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INVARIANT TRANSFORMATIONS FOR THE STURM–LIOUVILLE OPERATOR

ABSTRACT. In this paper we consider the Sturm–Liouville operator on a finite interval. For particular boundary conditions, a group of invariant transformations that preserve the operator's spectrum is constructed. This result permits us to reconsider some old problems for the Sturm–Liouville operator. In particular, the influence of this group on the inverse problem is discussed.

1. Introduction

In this paper we consider the differential equation

$$-y''(x) = \lambda p(x) y(x), \quad -1 < x < 1, \tag{1}$$

with the boundary conditions

$$y(-1) - ay'(-1) = 0, \quad y(1) + by'(1) = 0. \tag{2}$$

We view this differential expression as an operator in various Hilbert spaces, which shall be specified in each case.

It is known that for some special boundary conditions, see [5], there exists a group of transformations of (1), (2) that preserve the associated operator's spectrum. This result was published in [12], and as the referee kindly informed me, was also discussed in [13]. In a natural way, the above property gives a new representation for the operator (1), (2). Using these representation, we find conditions on the potential $p(x)$ under which the operator's spectrum is discrete. Hence, we are able to present new information about Sturm–Liouville operators with discrete spectrum and establish an estimate for the ground state.

In this paper we also consider the spectral inverse problem for Sturm–Liouville operators. In particularly, a generalization of M. Krein's [2] result is given here.

2. Construction of an invariant group

Let G denote the set $\{(x,a)\colon x\in(-1,1), a=\pm 1\}$. We define a binary operation on G as follows:

$$(x_1,a_1)*(x_2,a_2)=\left(\frac{x_1+a_1x_2}{1+x_1a_1x_2},a_1a_2\right).$$

It is easy to check that $(0,1)$ is the unit element, i.e., for arbitrary $(x,a)\in G$ we have

$$(x,a)*(0,1)=(0,1)*(x,a)=(x,a).$$

For each $(x,a)\in G$ there is an inverse element, i.e.,

$$(x,a)*(-ax,a)=(-ax,a)*(x,a)=(0,1).$$

In fact, G is a group with the binary operation $*$. Note that $*$ is not commutative, but the subset $G_0=\{(x,1)\in G\}$ is an Abelian subgroup.

Each element (x_0,a_0) of G generates a mapping of the interval $(-1,1)$ onto itself acting by the formula

$$g_{(x_0,a_0)}(x)=a_0\frac{x-x_0}{1-xx_0},\quad -1<x<1.$$

It is easy to check that

$$g_{(x_0,a_0)}(g_{(x_1,a_1)}(x))=g_{(x_1,a_1)*(x_0,a_0)}(x).$$

Definition 1. *By $L^2((1-x^2)^m dx)$, where m is a real number, we denote the space of functions $y(x)$ defined on $x\in(-1,1)$ that satisfy the condition*

$$||y||_m^2=\int_{-1}^{1}|y(x)|^2(1-x^2)^m dx<\infty.$$

Here we give two representations for the group G, which play an essential role in the subsequent sections.

With each element $(x_0,a_0)\in G$, we associate a formal operator $A_{(x_0,a_0)}$ that acts by the following rule:

$$\left(A_{(x_0,a_0)}f\right)(x)=\frac{(1-x_0^2)^2}{(1-xx_0)^4}f\left(a_0\frac{x-x_0}{1-xx_0}\right).$$

For arbitrary $f(x) \in L^2((1-x^2)^3 dx)$ we have $||A_{(x_0,a_0)}f||_3 = ||f||_3$. It is useful to introduce the transformation

$$\widehat{f}(\lambda) = \int_{-1}^{1} (1-t^2) f(t) \left(\frac{1-t}{1+t} \right)^{i\lambda} dt, \quad -\infty < \lambda < \infty.$$

First, we define this transformation on the functions with $\mathrm{supp}(f) \subset (-1,1)$. For these functions, a change of variables yields

$$\widehat{f}(\lambda) = \int_{-\infty}^{\infty} \frac{8}{(e^{y/2}+e^{-y/2})^4} f\left(\frac{e^{y/2}-e^{-y/2}}{e^{y/2}+e^{-y/2}} \right) e^{-i\lambda y} dy.$$

Consequently, $\widehat{f}$ is precisely the Fourier transform of the function $\frac{1}{2}\mathrm{sech}^4(\frac{t}{2})\, f\big(\tanh(\frac{t}{2})\big)$. So, by Parseval's identity,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |\widehat{f}(\lambda)|^2 d\lambda = \int_{-\infty}^{\infty} \frac{64}{(e^{y/2}+e^{-y/2})^8} \left| f\left(\frac{e^{y/2}-e^{-y/2}}{e^{y/2}+e^{-y/2}} \right) \right|^2 dy =$$

$$= \frac{1}{2} \int_{-1}^{1} |f(x)|^2 (1-x^2)^3 dx.$$

This formula permits us to extend the transformation $f \mapsto \widehat{f}$ to the entire space $L^2((1-x^2)^3 dx)$. We have

$$(A_{(x,a)}f)^{\wedge}(\lambda) = \left(\frac{1-x}{1+x} \right)^{i\lambda} \widehat{f}(a\lambda), \quad -\infty < \lambda < \infty.$$

With each element $(x_0, a_0) \in G$, we can associate another a formal operator $B_{(x_0,a_0)}$. This operator acts by the formula

$$\left(B_{(x_0,a_0)}y\right)(x) = \frac{1-x_0 x}{\sqrt{1-x_0^2}}\, y\left(a_0 \frac{x-x_0}{1-x x_0} \right).$$

We denote

$$\widetilde{y}(\lambda) = \int_{-1}^{1} \frac{y(t)}{(1-t^2)^{3/2}} \left(\frac{1-t}{1+t} \right)^{i\lambda} dt, \quad -\infty < \lambda < \infty.$$

Since

$$\widetilde{y}(\lambda)=\frac{1}{4}\int_{-\infty}^{\infty}(e^{x/2}+e^{-x/2})y\left(\frac{e^{x/2}-e^{-x/2}}{e^{x/2}+e^{-x/2}}\right)e^{-i\lambda x}dx,$$

from Parseval's identity we deduce that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|\widetilde{y}(\lambda)|^2d\lambda=$$

$$=\frac{1}{8}\int_{-\infty}^{\infty}(e^{y/2}+e^{-y/2})^2\left|y\left(\frac{e^{x/2}-e^{-x/2}}{e^{x/2}+e^{-x/2}}\right)\right|^2dx=\frac{1}{2}\int_{-1}^{1}|y(x)|^2\frac{dx}{(1-x^2)^2}.$$

We also have

$$\int_{-1}^{1}|y'(x)|^2dx=\frac{1}{\pi}\int_{-\infty}^{\infty}\left|-2i\lambda\widetilde{y}(\lambda)+\widetilde{y}_1(\lambda)\right|^2d\lambda,$$

where $y_1(x)=xy(x)$. Consequently,

$$\int_{-1}^{1}|y'(x)|^2dx\simeq\frac{5}{\pi}\int_{-\infty}^{\infty}|\widetilde{y}(\lambda)|^2(1+\lambda^2)d\lambda.$$

For arbitrary function $y(x)$ in $L^2((1-x^2)^{-2}dx)$, we have $||B_{(x_0,a_0)}y||_{-2}=||y||_{-2}$.

We also have the following formula:

$$\left(\widetilde{B_{(x_0,a_0)}y}\right)(\lambda)=\left(\frac{1-x_0}{1+x_0}\right)^{i\lambda}\widetilde{y}(a_0\lambda).$$

3. Invariant transformations

In this section, we describe a group of transformations, which preserve the operator's spectrum.

Theorem 1. *Suppose the differential equation*

$$-y''(x)=\lambda p(x)y(x),\quad -1<x<1,\tag{3}$$

with the boundary conditions

$$y(-1)-ay'(-1)=0,\quad y(1)+by'(1)=0,\tag{4}$$

has a nontrivial solution. If $2a+a^2=0$ and $2b+b^2=0$, then for arbitrary $x_0 \in (-1,1)$ the potential function

$$\left(A_{(x_0,1)}p\right)(x) = \frac{(1-x_0^2)^2}{(1-xx_0)^4} p\left(\frac{x-x_0}{1-xx_0}\right) \tag{5}$$

generates an equation with a nontrivial solution satisfying the same boundary conditions (4). If $a=b$, then the boundary value problem (3), (4) with $p(x)$ replaced by

$$A_{(x_0,a_0)}p(x)$$

has a nontrivial solution for each $(x_0,a_0) \in G$.

Proof. Let $y(x)$, $-1<x<1$, be a nontrivial solution for the boundary value problem (4), (5). We denote

$$z(x) = (1-xx_0)y\left(\frac{x-x_0}{1-xx_0}\right).$$

An elementary calculation shows that

$$-z''(x) = \lambda \frac{(1-x_0^2)^2}{(1-xx_0)^4} p\left(\frac{x-x_0}{1-xx_0}\right) z(x), \quad -1<x<1.$$

From the boundary conditions on y we have

$$z(-1) - \frac{a(1+x_0)}{1-x_0-ax_0} z'(-1) = 0, \quad z(1) + \frac{b(1-x_0)}{1+x_0+bx_0} z'(1) = 0.$$

By the conditions $2a+a^2=0$ and $2b+b^2=0$, we have

$$a = \frac{a(1+x_0)}{1-x_0-ax_0}, \quad b = \frac{b(1-x_0)}{1+x_0+bx_0}$$

for arbitrary $x_0 \in (-1,1)$. □

Lemma 1. *Let $p(x)$, $-1<x<1$, be a nonnegative integrable function. Then in the family of functions $A_{(x_0,1)}p(x)$, $\quad -1<x_0<1$, there exists one with*

$$\int_{-1}^{1} x\left(A_{(x_1,1)}p\right)(x)dx = 0.$$

Proof. The function

$$M(x_0)=\int_{-1}^{1}\left(A_{(x_0,1)}p\right)(x)dx=\int_{-1}^{1}\frac{(1+tx_0)^2}{1-x_0^2}p(t)dt,\quad -1<x_0<1,$$

attains its minimum for some $x_1\in(-1,1)$, and for this x_1 we have

$$0=M'(x_1)=\frac{2}{1-x_1^2}\int_{-1}^{1}x\left(A_{(x_1,1)}p\right)(x)dx.\qquad\square$$

Theorem 2. *Let $p(x)$, $-1\le x\le 1$, be a continuous function, and let $2a+a^2\neq 0$ or $2b+b^2\neq 0$. Then in the family of functions $A_{(x_0,1)}p(x)$, $-1<x_0<1$, there exist two for which the corresponding operators have different spectra.*

Proof. Indeed, if $a+b+2\neq 0$, then Eq. (3) with the boundary conditions (4) can be rewritten in the form

$$y(x)=-\lambda\int_{-1}^{x}(x-t)p(t)y(t)dt+\lambda\frac{a+x+1}{a+b+2}\int_{-1}^{1}(1+b-t)p(t)y(t)dt,\ -1<t<1.$$

Consequently,

$$(2+a+b)\sum_{n=1}^{\infty}\frac{1}{\lambda_n(a,b)}=\int_{-1}^{1}(1+b-x)(1+a+x)p(x)dx=$$

$$=\int_{-1}^{1}(1-x^2)p(x)dx+(a+b+ab)\int_{-1}^{1}p(x)dx+(b-a)\int_{-1}^{1}xp(x)dx,$$

where the $\lambda_n(a,b)$ are the eigenvalues of the problem (3), (4). For the particular cases where $a=b=0$ and $a=b=-2$, by the trace formula, we have

$$\frac{1}{2}\int_{-1}^{1}(1-x^2)p(x)dx=\sum_{n=1}^{\infty}\frac{1}{\lambda_n(0,0)}=-\sum_{n=1}^{\infty}\frac{1}{\lambda_n(-2,-2)}.$$

Note that there is only one linear functional invariant for all the transformations (5) up to a constant factor. That functional is

$$\int_{-1}^{1} p(x)(1-x^2)dx.$$

We have

$$(2+a+b)\sum_{n=1}^{\infty}\frac{1}{\lambda_n(a,b)}=\int_{-1}^{1}(1+b-x)(1+a+x)\frac{(1-x_0^2)^2}{(1-xx_0)^4}p\left(\frac{x-x_0}{1-xx_0}\right)dx=$$

$$=\int_{-1}^{1}(1-x^2)p(x)dx+\frac{1}{1-x_0^2}\left[\left((a+b+ab)\int_{-1}^{1}p(x)dx+(b-a)\int_{-1}^{1}xp(x)dx\right)+\right.$$

$$+x_0\left(2(a+b+ab)\int_{-1}^{1}xp(x)dx+(b-a)\int_{-1}^{1}(1+x^2)p(x)dx\right)+$$

$$\left.+x_0^2\left((a+b+ab)\int_{-1}^{1}x^2p(x)dx+(b-a)\int_{-1}^{1}xp(x)dx\right)\right].$$

This rational function is constant for all $-1<x_0<1$ if and only if the numerator is proportional to $1-x_0^2$. So,

$$(a+b+ab)\int_{-1}^{1}p(x)dx+(b-a)\int_{-1}^{1}xp(x)dx\pm$$

$$\pm\left(2(a+b+ab)\int_{-1}^{1}xp(x)dx+(b-a)\int_{-1}^{1}(1+x^2)p(x)dx\right)+$$

$$+(a+b+ab)\int_{-1}^{1}x^2p(x)dx+(b-a)\int_{-1}^{1}xp(x)dx=0,$$

which gives $b(a+2)=0$ and $a(b+2)=0$. Consequently, if $a+b+2\neq 0$, then only for the values $a=b=0$ and $a=b=-2$ may the spectrum be invariant.

Now we consider the case where $a+b+2=0$. In this case, the point $\lambda_1 = 0$ is an eigenvalue. The corresponding eigenfunction is $y_1(x) = 1+a+x = x-1-b$.

For the solutions of the equation

$$-y''(x) = \lambda p(x)y(x), \quad -1 < x < 1,$$

with the boundary condition $y(-1) - ay(-1) = 0$ only at the end point -1, we have

$$y(x) = \lambda \int_{-1}^{x} (x-t)p(t)y(t)dt + c(1+a+x),$$

where c is an arbitrary constant. From the boundary condition $y(1) + by'(1) = 0$ and the fact that $a+b+2=0$, it follows that

$$\lambda_n \int_{-1}^{1} (1+b-t)p(t)y_n(t)dt = 0, \quad n = 1, 2, \ldots.$$

So, for the eigenfunctions $y_n(x)$ corresponding to the eigenvalues $\lambda_n \neq 0$, we have

$$\int_{-1}^{1} (1+b-t)p(t)y_n(t)dt = 0.$$

We introduce an operator P acting by the following formula:

$$(Py)(x) = y(x) - \frac{(1+b-x)}{d} \int_{-1}^{1} (1+b-t)p(t)y(t)dt,$$

where

$$d = \int_{-1}^{1} (1+b-t)^2 p(t)dt.$$

It is easy to check that $Py_1 = 0$, $Py_n = y_n$, $n = 2, 3, \ldots$. Next, we introduce an operator A acting by the following formula:

$$(Ay)(x) = \int_{-1}^{x} (x-t)p(t)y(t)dt.$$

Note that for $n = 2, 3, \ldots$ we have

$$(Ay_n)(x) = -\frac{1}{\lambda_n}(y_n(x) - y_n'(-1)y_1(x)).$$

Consequently, for the operator $B = PAP$ we have

$$By_1 = 0, \quad By_n = -\frac{1}{\lambda_n} y_n, \quad n = 2, 3, \ldots .$$

Since the family of functions $y_n(x)$, $n = 1, 2, \ldots$, is dense in the space $L_2(-1, 1)$, we see that B has no other eigenfunction and we have

$$-\operatorname{tr} B = \sum_{n=1}^{\infty} \frac{1}{\lambda_n}.$$

Elementary calculations yield

$$(By)(x) = \int_{-1}^{x} (x-t)p(t)y(t)dt -$$

$$-\frac{1}{d}\int_{-1}^{1}\int_{-1}^{1} y(s)p(s)p(t)(1+b-t)[(1+b-s)(x-t)^+ + (1+b-x)(t-s)^+]dtds +$$

$$\frac{1+b-x}{d^2}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} y(s)p(s)p(t)p(v)(1+b-t)(1+b-s)(1+b-v)(s-t)^+ dtdsdv,$$

where $a^+ = \max(a, 0)$.

Consequently,

$$-\operatorname{tr}(B) = \sum_{n=2}^{\infty} \frac{1}{\lambda_n(a,b)} =$$

$$= \frac{1}{d}\int_{-1}^{1}\int_{-1}^{1} (1+b-x)(1+b-t)(t-x)^+ p(x)p(t)dxdt. \qquad (6)$$

We observe that only in the cases of $b = 0$ or $b = -2$ the right-hand side of (6) is invariant under all the transformations (5). Indeed, by

Lemma 1, without loss of generality we may assume that $\int_{-1}^{1} xp(x)dx = 0$. Elementary calculations yield $-\operatorname{tr} B = \frac{A(x_0)}{B(x_0)}$, where

$$A(x_0) = \frac{(2+b)^2}{4}\frac{1-x_0}{1+x_0}\int_{-1}^{1}\int_{-1}^{1} p(s)p(t)(1-t)(1-s)(t-s)^{+}dtds+$$

$$+\frac{b^2}{4}\frac{1+x_0}{1-x_0}\int_{-1}^{1}\int_{-1}^{1} p(s)p(t)(1+t)(1+s)(t-s)^{+}dtds+$$

$$+(2+b)b\int_{-1}^{1}\int_{-1}^{1} p(s)p(t)(1-ts)(s-t)^{+}dtds,$$

and

$$B(x_0) = \frac{(2+b)^2}{4}\frac{1-x_0}{1+x_0}\int_{-1}^{1} p(t)(1-t)^2dt + \frac{b^2}{4}\frac{1+x_0}{1-x_0}\int_{-1}^{1} p(t)(1+t)^2dt+$$

$$+(2+b)b\int_{-1}^{1} p(t)(1-t^2)dt.$$

It is easy to check that this quantity is invariant if $b(2+b) = 0$. In the case where $b(2+b) \neq 0$, this quantity may be invariant only if

$$\frac{\int_{-1}^{1}\int_{-1}^{1} p(s)p(t)(1-t)(1-s)(t-s)^{+}dtds}{\int_{-1}^{1} p(t)(1-t)^2dt} =$$

$$= \frac{\int_{-1}^{1}\int_{-1}^{1} p(s)p(t)(1+t)(1+s)(t-s)^{+}dtds}{\int_{-1}^{1} p(t)(1+t)^2dt} =$$

$$= \frac{\int_{-1}^{1}\int_{-1}^{1} p(s)p(t)(1-ts)(t-s)^{+}dtds}{\int_{-1}^{1} p(t)(1-t^2)dt}.$$

Since

$$\int_{-1}^{1} p(t)(1-t)^2dt = \int_{-1}^{1} p(t)(1+t)^2dt = \int_{-1}^{1} p(t)(1+t^2)dt,$$

we have

$$\int_{-1}^{1}\int_{-1}^{1} p(s)p(t)(1-t)(1-s)(t-s)^{+}\,dtds =$$

$$= \int_{-1}^{1}\int_{-1}^{1} p(s)p(t)(1+t)(1+s)(t-s)^{+}\,dtds =$$

$$= \int_{-1}^{1}\int_{-1}^{1} p(s)p(t)(1+ts)(t-s)^{+}\,dtds.$$

Thus, we arrive at the condition

$$\frac{\int_{-1}^{1}\int_{-1}^{1} p(s)p(t)(1-ts)(t-s)^{+}\,dtds}{\int_{-1}^{1} p(t)(1-t^2)dt} =$$

$$= \frac{\int_{-1}^{1}\int_{-1}^{1} p(s)p(t)(1+ts)(t-s)^{+}\,dtds}{\int_{-1}^{1} p(t)(1+t^2)dt}.$$

Consequently,

$$\int_{-1}^{1}\int_{-1}^{1} p(s)p(t)(t-s)^{+}\,dtds \int_{-1}^{1} p(t)t^2 dt =$$

$$= \int_{-1}^{1}\int_{-1}^{1} p(s)p(t)st(t-s)^{+}\,dtds \int_{-1}^{1} p(t)dt.$$

This formula cannot be true in our case, because all the integrals are positive except for the following one:

$$\int_{-1}^{1}\int_{-1}^{1} p(s)p(t)st(t-s)^{+}\,dtds.$$

Indeed, we have

$$\int_{-1}^{1}\int_{-1}^{1} p(s)p(t)st(t-s)^{+}\,dtds =$$

$$= \int_{-1}^{1} sp(s) \left(\int_{-1}^{s} p(t)t(s-t)dt \right) ds = -\int_{-1}^{1} \left(\int_{-1}^{s} tp(t)dt \right)^2 ds < 0.$$

So, the claim of Theorem 1 fails for the other boundary conditions. □

Remark 1. The differential equation

$$-u''(t) + q(t)u(t) = \lambda u(t), \quad 0 \le t \le l \tag{7}$$

with the boundary conditions

$$u(0) = 0, \quad u(l) = 0 \tag{8}$$

is very important in applications. Here $q(t)$ is a real-valued integrable function and λ is a complex number.

Liouville suggested certain transformations, which relate the operators (7), (8) and (1), (2). Those transformations are:

$$t = \int_{-1}^{x} \sqrt{p(s)}\, ds, \quad u(t) = \sqrt[4]{p(x)}\, y(x), \tag{9}$$

$$q(t) = -\frac{1}{\sqrt[4]{p^3(x)}} \frac{d^2}{dx^2} \left(\frac{1}{\sqrt[4]{p(x)}} \right). \tag{10}$$

We note that the boundary conditions are preserved and

$$\int_{0}^{l} u^2(t)dt = \int_{-1}^{1} p(x)y^2(x)dx.$$

Long, but elementary calculations show that if $p(x)$ and $q(t)$ are related by formulas (10), then for each $-1 < x_0 < 1$ the function $A_{(x_0,1)}p(x)$ (instead of $p(x)$) and the same $q(t)$ also satisfy (10). Indeed,

$$-q(t) = \frac{1}{16} \left(5(p'(x))^2 - 4p(x)p''(x) \right) p^{-3}(x)$$

so, for the function

$$p_1(x) = \frac{(1-x_0)^2}{(1-xx_0)^4} p \left(\frac{x-x_0}{1-xx_0} \right),$$

we have

$$-q_1(t_1) = \frac{1}{16}\left(5(p_1'(x))^2 - 4p_1(x)p_1''(x)\right)p_1^{-3}(x) =$$

$$= \frac{1}{16}\left(5\left(p'\left(\frac{x-x_0}{1-xx_0}\right)\right)^2 -\right.$$

$$\left.-4p\left(\frac{x-x_0}{1-xx_0}\right)p''\left(\frac{x-x_0}{1-xx_0}\right)\right)p^{-3}\left(\frac{x-x_0}{1-xx_0}\right),$$

where

$$t_1 = \int_{-1}^{x} \sqrt{p_1(s)}\,ds.$$

So, $q(t) = q_1(t)$. $\square$

This remark shows that for Eq. (7) we have no invariant transformation that can be generated by the transformations constructed above.

4. Discrete spectrum

In this section we consider some conditions on the potential function that guarantee that the spectrum of the operator (1), (2) is discrete in the case where $a = b = 0$.

Certainly, the choice of the space where the operator is considered plays an essential role. Here we give some results of the string theory, in order to underline that it is natural to consider those spaces.

The further comments about the string theory are taken from B. Sz.-Nagy's paper [8].

Suppose a string oscillates in the plane (x, y). The ends of the string are situated at the points $(-1, 0)$ and $(1, 0)$. Let the point of the string with abscissa x have the ordinate $y(x, t)$ at the time moment t. So,

$$y(-1, t) = y(1, t) = 0.$$

At the time moment t the string's kinetic energy is equal to

$$E_c(t) = \frac{1}{2}\int_{-1}^{1}\left(\frac{\partial y(x,t)}{\partial t}\right)^2 p(x)dx,$$

where $p(x)$ is the string's density, and the potential energy is defined by the formula

$$E_p(t) = \frac{1}{2}\int_{-1}^{1}\left(\frac{\partial y(x,t)}{\partial x}\right)^2 dx.$$

For the string's free motions we have the equation

$$p(x)\frac{\partial^2 y(x,t)}{\partial t^2} - \frac{\partial^2 y(x,t)}{\partial x^2} = 0.$$

For the solutions of this equation that have the form $y(x,t) = e^{i\lambda t}y(x)$, the conditions of finite kinetic and potential energy are

$$\int_{-1}^{1} y^2(x)p(x)dx < \infty, \quad \int_{-1}^{1}(y'(x))^2 dx < \infty.$$

Here the function $y(x)$ satisfies the following equation:

$$-y''(x) = \lambda p(x)y(x), \quad -1 < x < 1,$$

with the boundary conditions

$$y(-1) = y(1) = 0.$$

The choice of the spaces is justified by this remark.

In the paper [1], this operator was considered in the space of functions y with

$$\int_{-1}^{1} p(x)y^2(x)dx < \infty.$$

We note that if a function $y(x)$ satisfies Eq. (1) with boundary conditions $y(-1) = y(1) = 0$, then for some constant $0 < A < \infty$ we have the inequality

$$A\int_{-1}^{1}\frac{|y(t)|^2}{(1-t^2)^2}dt \le \int_{-1}^{1}|y'(x)|^2 dx = -\int_{-1}^{1} y''(x)\overline{y}(x)dx = \int_{-1}^{1}\lambda p(t)|y(t)|^2 dt.$$

So, the eigenfunctions of Eq. (1) with the boundary conditions $y(-1) = y(1) = 0$ have finite norms in the space

$$\int_{-1}^{1} p(t)|y(t)|^2 dt < \infty$$

and in the space

$$\int_{-1}^{1} |y'(x)|^2 dx < \infty$$

simultaneously. Note that this is a statement about eigenvalues, and it tells nothing about the continuous part of the spectrum.

We shall use the following auxiliary result.

Theorem 3. *A function $y(x) \in L^2((1-x^2)^{-2}dx)$ is a solution of the equation*

$$-y''(t) = \lambda p(t)y(t), \quad y(-1) = 0, \quad y(1) = 0,$$

if and only if $\widetilde{y}(x) \in L^2(-\infty, \infty)$ and

$$(2x^2 + 1/2)\widetilde{y}(x) = \lambda \int_{-\infty}^{\infty} \widehat{p}(x-t)\widetilde{y}(t)dt.$$

Proof. For the convolution of the functions $\widetilde{y}(\lambda)$ and $\widehat{p}(\lambda)$ we have

$$(\widehat{p} * \widetilde{y})(x) = \int_{-\infty}^{\infty} \widehat{p}(x-t)\widetilde{y}(t)dt =$$

$$= 2\int_{-\infty}^{\infty} p\left(\frac{e^{t/2}-e^{-t/2}}{e^{t/2}+e^{-t/2}}\right) y\left(\frac{e^{t/2}-e^{-t/2}}{e^{t/2}+e^{-t/2}}\right) \frac{e^{-itx}}{(e^{t/2}+e^{-t/2})^3} dt =$$

$$= \frac{1}{2}\int_{-1}^{1} p(s)y(s)\sqrt{1-s^2}\left(\frac{1-s}{1+s}\right)^{ix} ds.$$

For the functions vanishing near the endpoints $-1, 1$ we have

$$\frac{1}{2}\int_{-1}^{1} y''(t)\sqrt{1-t^2}\left(\frac{1-t}{1+t}\right)^{ix} dt =$$

$$= -(1/2 + 2x^2)\int_{-1}^{1} \frac{y(t)}{(1-t^2)^{3/2}}\left(\frac{1-t}{1+t}\right)^{ix} dt = -(2x^2 + 1/2)\widetilde{y}(x).$$

In the general case, i.e., if $y(x) \in L^2((1-x^2)^{-2}dx)$, we approximate $y(x)$ by functions with compact support. $\square$

In the proof of the next two theorems we use the following result, see [6, pp. 24–25].

Lemma 2. *Let m be a natural number and $\mu(dx)$ a positive measure on $(-\infty,\infty)$; then the best constant in the inequality*

$$\int_{-\infty}^{\infty} |u(x)|^2 \mu(dx) \le C \sum_{k=0}^{m} \int_{-\infty}^{\infty} |u^{(k)}(x)|^2 dx$$

is equivalent to the number

$$C \simeq \sup_{-\infty<x<\infty} \int_{x}^{x+1} \mu(dx).$$

Theorem 4. *Let $p(x) > 0$, $-1 \le x \le 1$, be a measurable function. Then the spectrum of the operator*

$$-y''(x) = \lambda p(x) y(x), \quad -1 \le x \le 1,$$

with the boundary conditions $y(-1) = y(1) = 0$ in the space of functions for which

$$\int_{-1}^{1} |y'(x)|^2 \, dx < \infty$$

is discrete if and only if

$$\sup_{|x|<1} (1-|x|) \int_{2|t-x|<1-|x|} p(t)dt < \infty$$

and

$$\lim_{|x|\to 1^-} (1-|x|) \int_{2|t-x|<1-|x|} p(t)dt = 0.$$

Proof. It suffices to prove that the operator

$$(Cy)(x) = \frac{2}{4x^2+1} \int_{-\infty}^{\infty} \widehat{p}(x-t)\widetilde{y}(t)dt$$

acting in the space $L^2((1+x^2)dx)$ is compact. This is equivalent to proving that the operator B that acts by the formula

$$B\widetilde{y}(x) = \int_{-\infty}^{\infty} \widehat{p}(x-t)\widetilde{y}(t)dt$$

from the space $L^2((1+x^2)dx)$ to $L^2((1+x^2)^{-1}dx)$ is compact.

First, we find conditions ensuring the boundedness of the operator B. The Fourier transform (which we denote by $\mathbf{F}$) of the function $\widehat{p}(x)$ is

$$\mathbf{F}\widehat{p}(y) = G(y) = (e^{y/2}+e^{-y/2})^{-4} p\left(\frac{e^{y/2}-e^{-y/2}}{e^{y/2}+e^{-y/2}}\right) \geq 0.$$

Then for the operator $\mathbf{F}B\mathbf{F}$ we have

$$\mathbf{F}B\mathbf{F} : W_2^1 \to \left(W_2^1\right)^*.$$

Estimation of the norm of this operator is equivalent to finding the best constant M in the following inequality:

$$\left|\int_{-\infty}^{\infty} f_1(x)f_2(x)G(x)dx\right| \leq$$

$$\leq M\left(\int_{-\infty}^{\infty} (f_1^2(x)+(f_1'(x))^2)dx\right)^{1/2}\left(\int_{-\infty}^{\infty} (f_2^2(x)+(f_2'(x))^2)dx\right)^{1/2},$$

or in

$$\int_{-\infty}^{\infty} \left(f(x)\sqrt{G(x)}\right)^2 dx \leq M\left(\int_{-\infty}^{\infty} (f^2(x)+(f'(x))^2)dx\right).$$

By Lemma 2, we have

$$M \simeq \sup_x \int_x^{x+1} G(t)dt \simeq \sup_{|x|<1} (1-|x|) \int_{2|t-x|<1-|x|} p(t)dt.$$

Consequently, the norm $||B||$ of the operator B is equivalent to the quantity

$$\sup_{|x|<1} (1-|x|) \int_{2|t-x|<1-|x|} p(t)dt.$$

For arbitrary natural n, we denote by Q_n the projection acting by the formula

$$Q_n f(x) = f(x), \quad |x| < n,$$

and

$$Q_n f(x) = 0, \quad |x| \geq n.$$

The family $B_n = BQ_n, \quad n = 1, 2, \ldots$, is an increasing sequence of compact operators, and B is its limit in the weak topology. So, B is compact if and only if $\lim_{n\to\infty} ||B_n - B|| = 0$, or if and only if

$$\lim_{|x|\to\infty} \sup_x \int_x^{x+1} G(t)dt = 0,$$

which is equivalent to the assumption of the theorem. □

Theorem 5. *Let $p(x) > 0$, $-1 \leq x \leq 1$, be a measurable function. Then the spectrum of the operator*

$$-y''(x) = \lambda p(x)y(x), \quad -1 \leq x \leq 1,$$

with the boundary conditions

$$y(-1) = y(1) = 0$$

in the space of functions for which

$$\int_{-1}^{1} \frac{y^2(x)}{(1-x^2)^2} dx < \infty$$

is discrete if and only if

$$\sup_{|x|<1} \left((1-|x|)^3 \int_{2|t-x|<1-|x|} p^2(t)dt \right) < \infty$$

and

$$\lim_{|x|\to 1^-} \left((1-|x|)^3 \int_{2|t-x|<1-|x|} p^2(t)dt \right) = 0.$$

Proof. The proof is similar to the proof of the preceding theorem. In the present case, we need to estimate the norm $|B|$ of the operator B acting from the space $L^2 = L^2(-\infty,\infty)$ to $L^2((1+x^2)^{-2}dx)$.

Presently, for the operator $\mathbf{F}B$, we have

$$\mathbf{F}B\mathbf{F} : L_2(\infty,\infty) \to \left(W_2^2\right)^*.$$

We need to estimate the constant M in the following inequality:

$$\left| \int_{-\infty}^{\infty} G(x) f_1(x) f_2(x) dx \right| \le$$

$$\le M \left(\int_{-\infty}^{\infty} f_1^2(x) dx \right)^{1/2} \left(\int_{-\infty}^{\infty} (f_2^2(x) + (f_2'(x))^2 + (f_2''(x))^2) dx \right)^{1/2}.$$

Observe that

$$M^2 \simeq \sup_x \int_x^{x+1} G^2(t)dt. \quad \square$$

Remark 2. If $p(x) = 1/(1-x^2)^2$, then our operator becomes

$$(2x^2 + 1/2)\widetilde{y}(x) = \lambda\left(\widehat{p} * \widetilde{y}\right)(x) = \lambda\widetilde{y}(x), \quad -\infty < x < \infty.$$

The spectrum of this operator is the set $[1/2, \infty)$.

Remark 3. Actually, in Theorems 4 and 5, we use the characterization of the multipliers acting in weighted Hilbert spaces. More general results can be found in [6]. In the general case the concept of capacity must be introduced. In our situation we can do without it. Here we have presented the principal moments of the proof about multipliers, in order to underline the difference between the assumptions of Theorems 4 and 5.

5. The ground state estimation

In this section we estimate the smallest eigenvalue. The classical results on this topic can be found in [10].

By Theorem 1, all the potentials $A_{(x_0,a_0)}p(x)$ generate an operator with the same spectrum. So, the condition

$$\int_{-1}^{1} xp(x)dx = 0$$

in the next theorem does not result in any loss of generality (see Lemma 1).

Theorem 6. *Let $0 < \lambda_1 < \cdots < \lambda_n < \dots$ be the spectrum of the problem*

$$-y''(x) = \lambda p(x) y(x), \quad -1 < x < 1,$$

with the boundary conditions $y(-1) = y(1) = 0$ in the space $L_2(p(x)dx)$, where $0 < p(x) \in L_1(-1,1)$, $-1 < x < 1$, and

$$\int_{-1}^{1} x p(x) dx = 0.$$

Then

$$\lambda_1 < \frac{4e}{(e-1)\int_{-1}^{1}(1-x^2)p(x)dx} \ln\left(\frac{2\int_{-1}^{1} p(x)dx}{\int_{-1}^{1}(1-x^2)p(x)dx}\right).$$

Proof. Let $\lambda_1 > 0$ be the smallest eigenvalue and $y(x) > 0$, $-1 < x < 1$, the corresponding eigenfunction. Since $y'(-1) > 0$, $y'(1) < 0$, we have

$$\frac{\lambda_1}{2}\int_{-1}^{1} p(x)(1-x^2)dx = -\frac{1}{2}\int_{-1}^{1} \frac{(1-x^2)y''(x)}{y(x)} dx =$$

$$= 2 + \frac{1}{2}\int_{-1}^{1} y'(x) \frac{d}{dx}\left(\frac{1-x^2}{y(x)}\right) dx =$$

$$= 2 - \frac{1}{2}\int_{-1}^{1}\left(\sqrt{1-x^2}\frac{y'(x)}{y(x)} + \frac{2x}{\sqrt{1-x^2}}\right)^2 dx +$$

$$+ \int_{-1}^{1} x\left(\frac{y'(x)}{y(x)} + \frac{2x}{1-x^2}\right) dx < 2 + \ln\left(\frac{-y'(-1)y'(1)}{4}\right) - \int_{-1}^{1} \ln\frac{y(x)}{1-x^2} dx.$$

For some point $-1 < x_0 < 1$ we have $y(x_0) = \max_{|x|\le 1} y(x)$. The inequality $y''(x) \le 0$, $-1 < x < 1$, implies

$$y(x) \ge y(x_0)\frac{1+x}{1+x_0}, \quad -1 \le x \le x_0,$$

and

$$y(x) \geq y(x_0)\frac{1-x}{1-x_0}, \quad x_0 \leq x \leq 1.$$

So,

$$\frac{1}{2}\int_{-1}^{1} \ln \frac{(1-x^2)}{y(x)} dx \leq$$

$$\leq \frac{1}{2}\int_{-1}^{x_0} \ln \frac{(1-x)(1+x_0)}{y(x_0)} dx + \frac{1}{2}\int_{x_0}^{1} \ln \frac{(1+x)(1-x_0)}{y(x_0)} dx =$$

$$= \ln \frac{1}{y(x_0)} + 2\ln 2 - 1.$$

We have also

$$2y'(-1) = -\int_{-1}^{1} y''(x)(1-x)dx = \lambda_1 \int_{-1}^{1} (1-x)p(x)y(x)dx$$

and

$$-2y'(1) = -\int_{-1}^{1} y''(x)(1+x)dx = \lambda_1 \int_{-1}^{1} (1+x)p(x)y(x)dx.$$

Consequently,

$$2\exp\left\{\frac{\lambda_1}{4}\int_{-1}^{1} p(x)(1-x^2)dx\right\} < 2\frac{\sqrt{-y'(1)y'(-1)}}{y(x_0)} =$$

$$= \lambda_1 \left(\int_{-1}^{1} (1-x)p(x)\frac{y(x)}{y(x_0)}dx \int_{-1}^{1} (1+x)p(x)\frac{y(x)}{y(x_0)}dx\right)^{1/2} <$$

$$< \lambda_1 \left(\int_{-1}^{1} p(x)dx\right). \tag{11}$$

Denoting

$$s = \frac{\lambda_1}{4}\int_{-1}^{1} p(x)(1-x^2)dx$$

and

$$C = \left(2\int_{-1}^{1} p(x)dx\right)\left(\int_{-1}^{1}(1-x^2)p(x)dx\right)^{-1},$$

we can rewrite (11) in the form

$$\frac{e^s}{s} < C, \quad 0 < s.$$

We have $\inf_s \dfrac{e^s}{s} = e < C$, and the function e^s/s is monotone increasing for $1 \le s$. For the value

$$s_0 = \frac{e}{e-1}\ln C \ge 1$$

we have $\dfrac{e^{s_0}}{s_0} > C$. So, $s \le s_0 = \dfrac{e}{e-1}\ln C$, which is equivalent to the claim of the theorem. □

6. Inverse problem

In this section, we generalize the following result of Krein [2].

Theorem 7. *Let $\lambda_1 < \lambda_2 < \ldots$ be the eigenvalues of the problem*

$$-y''(x) = \lambda p(x)y(x), \quad -1 \le x \le 1, \quad y(-1) = y(1) = 0,$$

where $0 < p(x)$, $p(-x) = p(x)$, and $p(x) \in L_1(-1,1)$. Then there is no other function satisfying the same conditions and such that it generates an operator with the same spectrum.

The main results of this section are described in the following two theorems.

Theorem 8. *Suppose $p(x) > 0$ and $p(x) \in L_1(-1,1)$. Let two real numbers a, b satisfy the condition*

$$(1-(a+1)^2)(1-(b+1)^2) > 0.$$

Let $\lambda_1 < \lambda_2 < \ldots$ be the spectrum of the problem

$$-y''(x) = \lambda p(x)y(x), \quad y(-1) - ay'(-1) = 0, \quad y(1) + by'(1) = 0. \tag{12}$$

Then the potential function

$$p_1(x) = \frac{(1-x_0^2)^2}{(1-xx_0)^4}\, p\left(\frac{x_0 - x}{1 - xx_0}\right), \quad -1 < x < 1,$$

where $x_0 = (a-b)/(a+b+ab)$ *(we put* $x_0 = 0$ *if* $a = b$*) generates an operator with the same spectrum.*

Proof. We note that $x_0 \in (-1, 1)$. Indeed,

$$1 - x_0^2 = \frac{(1-(a+1)^2)(1-(b+1)^2)}{(a+b+ab)^2} > 0.$$

If $y_n(x)$ is an eigenfunction of the operator (1), (2), then the function

$$z_n(x) = (1 - xx_0)y_n\left(\frac{x_0 - x}{1 - xx_0}\right)$$

is an eigenfunction of the operator with the potential function $p_1(x)$ in place of $p(x)$. Indeed, for the boundary conditions we have

$$z(-1) - \frac{b(1+x_0)}{1 - x_0 - bx_0}z'(-1) = 0, \quad z(1) + \frac{a(1-x_0)}{1 + x_0 + ax_0}z'(1) = 0.$$

Note that

$$a = \frac{b(1+x_0)}{1 - x_0 - bx_0}, \quad b = \frac{a(1-x_0)}{1 + x_0 + ax_0}$$

if

$$x_0 = \frac{a-b}{a+b+ab}. \qquad \square$$

Theorem 9. *Suppose two real numbers* a, b *satisfy the condition*

$$(1-(a+1)^2)(1-(b+1)^2) > 0,$$

and put

$$x_0 = \frac{a-b}{a+b+ab} \in (-1, 1)$$

and $x_0 = 0$ *if* $a = b$*. Let* $p(x) > 0$ *be a continuous differentiable function satisfying the condition*

$$(1-x^2)^2 p(x) = (1-y^2)^2 p(y) \tag{13}$$

for arbitrary $x, y \in (-1, 1)$ *with* $\dfrac{x+y}{1+xy} = x_0$*. Let* $\lambda_1 < \lambda_2 < \ldots$ *be the spectrum of the operator*

$$-y''(x) = \lambda p(x)y(x), \quad y(-1) - ay'(-1) = 0, \quad y(1) + by'(1) = 0.$$

Then there is no other potential function $p(x)$ satisfying the above conditions and generating an operator with the same spectrum.

Proof. Let $p_1(x)$ and $p_2(x)$ be two potential functions that generate the same spectrum $\lambda_1 < \lambda_2 < \ldots$. We denote by $\varphi_j(x,\lambda)$, $j = 1,2$, unique solutions of the equations

$$-\varphi_j''(x,\lambda) = \lambda p_j(x)\varphi_j(x,\lambda), \quad -1 \le x \le 1, \quad j = 1,2,$$

with the same boundary conditions $\varphi_j(-1,\lambda) = a$, $\varphi_j'(-1,\lambda) = 1$. We denote by $\psi_j(x,\lambda)$, $j = 1,2$, a unique solution of the equation

$$-\psi_j''(x,\lambda) = \lambda p_j(x)\psi_j(x,\lambda), \quad -1 \le x \le 1,$$

with the boundary conditions $\psi_j(1,\lambda) = -b$, $\psi_j'(1,\lambda) = 1$. Since for both cases $j = 1,2$ we have the same spectrum, we must have

$$\varphi_j(1,\lambda_n) + b\varphi_j'(1,\lambda_n) = 0, \;\; \psi_j(-1,\lambda_n) - a\psi'(-1,\lambda_n) = 0, \;\; n = 0,1,2,\ldots.$$

The functions $\psi_j(x,\lambda_n)$ and $\psi_j(-1,\lambda_n)\varphi_j(x,\lambda_n)$ satisfy the same boundary conditions at the endpoint -1. So,

$$\psi_j(x,\lambda_n) = \psi_j'(-1,\lambda_n)\varphi_j(x,\lambda_n), \quad n = 0,1,2,\ldots.$$

Comparing the boundary conditions at the endpoint 1, we get

$$\varphi_j(x,\lambda_n) = \varphi_j'(1,\lambda_n)\psi_j(x,\lambda_n), \quad n = 0,1,2,\ldots.$$

By assumption, we have

$$p_j(x) = \frac{(1-x_0^2)^2}{(1-xx_0)^4} p_j\left(\frac{x_0 - x}{1 - xx_0}\right), \quad -1 < x < 1.$$

So, the function

$$(1 - xx_0)\varphi_j\left(\frac{x_0 - x}{1 - xx_0}, \lambda_n\right), \quad -1 < x < 1,$$

is also an eigenfunction corresponding to the eigenvalue λ_n. Since each eigenvalue λ_n is simple, there is a constant c_n such that

$$c_n\varphi_j(x,\lambda_n) = (1 - xx_0)\varphi_j\left(\frac{x_0 - x}{1 - xx_0}, \lambda_n\right), \quad -1 < x < 1.$$

Since $\varphi_j(-1,\lambda_n) = a$, we have $ac_n = (1 + x_0)\varphi_j(1,\lambda_n)$ and $\varphi_j(1,\lambda_n)c_n = (1 - x_0)a$. Consequently,

$$\varphi_j^2(1,\lambda_n) = a^2\frac{1 - x_0}{1 + x_0}.$$

By Sturm's comparison theorem, the eigenfunction $\varphi_j(x, \lambda_n)$ has exactly n, $n = 0, 1, 2, \ldots$, zeros in the interval $(-1, 1)$. So, $\varphi_j(1, \lambda_n) = -\varphi_j(1, \lambda_{n+1})$, $n = 0, 1, 2, \ldots$. Consequently,

$$\varphi_j(x, \lambda_n) = (-1)^n \varphi_j'(1, \lambda_0)\psi_j(x, \lambda_n), \quad n = 0, 1, 2, \ldots.$$

Since the two functions

$$\varphi_1(x, \lambda_0), \quad \varphi_2(x, \lambda_0), \quad x \in (-1, 1)$$

have no zeros, $\varphi_1(-1, \lambda_0) = \varphi_2(-1, \lambda_0)$, and $|\varphi_1(1, \lambda_0)| = |\varphi_2(1, \lambda_0)|$, we see that $\varphi_1(1, \lambda_0) = \varphi_2(1, \lambda_0)$. Denote

$$D(\lambda) = (\lambda - \lambda_0) \prod_{n=1}^{\infty} \left(\frac{\lambda_n - \lambda}{\pi^2 n^2} \right).$$

Following E. Trubowitz [3], we consider the integral

$$I_R(x, y) = \int_{C_R} \frac{(\varphi_1(x, \lambda) - \varphi_2(y, \lambda))(\psi_1(x, \lambda) - \psi_2(y, \lambda))}{D(\lambda)} d\lambda,$$

where $C_R = \{\lambda;\ |\lambda| = R\}$. By Cauchy's residue theorem, we have

$$I_R(x, y) = \sum_{|\lambda_n| < R} \frac{(\varphi_1(x, \lambda_n) - \varphi_2(y, \lambda_n))(\psi_1(x, \lambda_n) - \psi_2(y, \lambda_n))}{D'(\lambda_n)} =$$

$$= \varphi_1'(1, \lambda_0) \sum_{|\lambda_n| < R} \frac{(\varphi_1(x, \lambda_n) - \varphi_2(y, \lambda_n))^2}{(-1)^n D'(\lambda_n)}.$$

Since the zeros of the entire function $D(\lambda)$ are simple, we obtain

$$(-1)^n D'(\lambda_n) < 0, \quad n = 0, 1, 2, \ldots.$$

By some known estimates (see [4, p. 13]), we have

$$|((\varphi_1(x, \lambda) - \varphi_2(y, \lambda))(\psi_1(x, \lambda) - \psi_2(y, \lambda))| =$$

$$= O\left(\frac{1}{\sqrt{|\lambda|^3}} \exp\left\{ \sqrt{|\operatorname{Im}(\lambda)|} \int_{-1}^{1} \sqrt{p_1(t)}\, dt \right\} \right),$$

where the numbers x, y are related by the formula

$$\int_{-1}^{x} \sqrt{p_1(t)}\, dt = \int_{-1}^{y} \sqrt{p_2(t)}\, dt. \tag{14}$$

Taking the asymptotic behavior of the eigenvalues λ_n into account, we get $\liminf_{R\to\infty} |I_R(x,y)| = 0$. Consequently, $\varphi_1(x,\lambda_n) = \varphi_2(y,\lambda_n)$, $n = 0,1,2,\ldots$, for arbitrary pares $x, y = y(x)$ satisfying (14). So, we have

$$\varphi_1'(x,\lambda_n) = \varphi_2'(y(x),\lambda_n)y'(x), \quad n = 0,1,2,\ldots$$

Since $y'(x)\sqrt{p_2(y(x))} = \sqrt{p_1(x)}$, it follows that

$$\frac{\varphi_1'(x,\lambda_n)}{\sqrt{p_1(x)}} = \frac{\varphi_2'(y(x),\lambda_n)}{\sqrt{p_2(y(x))}}, \quad n = 0,1,2,\ldots.$$

Consequently,

$$\frac{2\varphi_1''(x,\lambda_n)p_1(x) - \varphi_1(x,\lambda_n)p_1'(x)}{p_1^2(x)} =$$

$$= \frac{2\varphi_2''(y(x),\lambda_n)p_2(y(x)) - \varphi_2(y(x),\lambda_n)p_2'(y(x))}{p_2^2(y(x))}, \quad n = 0,1,2,\ldots,$$

or

$$\frac{-2\lambda_n\varphi_1(x,\lambda_n)p_1^2(x) - \varphi_1(x,\lambda_n)p_1'(x)}{p_1^2(x)} =$$

$$= \frac{-2\lambda_n\varphi_2(y(x),\lambda_n)p_2^2(y(x)) - \varphi_2(y(x),\lambda_n)p_2'(y(x))}{p_2^2(y(x))}, \quad n = 0,1,2,\ldots.$$

But $\varphi_1'(x,\lambda_n) = \varphi_2'(y,\lambda_n)y'(x)$, $n = 0,1,2,\ldots$, whence

$$\frac{d\ln p_1(x)}{dx} = \frac{d\ln p_2(y(x))}{dx}, \quad -1 < x < 1,$$

and $p_1(x) = cp_2(y(x))$, $-1 < x < 1$, for some constant c. From the definition of the function $y(x)$, we have

$$c\int_{-1}^{x} \sqrt{p_2(y(t))}\,dt = \int_{-1}^{y(x)} \sqrt{p_2(t)}\,dt, \quad -1 < x < 1.$$

Differentiating, we get

$$c\sqrt{p_2(y(x))} = y'(x)\sqrt{p_2(y(x))}, \quad -1 < x < 1.$$

Consequently, $y(x)$ is a linear function. Since $y(-1) = -1$, $y(1) = 1$, we see that $y(x) = x$, $-1 < x < 1$. Thus, $p_1(x) = p_2(x)$, $-1 < x < 1$. $\square$

7. Examples for the exceptional cases of the boundary conditions

In the examples given below the next lemma plays an essential role.

Lemma 3. *Suppose that $p(x)$ and $\psi(t)$ satisfy the following conditions:*
(1) *the function $p(x) \in L_1(-1,1)$ is differentiable and nonnegative;*
(2) *the function $\psi(t)$, $-1 < t < 1$, has positive derivative at each point $t \neq 0$ and $\psi(-1) = -1$, $\psi(1) = 1$;*
(3) *we have*

$$2t^2 p(t) = \psi'(t) \left(\int_{-1}^{1} x^2 p(x) dx \right), \quad -1 < t < 1.$$

Suppose the second order derivative of the function $y(x)$, $-1 < x < 1$, is absolutely continuous and for some constant λ we have $-y''(x) = \lambda p(x) y(x)$, $-1 < x < 1$. Then the function

$$z(x) = y(\varphi(x)) - \varphi(x) y'(\varphi(x)), \quad -1 < x < 1,$$

satisfies the equation

$$-z''(x) = \lambda (\varphi'(x))^2 p(x) z(x), \quad -1 < x < 1,$$

where $\varphi(x)$ is the inverse of $\psi(t)$.

Proof. Since $\psi'(\varphi(x))\varphi'(x) = 1$, $-1 < x < 1$, assumption 3 of Lemma 3 can be rewritten in the form

$$\frac{d(\varphi^2(x)\varphi'(x)p(\varphi(x)))}{dx} = 0, \quad -1 < x < 1.$$

Consequently,

$$-z''(x) = \frac{d(\varphi(x)\varphi'(x)y''(\varphi(x)))}{dx} =$$

$$= -\lambda \frac{d(\varphi(x)\varphi'(x)p(\varphi(x))y(\varphi(x)))}{dx} = \lambda(\varphi'(x))^2 p(x) z(x), \quad -1 < x < 1. \ \square$$

Remark 5. At the boundary points we have $z(-1) = y(-1) + y'(-1)$, $z(1) = y(1) - y'(1)$. So, if for the function $y(x)$ we have $y(-1) = 0$, $y(1) = 0$, then the corresponding function $z(x)$ satisfies $z'(-1) = 0$, $z'(1) = 0$, because $y''(-1) = 0$ and $y''(1) = 0$.

Example 1. The functions $p(x) = 1$, $-1 < x < 1$, and $\psi(t) = t^3$, $-1 < t < 1$, satisfy the conditions of Lemma 3.

We have $\varphi(x) = \sqrt[3]{x}$, $-1 < x < 1$, and

$$\left(\varphi'(x)\right)^2 p(\varphi(x)) = \frac{1}{9\sqrt[3]{x^4}}, \quad -1 < x < 1.$$

It is well known that the eigenvalue problem

$$-y''(x) = \lambda y(x), \quad -1 < x < 1,$$

$$y(-1) = 0, \quad y(1) = 0,$$

has nontrivial solutions if $\lambda = \frac{\pi^2 n^2}{4}$, $n = 1, 2, \ldots$, and the corresponding eigenfunctions are

$$y_n(x) = \sin\left(\frac{\pi n(x+1)}{2}\right), \quad n = 1, 2, \ldots.$$

By Lemma 3, the same numbers $\lambda = \frac{\pi^2 n^2}{4}$, $n = 1, 2, \ldots$, are eigenvalues for the equation

$$-9\sqrt[3]{x^4} z''(x) = \lambda z(x), \quad -1 < x < 1, \quad z'(-1) = 0, \quad z'(1) = 0$$

with the following eigenfunctions:

$$z_n(x) = \sin\left(\frac{\pi n(\sqrt[3]{x}+1)}{2}\right) - \\ - \frac{\pi n\sqrt[3]{x}}{2}\cos\left(\frac{\pi n(\sqrt[3]{x}+1)}{2}\right), \quad n = 1, 2, \ldots.$$

We note that $\lambda = 0$ is also an eigenvalue, and the corresponding eigenfunction is $z_0(x) = 1$.

In $L_2(-1,1)$, the operator in question has no other eigenvalues at all.

On the other hand, the eigenvalue problem

$$-z''(x) = \lambda z(x), \quad -1 < x < 1, \quad z'(-1) = 0, \quad z'(1) = 0$$

has the same eigenvalues $\lambda = \dfrac{\pi^2 n^2}{4}$, $n = 0, 1, 2, \ldots$.

Consequently, in the space $L_2(-1,1)$, two different even potential functions with the same boundary conditions lead to the same spectrum. Note that the potential function $9\sqrt[3]{x^{-4}}$ does not belong to the space $L_1(-1,1)$.

Example 2. Consider the eigenvalue problem

$$-|x|^{\sigma} z''(x) = \lambda z(x), \quad -1 < x < 1,$$

with the boundary conditions

$$z(-1) = 0, \quad z(1) = 0,$$

where $1 < \sigma < 2$. It should be emphasized that the potential function in this example does not belong to the space $L_1(-1,1)$.

For an arbitrary $x_0 \in (-1,1)$, the eigenvalue problem

$$-z''(x) = \lambda \left(\frac{(1-x_0^2)^2}{(1-xx_0)^{4-\sigma}|x-x_0|^{\sigma}} \right) z(x), \quad -1 < x < 1,$$

with the boundary conditions

$$z(-1) = 0, \quad z(1) = 0,$$

has the same spectrum.

We define a function $\varphi(x)$, $-1 < x < 1$, by the formulas

$$\varphi(x) = \left(-1 - \int_{-1}^{x} \frac{(1-x_0^2)^2}{(1-tx_0)^{4-\sigma}|t-x_0|^{\sigma}} dt \right)^{-1}, \quad -1 < x < x_0,$$

and

$$\varphi(x) = \left(1 + \int_{x}^{1} \frac{(1-x_0^2)^2}{(1-tx_0)^{4-\sigma}|t-x_0|^{\sigma}} dt \right)^{-1}, \quad x_0 < x < 1.$$

Using this function, we find the potential functions $p_{x_0}(x)$, $-1 < x < 1$, from the formula

$$(\varphi'(x))^2 p_{x_0}(\varphi(x)) = \frac{(1-x_0^2)^2}{(1-xx_0)^{4-\sigma}|x-x_0|^{\sigma}}, \quad -1 < x < 1.$$

Note that the functions $p_{x_0}(x)$, $-1 < x < 1$, are not continuous on $[-1,1]$.

Consequently, the eigenvalue problems

$$-y''(x) = \lambda p_{x_0}(x) y(x), \quad -1 < x < 1,$$
$$y(-1) + y'(-1) = 0, \quad y(1) - y'(1) = 0,$$

have the same discrete spectrum for each $-1 < x_0 < 1$.

This example shows that, in spite of the condition $p_0(-x) = p_0(x)$, $-1 < x < 1$, there are different potential functions with the same discrete spectrum. This phenomenon is not possible for the operators (7), see [3].

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