{2.2}$$

We write $\mathcal{T}_1 \xrightarrow{\mathrm{LD}} \mathcal{T}_2$ to denote that $\mathcal{T}_2$ is LD from $\mathcal{T}_1$. If $\mathcal{T}_1 \xrightarrow{\mathrm{LD}} \mathcal{T}_2$ and $\mathcal{T}_2 \xrightarrow{\mathrm{LD}} \mathcal{T}_1$, then we say that $\mathcal{T}_1$ and $\mathcal{T}_2$ are mutually locally derivable (MLD).

Remark. It is clear that MLD is an equivalence relation. If $\mathcal{T}_1 \xrightarrow{\mathrm{LD}} \mathcal{T}_2$, then there is a factor map of the corresponding topological dynamical systems, see [PF]. It is the tiling analog of a finite block code in symbolic dynamics. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are MLD, then the associated tiling dynamical systems are topologically conjugate, but the converse is not true [8, 12].

Let $\phi : \mathbb{R}^d \to \mathbb{R}^d$ be an expanding linear map, that is, all its eigenvalues are greater than 1 in modulus. Then there is a norm $|\cdot|$ and $\lambda > 1$ such that

$$|\phi g| \geqslant \lambda |g|, \quad g \in \mathbb{R}^d. \tag{2.3}$$

We fix such a norm, and use the corresponding metric, whenever we have an expanding linear map. (Alternatively, we can pass from ϕ to ϕ^ℓ for appropriate $\ell \in \mathbb{N}$, and use the Euclidean norm.)

Definition 2.3. *Let $\phi : \mathbb{R}^d \to \mathbb{R}^d$ be an expanding linear map. A repetitive FLC tiling $\mathcal{T}$ is called a pseudo-self-affine tiling with expansion ϕ if $\phi\mathcal{T} \xrightarrow{\mathrm{LD}} \mathcal{T}$.*

A repetitive FLC tiling $\mathcal{T}$ is called a self-affine tiling with expansion ϕ if

(i) *for any tile $T = (A, i) \in \mathcal{T}$, there is a $\mathcal{T}$-patch $\omega(T)$ such that $\operatorname{supp}(\omega(T)) = \phi A$;*

(ii) *for any tile T and $g \in \mathbb{R}^d$,*

$$(T \in \mathcal{T},\ T + g \in \mathcal{T}) \;\Rightarrow\; \omega(T + g) = \omega(T) + \phi g.$$

It is easy to see that a self-affine tiling is pseudo-self-affine. Also note that the property of being pseudo-self-affine is preserved under MLD.

3. New results

Theorem 3.1. *Let $\mathcal{T}$ be a pseudo-self-affine tiling of $\mathbb{R}^d$ with expansion ϕ. Then for any $k \in \mathbb{N}$ sufficiently large, there exists a tiling $\mathcal{T}'$ which is self-affine with expansion ϕ^k, such that $\mathcal{T}$ is MLD with $\mathcal{T}'$.*

It is perhaps possible to make sure that the tiles of $\mathcal{T}'$ are connected and even homeomorphic to a ball, but this would require additional work.

Theorem 3.1 implies that the dynamical systems corresponding to pseudo-self-affine tilings have the same properties as those corresponding to self-affine tilings. In particular, they are uniquely ergodic, not strongly mixing; there are sufficient conditions for weak mixing, etc., see [17].

Remark. One can consider tilings that are not translationally finite (but have FLC with respect to a larger group, e.g., with respect to all orientation-preserving Euclidean isometries), such as the "pinwheel tiling" and its relatives, see [3] and references there. Recently, B. Rand [13] generalized the planar result of [11] to this setting. It would be interesting to know if this can also be done in higher dimensions.

A pseudo-self-affine tiling with expansion ϕ is said to be *pseudo-self-similar* if ϕ is a similitude, i.e., if $|\phi(x) - \phi(y)| = \lambda |x - y|$ for all $x, y \in \mathbb{R}^d$ for some $\lambda > 1$. Theorem 3.1 allows us to complete the characterization of pseudo-self-affine tilings in $\mathbb{R}^d$ for $d \geqslant 3$ (the planar case was done in [11]).

Definition 3.2 (**See** [10]). *Suppose that $\mathcal{T}$ is a repetitive tiling of $\mathbb{R}^d$. Let $r > 0$, $P_r = [B_r(0)]^{\mathcal{T}}$ and create the locator set*

$$\mathcal{L}_r = \{q \in \mathbb{R}^d \quad \text{such that there exists} \quad P \subset \mathcal{T} \quad \text{with} \quad P_r = P - q\}.$$

Let $R(r)$ be the repetitivity radius of P_r so that every ball of radius $R(r)$ in $\mathcal{T}$ contains a translate of P_r. The derived Voronoï tiling $\mathcal{T}_r$ has a tile t_q for each $q \in \mathcal{L}_r$ with support

$$\operatorname{supp}(t_q) = \{x \in \mathbb{R}^d : \ |q - x| \leqslant |q' - x| \text{ for all } q' \in \mathcal{L}_r\};$$

t_q is labeled by the translational equivalence class of the patch $[B_{2R(r)}(q)]^{\mathcal{T}}$. The derived Voronoï family is defined by $\mathcal{F}(\mathcal{T}) = \{\mathcal{T}_r : r > 0\}$.

Given an expanding similitude $\phi : \mathbb{R}^d \to \mathbb{R}^d$, we say a family of tilings $\mathcal{F}$ is ϕ-finite if there exist $\mathcal{S}_1, \dots, \mathcal{S}_M$ in $\mathcal{F}$ so that for any $\mathcal{T} \in \mathcal{F}$, there is an $i \in \{1, \dots, M\}$ and $j \in \mathbb{Z}^+$ with $\mathcal{T} = \phi^j \mathcal{S}_i$. (Here we identify two *tilings if they are equal up to a one-to-one correspondence between the label sets.)*

Theorem 3.3. *A non-periodic, repetitive tiling of $\mathbb{R}^d$ is pseudo-self-similar if and only if its derived Voronoï family is ψ-finite for an expanding, orientation-preserving similitude ψ.*

The reader is referred to [11, Sect. 6] where the proof is given in the 2-dimensional case. In fact, the theorem about a pseudo-self-similar tiling being MLD to a self-similar one was the only place where dimension 2 was used in [11]. In view of Theorem 3.1, the result now transfers to $\mathbb{R}^d$ for arbitrary d.

4. Proof of Theorem 3.1.

The main difference with [11] is that there we applied a map to "redraw the boundaries" of tiles, whereas here we apply a map to the tiles themselves. The prototiles of the self-affine tiling will be obtained as attractors of a graph-directed iterated function system (a solution of a system of set equations). The proof then proceeds via the theory of substitution Delone sets developed by Lagarias and Wang [5].

4.1. Reduction. We are going to "recode" the tiling keeping the supports unchanged, but increasing the number of labels (this is similar to the "higher block presentation" in Symbolic Dynamics, see [7]. The new label of a $\mathcal{T}$-tile T will be the equivalence class of the patch $[N_L(\mathrm{supp}(T))]^{\mathcal{T}}$ with the marked tile T, for some $L > 0$. This allows us to prove the following statement.

Proposition 4.1. *Let $\mathcal{S}$ be a pseudo-self-affine tiling with expansion ϕ. Then there exists a tiling $\mathcal{T}$ which is MLD with $\mathcal{S}$, has the same tile supports as $\mathcal{S}$ (differs only in labels), such that for any $T, T' \in \mathcal{T}$, $g \in \mathbb{R}^d$, for any $k \geqslant 1$,*

$$T' = T + g \ \Rightarrow\ [\mathrm{supp}(T')]^{\phi^{-k}\mathcal{T}} = [\mathrm{supp}(T)]^{\phi^{-k}\mathcal{T}} + g. \qquad (4.1)$$

The proof of the proposition is straightforward but lengthy, so we postpone it to the next section. In view of the proposition, we can assume that the pseudo-self-affine tiling $\mathcal{T}$ in Theorem 3.1 satisfies (4.1).

Let $d_M = d_M(\mathcal{T}) = \sup\{\mathrm{diam}(\mathrm{supp}(T)) : T \in \mathcal{T}\}$, which is finite by finite local complexity. Similarly, we can find $\eta = \eta(\mathcal{T}) > 0$ such that every tile support contains a closed ball of radius η in its interior. Recall that $\lambda > 1$ is the lower bound on the expansion under the linear map ϕ. Fix $k \in \mathbb{N}$ such that

$$\lambda^k > 2 + \eta^{-1} d_M. \tag{4.2}$$

This will be the k in Theorem 3.1, and we fix it for the rest of the proof.

Substitution map. We are going to define a map

$$f : \mathcal{T} \to (\phi^{-k}\mathcal{T})^*$$

with the following properties:

(S1)

$$(T \in \mathcal{T},\ T + g \in \mathcal{T}) \ \Rightarrow\ f(T+g) = f(T) + g;$$

(S2) $\{f(T) :\ T \in \mathcal{T}\}$ is a tiling of $\mathbb{R}^d$, that is, the $\phi^{-k}\mathcal{T}$-patches $f(T)$, for $T \in \mathcal{T}$, have supports with disjoint interiors and whose union is all of $\mathbb{R}^d$;

(S3) $T, S \in \mathcal{T},\ \phi^{-k}S \in f(T) \ \Rightarrow\ \mathrm{supp}(\phi^{-k}S) \cap \mathrm{supp}(T) \neq \emptyset$;

(S4) $T, S \in \mathcal{T},\ \mathrm{supp}(\phi^{-k}S) \subset \mathrm{int}(\mathrm{supp}(T)) \ \Rightarrow\ \phi^{-k}S \in f(T)$.

Observe that the maximal diameter of $\phi^{-k}\mathcal{T}$-tiles is not greater than $\lambda^{-k}d_M$, hence (4.2) and (S4) will guarantee that $f(T)$ is non-empty.

Let $\partial\mathcal{T} = \bigcup\{\partial(\mathrm{supp}(T)) :\ T \in \mathcal{T}\}$. Note that $\partial\mathcal{T}$ is nowhere dense since each tile support is the closure of its interior. Let $\{T_1, \dots, T_m\}$ be a prototile set for $\mathcal{T}$. Then $\mathcal{T} = \bigcup_{i=1}^m (T_i + \Lambda_i)$ for some Delone sets Λ_i. For each T_i choose a "reference point"

$$c(T_i) \in \mathrm{int}(\mathrm{supp}(T_i)) \setminus \bigcup_{g\in\Lambda_i} (\phi^k(\partial\mathcal{T}) - g), \tag{4.3}$$

which is possible by the Baire Category Theorem. Then define $c(T_i+g) = c(T_i) + g$ for $g \in \Lambda_i$ and let $c(\mathcal{T}) = \{c(T) :\ T \in \mathcal{T}\}$. Now for $T \in \mathcal{T}$ we define

$$f(T) = \{\phi^{-k}S :\ \phi^{-k}c(S) \in \mathrm{int}(\mathrm{supp}(T))\}.$$

In words, $f(T)$ is the patch of $\phi^{-k}\mathcal{T}$-tiles whose reference points lie in the interior of $\mathrm{supp}(T)$. Observe that $\phi^{-k}c(\mathcal{T}) \cap \partial\mathcal{T} = \emptyset$ by (4.3), so f is well-defined. Conditions (S2)–(S4) hold by construction. Since $f(T)$ depends only on the $\phi^{-k}\mathcal{T}$ patch of tiles intersecting T, the condition (S1) holds by (4.1). Thus, we have defined the desired map $f :\ \mathcal{T} \to (\phi^{-k}\mathcal{T})^*$.

Iterating $\phi^k f$. The substitution map f, defined above, naturally extends to a map (also denoted by f) from $\mathcal{T}^*$ to $(\phi^{-k}\mathcal{T})^*$ as follows: for a patch $P \subset \mathcal{T}$ let

$$f(P) = \bigcup\{f(T) :\ T \in P\}.$$

The properties (S3) and (S4) of the map f imply the analog of (S4) for patches:

$$P \subset \mathcal{T},\ S \in \mathcal{T},\ \mathrm{supp}(\phi^{-k}S) \subset \mathrm{int}(\mathrm{supp}(P)) \ \Rightarrow\ \phi^{-k}S \in f(P). \quad (4.4)$$

Note that $\phi^k f$ maps $\mathcal{T}^*$ to itself, so we can iterate it and obtain the maps $(\phi^k f)^n : \mathcal{T}^* \to \mathcal{T}^*$.

Now the rough idea for the rest of the proof is that iterating $\phi^k f$ and rescaling we obtain a self-affine tiling in the limit. More precisely, one can show that $\mathrm{supp}(\phi^{-kn}(\phi^k f)^n(T))$ converges to a compact set A' in the Hausdorff metric, for any tile $T \in \mathcal{T}$. Then one can prove that A' is the closure of its interior, and we may consider the tile $T' = (A', \ell(T))$. It turns out that $\{T' :\ T \in \mathcal{T}\}$ is the desired self-affine tiling. There are a number of technical obstacles on this route, and we are going to proceed indirectly, utilizing the theory of substitution Delone sets developed by Lagarias and Wang, with an extra step coming from [6].

We will need the following simple fact, which is immediate from the definitions: for any tiling $\mathcal{S}$, compact set F, and invertible linear map ψ,

$$\psi\left([F]^{\mathcal{S}}\right) = [\psi F]^{\psi\mathcal{S}}. \quad (4.5)$$

Lemma 4.2. *Let $R \geqslant \eta$ and $x \in \mathbb{R}^d$. Then*

$$(\phi^k f)([N_R(x)]^{\mathcal{T}}) \supset [N_{2R}(\phi^k x)]^{\mathcal{T}}.$$

Proof. Observe that the maximal diameter of a $\phi^{-k}\mathcal{T}$-tile is at most $d_M \lambda^{-k}$. Thus, by (4.4),

$$f([N_R(x)]^{\mathcal{T}}) \supset [N_{R-d_M\lambda^{-k}}(x)]^{\phi^{-k}\mathcal{T}}.$$

Then by (4.5),

$$(\phi^k f)([N_R(x)]^{\mathcal{T}}) \supset [\phi^k N_{R-d_M\lambda^{-k}}(x)]^{\mathcal{T}} \supset [N_{\lambda^k R-d_M}(\phi^k x)]^{\mathcal{T}}.$$

Now the desired statement follows since $\lambda^k R - d_M > 2R$ for $R \geqslant \eta$ by (4.2). □

Substitution Delone set family. For each prototile T_j, we have a $\phi^{-k}\mathcal{T}$ patch $f(T_j)$. Thus, we can write

$$f(T_j) = \bigcup_{i \leqslant m} (\phi^{-k}T_i + \phi^{-k}D_{ij}), \quad (4.6)$$

where $D_{ij} \subset \Lambda_i$ is a finite set. We have by (S2), (S1), and (4.6),

$$\begin{aligned}\bigcup_{i\leqslant m}(\phi^{-k}T_i+\phi^{-k}\Lambda_i)=\phi^{-k}\mathcal{T}&=\bigcup_{T\in\mathcal{T}}f(T)\\&=\bigcup_{j\leqslant m}(f(T_j)+\Lambda_j)\\&=\bigcup_{j\leqslant m}\Big(\bigcup_{i\leqslant m}(\phi^{-k}T_i+\phi^{-k}D_{ij})+\Lambda_j\Big)\\&=\bigcup_{i\leqslant m}\Big(\phi^{-k}T_i+\bigcup_{j\leqslant m}(\Lambda_j+\phi^{-k}D_{ij})\Big),\end{aligned}$$

where all the unions are essentially disjoint, that is, the tiles and patches in the unions have disjoint support interiors. It follows that

$$\Lambda_i=\bigcup_{j\leqslant m}(\phi^k\Lambda_j+D_{ij}),\quad i=1,\dots,m, \tag{4.7}$$

where the unions are disjoint.

Now we need some definitions from [7, 5, 6]. A *multiset*[1] in $\mathbb{R}^d$ is a subset $\mathbf{X}=X_1\times\cdots\times X_m\subset(\mathbb{R}^d)^m$ where $X_i\subset\mathbb{R}^d$. We also write $\mathbf{X}=(X_1,\dots,X_m)=(X_i)_{i\leqslant m}$. Although $\mathbf{X}$ is a product of sets, it is convenient to think of it as a set with types or colors, i being the color of points in X_i. A *Delone multiset* is a multiset $\mathbf{X}=(X_i)_{i\leqslant m}$ where each X_i is a Delone set.

Consider the following mapping on multisets:

$$\Phi\left((X_i)_{i\leqslant m}\right)=\left(\bigcup_{j\leqslant m}(\phi^kX_j+D_{ij})\right)_{i\leqslant m}. \tag{4.8}$$

Then the Delone multiset $\mathbf{\Lambda}:=(\Lambda_i)_{i\leqslant m}$ is a fixed point of Φ by (4.7), that is, $\Phi(\mathbf{\Lambda})=\mathbf{\Lambda}$. We say that $\mathbf{\Lambda}$ is a *substitution Delone multiset* (see [6, Def. 3.4]). The *substitution matrix* is defined by $\mathrm{S}:=[|D_{ij}|]_{i,j\leqslant m}$. The substitution Delone set is said to be *primitive* if S^ℓ is strictly positive (entry-wise) for some $\ell\in\mathbb{N}$.

Lemma 4.3. *The substitution Delone multiset $\mathbf{\Lambda}:=(\Lambda_i)_{i\leqslant m}$ is primitive.*

Proof. By (4.6), $\mathrm{S}(i,j)=D_{ij}$ counts the number of tiles equivalent to T_i in $(\phi^kf)(T_j)$. It is easy to see that $(\mathrm{S})^\ell(i,j)$ counts the number of tiles

[1] Caution: in [5], the word multiset refers to a set with multiplicities.

equivalent to T_i in $(\phi^k f)^\ell(T_j)$. Recall that every $\mathcal{T}$-tile support contains a closed ball of radius $\eta > 0$ in its interior. Thus, $\{T_j\} = [N_\eta(x)]^{\mathcal{T}}$ for some $x \in \mathrm{supp}(T_j)$. Then we can apply Lemma 4.2 ℓ times to obtain that

$$(\phi^k f)^\ell(T_j) = (\phi^k f)^\ell([N_\eta(x)]^{\mathcal{T}}) \supset [N_{2^\ell\eta}(\phi^{k\ell}x)]^{\mathcal{T}}. \qquad (4.9)$$

By the repetitivity of $\mathcal{T}$, the right-hand side contains tiles of all types for ℓ sufficiently large, and the claim follows. $\square$

Self-affine tiling. For our substitution Delone multiset $\mathbf{\Lambda}$ there is an *adjoint system of set equations*

$$\phi^k F_j = \bigcup_{i\leqslant m}(F_i + D_{ij}), \quad j \leqslant m. \qquad (4.10)$$

It is a well-known fact in the theory of graph-directed iterated function systems that (4.10) has a unique solution for which $(F_i)_{i\leqslant m}$ is a family of non-empty compact sets in $\mathbb{R}^d$. It is proved in Theorems 2.4 and 5.5 of [5] that if $\mathbf{\Lambda}$ is a primitive substitution Delone multiset, then each F_i from (4.10) has non-empty interior and is the closure of its interior.

Proposition 4.4. *Let $(F_i)_{i\leqslant m}$ be the solution of (4.10). Then*

$$\mathbb{R}^d = \bigcup_{i\leqslant m}(F_i + \Lambda_i),$$

and the sets in the right-hand side have disjoint interiors. In other words, we obtain a tiling of $\mathbb{R}^d$ by translates of F_i.

In the terminology of [LMS], the proposition means that the substitution Delone multiset $(\Lambda_i)_{i\leqslant m}$ is *representable* by tiles (F_i, i).

Proof. We are going to use Theorem 3.7 from [6], which in turn is based on Theorem 7.1 of [5].

A $\mathbf{\Lambda}$-*cluster* is a finite multiset $(\Gamma_i)_{i\leqslant m}$ where $\Gamma_i \subset \Lambda_i$ for all i. We have $0 \in \Lambda_i$ for all i since $T_i \in \mathcal{T}$. For $i \leqslant m$ consider the $\mathbf{\Lambda}$-cluster $\mathbf{e}^{(j)} = (e_i^{(j)})_{i\leqslant m}$ where $e_i^{(j)} = \emptyset$ for $j \neq i$ and $e_j^{(j)} = \{0\}$. There is a natural 1-to-1 correspondence between $\mathcal{T}$-patches and $\mathbf{\Lambda}$-clusters: given a $\mathcal{T}$-patch P, we can consider the cluster $\mathbf{\Gamma}(P) := (\Gamma_i)_{i\leqslant m}$ where $P = \bigcup_{i\leqslant m}(T_i + \Gamma_i)$, and conversely, every $\mathbf{\Lambda}$-cluster arises this way.

Lemma 4.5. (i) *For any $\mathcal{T}$-patch P we have*

$$\Phi(\mathbf{\Gamma}(P)) = \mathbf{\Gamma}(\phi^k f(P)).$$

(ii) *We have*

$$\Phi^\ell(\mathbf{e}^{(j)}) = \mathbf{\Gamma}((\phi^k f)^\ell(T_j)), \quad j \leqslant m,\ \ell \in \mathbb{N}.$$

Proof of the lemma. (i) Let $P = \bigcup_{i\leqslant m}(T_i + \Gamma_i)$ and $\mathbf{\Gamma} = \mathbf{\Gamma}(P) = (\Gamma_i)_{i\leqslant m}$. We have, essentially repeating the calculation which follows (4.6),

$$\begin{aligned} f(P) &= \bigcup_{j\leqslant m} (f(T_j) + \Gamma_j) \\ &= \bigcup_{i\leqslant m} \Big(\phi^{-k}T_i + \bigcup_{j\leqslant m} (\Gamma_j + \phi^{-k}D_{ij})\Big). \end{aligned}$$

Multiplying by ϕ^k and comparing to (4.8) yields the desired equality.

(ii) For $\ell = 1$ this holds by (4.6) and (4.7); for $\ell > 1$ this follows from part (i). □

Conclusion of the proof of Proposition 4.4. A cluster is said to be *legal* if it occurs in $\Phi^\ell(\mathbf{e}^{(j)})$ for some $j \leqslant m$ and $\ell \in \mathbb{N}$. In Theorem 3.7 of [6] it is proved that $\mathbf{\Lambda}$ is representable by tiles (F_i, i) if and only if every $\mathbf{\Lambda}$-cluster is legal.

Let $\mathbf{\Gamma}$ be any $\mathbf{\Lambda}$-cluster. Then $\mathbf{\Gamma} = \mathbf{\Gamma}(P)$ for some $\mathcal{T}$-patch P. Fix any $j \leqslant m$ and $x \in \mathrm{supp}(T_j)$ as in (4.9). Since $\mathcal{T}$ is repetitive, (4.9) implies that there exists $\ell \in \mathbb{N}$ such that $(\phi^k f)^\ell(T_j)$ contains a $\mathcal{T}$-patch equivalent to P. By Lemma 4.5 (ii), it follows that $\Phi^\ell(\mathbf{e}^{(j)})$ contains a $\mathbf{\Lambda}$-cluster equivalent to $\mathbf{\Gamma}$, and the proof is complete. □

Proposition 4.6. *Let $(F_i)_{i\leqslant m}$ be the solution of (4.10). For $i \leqslant m$ let $T_i' = (F_i, i)$ and*

$$\mathcal{T}' = \bigcup_{i\leqslant m} (T_i' + \Lambda_i).$$

Then

(i) *$\mathcal{T}'$ is a self-affine tiling of $\mathbb{R}^d$ with expansion ϕ^k;*

(ii) *$\mathcal{T}'$ is MLD with $\mathcal{T}$.*

Proof. We already proved that $\mathcal{T}'$ is a tiling in Proposition 4.4. The fact that it is repetitive can be shown directly (as in the argument above dealing with legal clusters), but it will also follow from part (ii). Then for part (i) it remains to check the "geometric substitution" property, see Definition 2.2. We define

$$\omega(T_j' + x) = \omega(T_j') + x := \bigcup_{i \leqslant m} (T_i' + D_{ij}) + \phi^k x, \quad \text{for} \quad x \in \Lambda_j.$$

The right-hand side is a patch with support $\phi^k(\text{supp}(T_j' + x))$ by (4.10); and it is a $\mathcal{T}'$-patch since $\phi^k x + D_{ij} \subset \Lambda_i$ by (4.7). The property (ii) of Definition 2.2 holds by construction.

(ii) Recall that $A_j = \text{supp}(T_j)$ and $F_j = \text{supp}(T_j')$. Let

$$C = \max_{j \leqslant m} \rho_H(A_j, F_j),$$

where ρ_H denotes the Hausdorff metric. We claim that $\mathcal{T} \xrightarrow{\text{LD}} \mathcal{T}'$ with radius C. Indeed, suppose

$$[N_C(x)]^{\mathcal{T}} = [N_C(y)]^{\mathcal{T}} + (x - y), \tag{4.11}$$

and let $T' \in \mathcal{T}'$ be such that $\text{supp}(T')$ contains x. Then $T' = T_j' + x$ for some $x \in \Lambda_j$ and $T = T_j + x \in \mathcal{T}$ satisfies

$$\rho_H(\text{supp}(T), \text{supp}(T')) \leqslant C \;\Rightarrow\; N_C(\text{supp}(T)) \supset \text{supp}(T') \ni x.$$

It follows that $\text{supp}(T) \cap N_C(x) \neq \emptyset$, hence $T \in [N_C(x)]^{\mathcal{T}}$. By (4.11),

$$T + (y - x) \in \mathcal{T} \;\Rightarrow\; T' + (y - x) \in \mathcal{T}'.$$

This proves that $[x]^{\mathcal{T}'} \subset [y]^{\mathcal{T}'} + (x - y)$, and the opposite inclusion is proved reversing the roles of x and y.

In the argument above, we did not use any properties of $\mathcal{T}$ and $\mathcal{T}'$ other than the fact that there is a 1-to-1 correspondence between their prototiles and the Delone sets Λ_i are the same for both tilings. Thus, it also shows $\mathcal{T}' \xrightarrow{\text{LD}} \mathcal{T}$ with the same radius C. This completes the proof of the proposition and the proof of the main theorem, modulo the proof of Proposition 4.1 provided in the next section. $\square$

5. Proof of Proposition 4.1

In the next lemma we collect some elementary properties which will be needed. We will use the following notation for a compact set $F \subset \mathbb{R}^d$ and $r > 0$:

$$F^{-r} := \{x \in F : \text{dist}(x, \partial F) \geqslant r\}.$$

Lemma 5.1. **(i)** *If $\mathcal{T}_1 \xrightarrow{\text{LD}} \mathcal{T}_2$ with radius R, then for any $L > R$ and any compact set $F \subset \mathbb{R}^d$,*

$$[N_L(F)]^{\mathcal{T}_1} = [N_L(F-g)]^{\mathcal{T}_1} + g \;\Rightarrow\; [N_{L-R}(F)]^{\mathcal{T}_2} = [N_{L-R}(F-g)]^{\mathcal{T}_2} + g.$$

(ii) *If $\mathcal{T}_1 \xrightarrow{\text{LD}} \mathcal{T}_2$ with radius R, then for any compact set $F \subset \mathbb{R}^d$,*

$$[F]^{\mathcal{T}_1} = [F-g]^{\mathcal{T}_1} + g \;\Rightarrow\; [F^{-R}]^{\mathcal{T}_2} = [(F-g)^{-R}]^{\mathcal{T}_2} + g.$$

(iii) *If $\mathcal{T}_1 \xrightarrow{\text{LD}} \mathcal{T}_2$ with radius R_1 and $\mathcal{T}_2 \xrightarrow{\text{LD}} \mathcal{T}_3$ with radius R_2, then $\mathcal{T}_1 \xrightarrow{\text{LD}} \mathcal{T}_3$ with radius $R_1 + R_2$.*

(iv) *If ψ is a linear map such that $|\psi(x)| \geqslant \gamma |x|$ for all $x \in \mathbb{R}^d$, for some $\gamma > 0$, and $F \subset \mathbb{R}^d$ is a compact set, then*

$$\psi(F^{-R/\gamma}) \subset (\psi F)^{-R}.$$

Proof. (i) is immediate from the definition; the main point is that if $\mathrm{supp}(T)$ intersects $N_{L-R}(F)$ for some $T \in \mathcal{T}_2$, then $T \in [x]^{\mathcal{T}_2}$ for $x \in \mathrm{supp}(T) \cap N_{L-R}(F)$, and $[N_R(x)]^{\mathcal{T}_1} \subset [N_L(F)]^{\mathcal{T}_1}$.

(ii) follows from (i) since $N_R(F^{-R}) \subset F$.

(iii) follows from (i) as well.

(iv) holds since

$$\mathrm{dist}(\psi(x), \partial(\psi F)) = \mathrm{dist}(\psi(x), \psi(\partial F)) \geqslant \gamma\, \mathrm{dist}(x, \partial F). \qquad \square$$

Lemma 5.2. *Suppose that $\mathcal{T}$ is a pseudo-self-affine tiling with expansion ϕ, such that $\phi\mathcal{T} \xrightarrow{\text{LD}} \mathcal{T}$ with radius $R > 0$ and $|\phi x| \geqslant \lambda x$ for all $x \in \mathbb{R}^d$. Then for all $\ell \geqslant 0$ we have that*

(i) *$\phi^{-\ell}\mathcal{T} \xrightarrow{\text{LD}} \phi^{-\ell-1}\mathcal{T}$ with radius $R\lambda^{-\ell-1}$;*

(ii) *$\mathcal{T} \xrightarrow{\text{LD}} \phi^{-\ell}\mathcal{T}$ with radius $R(\lambda-1)^{-1}$.*

Proof. (i) Suppose

$$[N_L(x)]^{\phi^{-\ell}\mathcal{T}} = [N_L(y)]^{\phi^{-\ell}\mathcal{T}} + (x-y)$$

for $L \geqslant R\lambda^{-\ell-1}$. Then by (4.5),

$$[\phi^{\ell+1} N_L(x)]^{\phi\mathcal{T}} = [\phi^{\ell+1} N_L(y)]^{\phi\mathcal{T}} + \phi^{\ell+1}(x-y),$$

and Lemma 5.1(i) implies

$$[(\phi^{\ell+1} N_L(x))^{-R}]^{\phi\mathcal{T}} = [(\phi^{\ell+1} N_L(y))^{-R}]^{\phi\mathcal{T}} + \phi^{\ell+1}(x-y).$$

In view of Lemma 5.1 (iv),

$$\left[\phi^{\ell+1}\left((N_L(x))^{-R\lambda^{-\ell-1}}\right)\right]^{\mathcal{T}} = \left[\phi^{\ell+1}\left((N_L(y))^{-R\lambda^{-\ell-1}}\right)\right]^{\mathcal{T}} + \phi^{\ell+1}(x-y).$$

Finally, since $(N_L(x))^{-r} = N_{L-r}(x)$, applying (4.5) yields

$$[N_{L-R\lambda^{-\ell-1}}(x)]^{\phi^{-\ell-1}\mathcal{T}} = [N_{L-R\lambda^{-\ell-1}}(y)]^{\phi^{-\ell-1}\mathcal{T}} + (x-y),$$

as desired.

(ii) is immediate from part (i) and Lemma 5.1 (iii). □

Proof of Proposition 4.1. We are given a pseudo-self-affine tiling $\mathcal{S}$ such that $\phi\mathcal{S} \xrightarrow{\mathrm{LD}} \mathcal{S}$ with radius R. For every tile $S \in \mathcal{S}$, with $A = \mathrm{supp}(S)$ we consider a new tile

$$T_S := \left(A, \langle [N_L(A)]^{\mathcal{S}}, S\rangle\right),$$

where $\langle P, S\rangle$ denotes the equivalence class of a patch $P \subset \mathcal{S}$ with marked tile $S \in P$. The positive number L will be specified later. The new tiling is

$$\mathcal{T} := \{T_S : \ S \in \mathcal{S}\}.$$

There is a finite number of tile types in $\mathcal{T}$ by FLC (though, most likely, it is huge compared to the number of tile types in $\mathcal{S}$). It is clear that $T_{S'} = T_S + g$ implies $S' = S + g$, so $\mathcal{T} \xrightarrow{\mathrm{LD}} \mathcal{S}$ with radius 0. By definition, $\mathcal{S} \xrightarrow{\mathrm{LD}} \mathcal{T}$ with radius L. It remains to check that for $S, S' \in \mathcal{S}$ with supports A, A', for any $k \geqslant 1$,

$$T_{S'} = T_S + g \ \Rightarrow \ [A']^{\phi^{-k}\mathcal{T}} = [A]^{\phi^{-k}\mathcal{T}} + g. \tag{5.1}$$

Equivalently, we need to verify that for every $T'' = T_{S''} \in \mathcal{T}$ with support A'' satisfying $\phi^{-k}A'' \cap A' \neq \emptyset$, we have $\phi^{-k}T'' + g \in \phi^{-k}\mathcal{T}$. What we do know is that the labels of $T = T_S$ and $T' = T_{S'}$ are the same, hence

$$[N_L(A')]^{\mathcal{S}} = [N_L(A)]^{\mathcal{S}} + g.$$

Thus, by Lemma 5.2 (ii) and Lemma 5.1 (i),

$$[N_{L-R(\lambda-1)^{-1}}(A')]^{\phi^{-k}\mathcal{S}} = [N_{L-R(\lambda-1)^{-1}}(A)]^{\phi^{-k}\mathcal{S}} + g. \tag{5.2}$$

Thus, $\phi^{-k}S'' + g \in \phi^{-k}\mathcal{S}$, that is, $\phi^{-k}S'' + g = \phi^{-k}S'''$ for some $S''' \in \mathcal{S}$. Let A''' be the support of S'''. We need to show that $\phi^{-k}T'' + g = \phi^{-k}T'''$, where $T''' = T_{S'''}$. We already know the equality of supports, so it remains to show that the labels are the same. This will follow if we verify that $\phi^{-k}[N_L(A'')]^{\mathcal{S}} + g = \phi^{-k}[N_L(A''')]^{\mathcal{S}}$, or equivalently, that

$$[\phi^{-k} N_L(A''')]^{\phi^{-k}\mathcal{S}} = [\phi^{-k} N_L(A'')]^{\phi^{-k}\mathcal{S}} + g. \tag{5.3}$$

However,

$$[\phi^{-k} N_L(A'')]^{\phi^{-k}\mathcal{S}} \subset [N_{L\lambda^{-k}}(\phi^{-k}A'')]^{\phi^{-k}\mathcal{S}} \subset [N_{(L+d_M)\lambda^{-k}}(A')]^{\phi^{-k}\mathcal{S}},$$

since $d_M(\phi^{-k}\mathcal{S}) \leqslant \lambda^{-k} d_M$, and (5.3) follows from (5.2), provided $L - R(\lambda-1)^{-1} > (L+d_M)\lambda^{-1} > (L+d_M)\lambda^{-k}$. Thus, it is enough to choose

$$L > R\lambda(\lambda-1)^{-2} + d_M(\lambda-1)^{-1},$$

and the proof is complete. $\square$

Acknowledgment. I am grateful to Jeong-Yup Lee and Lorenzo Sadun for helpful discussions.

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Поступило 19 апреля 2005 г.