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ASYMPTOTIC OPTIMALITY OF THE BAYES ESTIMATOR ON DIFFERENTIABLE IN QUADRATIC MEAN MODELS

Abstract. This paper deals with the study of the Bayes estimator's asymptotic properties on differentiable in quadratic mean (DQM) models in the case of independent and identically distributed observations. The investigation is led in order to define weak assumptions on the model under which this estimator is asymptotically efficient, regular and asymptotically of minimal risk. The results of the paper are applied to models based on a mixture distribution, the Cauchy with location and scale parameter's and the Weibull's.

1. Introduction and notations

We consider independent and identically distributed observations on an Euclidian space $\mathfrak{X}$, from an unknown probability distribution $\mathbf{P}_\theta$ belonging to a parametric family $\mathcal{P} = \{\mathbf{P}_\theta : \theta \in \Theta\}$ where Θ is an open subset of $\mathbb{R}^r$. The models are supposed to be identifiable, that is the application $\theta \mapsto \mathbf{P}_\theta$ is a bijection from Θ to $\mathcal{P}$, and dominated by a σ-finite measure μ, equivalent to the familly $\mathcal{P}$. Let f_θ be the density of $\mathbf{P}_\theta$ with respect to μ. The distribution of a sample $X_1, \cdots, X_n$ is denoted by $\mathbf{P}_\theta^n$ and the model associated is thus $\mathcal{M}_n = (\mathfrak{X}^n, \mathcal{B}(\mathfrak{X}^n), \{\mathbf{P}_\theta^n : \theta \in \Theta\})$ where $\mathcal{B}(\mathfrak{X}^n)$ is the Borel σ-field of $\mathfrak{X}^n$. We denote by $\mathcal{M}$ the model based on one observation, by $\mathbf{E}_\theta$ the expectation under the $\mathbf{P}_\theta^n$-distribution, and by $D_\theta \varphi_{\theta_0}(X)$, the differential at θ_0, regarding to the variable θ, of the two variables map φ. We use on $\mathbb{R}^r$ the max-norm, that is $\|a\| = \|(a_1, \cdots, a_r)\| = \max\{|a_1|, \ldots, |a_r|\}$, where the associated balls are cubes. The notation $o_{\mathbf{P}}(1)$ designates a sequence of random vectors that convergences to zero in $\mathbf{P}$-probability and $O_{\mathbf{P}}(1)$, a sequence that is bounded in $\mathbf{P}$-probability. At last, by convention we pose $\sup_\emptyset \varphi = 0$ for any nonnegative function φ.

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For statistical models generated by i.i.d. observations, as common as those with scale and location ones, it is not always simple to study the asymptotic properties of classical estimators such as the maximum likelihood's or the Bayes'. The difficulty is so much more important on general models, notably when the parameter space Θ is multidimensionnal and/or not bounded. I. A. Ibragimov and R. Z. Hasminskii have introduced some conditions leading to the asymptotic optimality of the two previous estimators (see [4]). More precisely, under an assumption of quadratic mean continuity of the map $\theta \mapsto D_\theta\sqrt{f_\theta}$, and supposing that the Fisher information and his inverse are globally bounded on Θ, these authors notably show that some Bayes estimators relative to a loss function are regular and asymptotically efficient with respect to the same loss function. Although theorically important and founding this approach doesn't allow us to treat many models such as, for instance, those based on the Cauchy with scale and location parameter's distribution. The principal reason is that the condition (3.1) in [4, Chap. III] is not satisfied which means that the Fisher information is not globally bounded on Θ. This is due to its explosive behavior in a neighbourhood of the parameter space's boundary. In fact, most of the time, the problem is a finite distance one, as in the previous Cauchy's model for which the Fisher information is not bounded when the scale parameter is in a neighbourhood of 0. Moreover, the weaker condition (N3) in the same chapter fails too, which besides is a real difficulty in the study of the maximum likelihood estimator. However, this problem can be overcome in the framework of the Bayes estimator thanks to its "integral" character. This is the object of this article.

Another way to study the asymptotic properties of the Bayes estimator on DQM models consists in the utilization of the Bernstein–von Mises theorem, which is based notably on the existence of a sequence of tests asymptotically uniformly consistent, and shows that the posterior distribution is, for the total variation norm, asymptotically normal with an optimal asymptotic variance (see [9]). As a consequence of this theorem, it can be shown in particular that the Bayes estimators relative to some sub-convex loss functions are asymptotically efficient. Here again, the bad behavior of the Fisher information is a cause of the non existence of such a sequence of tests which makes this result inapplicable.

One of the authors has proposed, for DQM models, some weak assumptions under which the maximum likelihood estimator has optimal asymptotic properties, that is asymptotic efficiency, regularity and asymptotic minimaxity (see [ ]10). This paper proceeds to do a similar work on the Bayes estimator. We notably show that modulo assumptions on the behavior of the Fisher information, this estimator is asymptotically efficient, regular, asymptotically of minimal risk and asymptotically minimax. We conclude with an application of these results on three distributions: a mixture one, the Cauchy with scale and location parameter's and the Weibull's.

The well known notion of local asymptotic normality, introduced originally by L. Le Cam (see [5] and [6]), characterizes the models which have an asymptotic behavior close to the Gaussian's, in a neighbourhood of θ of size $O\left(\frac{1}{\sqrt{n}}\right)$. More precisely, the parameter θ is fixed and one wants to approximate, by a Gaussian distribution, the absolutely continuous component of the measure $\mathbf{P}^n_{\theta+\frac{1}{\sqrt{n}}t}$ with respect to $\mathbf{P}^n_\theta$ (where $t \in \mathbb{R}^r$ is the new parameter and $\theta_{n,t} = \theta + \frac{1}{\sqrt{n}}t$). Denote by $\Psi_{\theta,t}$ the local likelihood ratio $\prod_{i=1}^n \frac{f_{\theta_{n,t}}}{f_\theta}(X_i)$, defined on $\{\prod_{i=1}^n f_\theta(X_i) > 0\}$. Then for all $\theta \in \Theta$ and $t \in \sqrt{n}(\Theta - \theta)$, $\mathbf{P}^n_{\theta_{n,t}} = \Psi_{\theta,t}\mathbf{P}^n_\theta + 1_{\{\prod_{i=1}^n f_\theta(X_i)=0\}}\mathbf{P}^n_{\theta_{n,t}}$ and $\Psi_{\theta,t}$ is defined $\mathbf{P}^n_\theta$-almost everywhere. A model is called *locally asymptotically normal* (LAN) at $\theta \in \Theta$ if there exists a random vector $S_{n,\theta}$ on $(\mathfrak{X}^n, \mathcal{B}(\mathfrak{X}^n), \mathbf{P}^n_\theta)$, which takes his values in $\mathbb{R}^r$, and a nonsingular (r, r) matrix $M(\theta)$ such that for all $t \in \mathbb{R}^r$

$$\log \Psi_{\theta,t} = t' S_{n,\theta} - \frac{1}{2} t' M(\theta) t + R_{\theta,t}$$

where $S_{n,\theta}$ converges in $\mathbf{P}^n_\theta$-distribution to $\mathcal{N}(0, M(\theta))$ and $R_{\theta,t} = o_{\mathbf{P}^n_\theta}(1)$. Le Cam has proposed (see [7]) sufficient assumptions of local asymptotic normality, using an argument of differentiability in quadratic mean. More precisely, a model is called *differentiable in quadratic mean* (DQM) at $\theta \in \Theta$ if there exists a map $D_\theta\sqrt{f_\theta} : \mathfrak{X} \to \mathbb{R}^r$ such that, as $h \to 0$,

$$\left\| \sqrt{f_{\theta+h}} - \sqrt{f_\theta} - \frac{1}{2} h' \left(D_\theta \sqrt{f_\theta}\right) \sqrt{f_\theta} \right\|_{L^2(\mu)} = o\left(\|h\|^2\right), \tag{1.1}$$

where $L^2(\mu)$ denotes the space of measurable maps $\psi : \mathfrak{X} \to \mathbb{R}$ satisfying $\|\psi\|_{L^2(\mu)} := \int_{\mathfrak{X}} [\psi(x)]^2 \, d\mu(x) < \infty$. Le Cam has shown (see [7]) that if a model is DQM at $\theta \in \Theta$, it is LAN at θ. When the model is DQM, then for μ-almost every x in $\{x \in \mathfrak{X} : f_\theta(x) = 0\}$, $D_\theta\sqrt{f_\theta(x)} = 0$. It is thus possible, by analogy with the natural definition of the differential of the log function, to define μ-almost everywhere $D_\theta \log f_\theta = \frac{2}{\sqrt{f_\theta}} D_\theta \sqrt{f_\theta}$ (with the convention $\frac{0}{0} = 0$). Therefore, $S_{n,\theta} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} D_\theta \log f_\theta (X_i)$, $\mathbf{P}_\theta^n$-almost everywhere and $M(\theta) = I(\theta) = \mathbf{E}_\theta (D_\theta \log f_\theta (X))^{\otimes 2}$ is the Fisher information of the model $\mathcal{M}$.

Recall that the Hellinger distance on $\mathcal{M}$, denoted by d, is defined as

$$d(\theta_1, \theta_2) = \left\| \sqrt{f_{\theta_1}} - \sqrt{f_{\theta_2}} \right\|_{L^2(\mu)}, \quad \forall (\theta_1, \theta_2) \in \Theta^2.$$

Since $\mathcal{M}$ is identifiable, d is a distance on Θ, bounded above by $\sqrt{2}$. This distance is called *Lipschitz* on Θ (respectively, *locally Lipschitz* at $\theta \in \Theta$) if the map $u \mapsto \sqrt{f_u}$ is Lipschitz on Θ (respectively, locally Lipschitz at θ) according to the $L^2(\mu)$-norm. Note that if the Hellinger distance is locally Lipschitz, it is Lipschitz on all the compacts. We can note otherwise that if $\mathcal{M}$ is a DQM model and I is locally bounded then d is locally Lipschitz at any $\theta \in \Theta$. More precisely, by the mean value theorem, for all $(\theta_1, \theta_2) \in \Theta^2$ such that $[\theta_1, \theta_2] \subset \Theta$

$$d(\theta_1, \theta_2) \leq \frac{1}{2} \sup_{u \in]\theta_1, \theta_2[} \sqrt{\|I(u)\|}\, \|\theta_1 - \theta_2\| \tag{1.2}$$

To deal with the Bayes estimator, we consider on Θ a σ-finite measure ν that possesses an absolutely continuous and positive density π with respect to the restriction of the Lebesgue measure on Θ. It is well known that if ν is a probability measure (the case in our study), then the Bayes estimator relative to the quadratic loss function is $\mathbf{P}_\theta^n$-almost everywhere of the form $\widehat{\theta}_n = \mathbf{E}(\Delta | (X_1, \cdots, X_n))$ where Δ is a random vector of probability distribution ν and for all $\theta \in \Theta$, $\mathbf{P}^{(X_1, \cdots, X_n)|\Delta=\theta} = \mathbf{P}_\theta$. Since there is no confusion on the loss function, we call this estimator simply, the Bayes estimator.

2. Asymptotic optimality

The asymptotic efficiency is studied here in the Fisher's sense. More precisely, if the model $\mathcal{M}$ is DQM, an asymptotically Gaussian estimator δ_n on $\mathcal{M}_n$ of the parameter is called *asymptotically efficient* if for all $\theta \in \Theta$, its asymptotic variance, that is the variance of the limit distribution of $\sqrt{n}(\delta_n - \theta)$, equals the Rao–Cramér bound, $I^{-1}(\theta)$. Recall that this characteristic cannot be considered as asymptotically optimal if we don't restrict the study to the *regular* estimators. This term defines an estimator

δ_n such that for all $\theta \in \Theta$ and $t \in \mathbb{R}^r$, $\sqrt{n}\,(\delta_n - \theta_{n,t})$ converges in $\mathbf{P}^n_{\theta_{n,t}}$-distribution to a random vector of distribution depending on θ but not on t. In the regular class, the asymptotically efficient estimators have a minimal asymptotic variance.

The following proposition, which is a consequence of the Hájek–Le Cam's convolution theorem (see [3] and [8]), gives the exact expression with a precision of order $\frac{1}{\sqrt{n}}$, of asymptotically efficient and regular estimators in the DQM's context.

Proposition 1. *Let $\mathcal{M}$ be a DQM model such that I is nonsingular. An estimator δ_n on $\mathcal{M}_n$ is asymptotically efficient and regular iff*

$$\sqrt{n}\,(\delta_n - \theta) - I^{-1}(\theta)\, S_{n,\theta} = o_{\mathbf{P}^n_\theta}(1), \quad \forall \theta \in \Theta. \tag{2.1}$$

Proof. See [8] or [10]. □

According to Hájek's terminology in [2], an estimator which satisfies property (2.1) is called *best regular*. On DQM models this term is equivalent to asymptotic efficiency and regularity.

We do not use the previous form of the result but a version where we replace $S_{n,\theta}$, by the bounded statistic $S^{k_n}_{n,\theta} = S_{n,\theta} 1_{\{\|S_{n,\theta}\| < k_n\}}$ (where $(k_n)_n$ is an increasing sequence of natural numbers tending to the infinity). Since $S_{n,\theta} - S^{k_n}_{n,\theta}$ converges in $\mathbf{P}^n_\theta$-probability to zero, instead of showing assertion (2.1) we may prove the following one:

$$\sqrt{n}\,(\delta_n - \theta) - I^{-1}(\theta)\, S^{k_n}_{n,\theta} = o_{\mathbf{P}^n_\theta}(1), \quad \forall \theta \in \Theta. \tag{2.2}$$

The introduction of $S^{k_n}_{n,\theta}$ is inspired by Le Cam's works on the local approximation of experiments by exponential ones. This author shows notably that there exists an increasing sequence of natural numbers $(k_n)_n$ tending to the infinity such that for all $\theta \in \Theta$ and $t \in \mathbb{R}^r$

$$\lim_{n \to \infty} \mathbf{E}_\theta \left| \Psi_{\theta,t} - C_{k_n,\theta,t} \Phi_{k_n,\theta,t} \right| = 0, \tag{2.3}$$

where $\Phi_{k_n,\theta,t} = \exp\left(t' S^{k_n}_{n,\theta} - \frac{1}{2} t' I(\theta)\, t \right)$ and $C_{k_n,\theta,t}$ is the normalisation's constant of the density $C_{k_n,\theta,t}\Phi_{k_n,\theta,t}$ of a probability distribution $\mathbb{Q}_{k_n,\theta,t}$, with respect to $\mathbf{P}^n_\theta$. In other words, the models based on the two families $\left\{ \mathbf{P}^n_{\theta_{n,t}}, t \in \sqrt{n}\,(\Theta - \theta) \right\}$ and $\{\mathbb{Q}_{k_n,\theta,t}, t \in \sqrt{n}\,(\Theta - \theta)\}$ are asymptotically equivalent according to the total variation norm.

The two asymptotic characteristics which are asymptotic efficiency and regularity, although of practical interest in the research of good estimators, are not totally satisfactory. Indeed, they are only based on the asymptotic expectation and variance. To appreciate the asymptotic quality of an estimator, an approach using a loss function[1] is clearly more precise.

In this article we limit us to the loss functions ℓ_n on $\mathcal{M}_n$ of the form

$$\ell_n\left(d,\theta\right)=\ell\left(\sqrt{n}\left(d-\theta\right)\right),$$

where ℓ is belonging to $\mathcal{G}$, the set of functions which are nonnegative, even and sub-convex (that is the $A_\alpha=\{x\in\mathbb{R}^r:\ell(x)<\alpha\}$ is convex for all $\alpha>0$). We denote in this context by $\mathcal{G}_q$ $(q>0)$ the subset of $\mathcal{G}$ containing the functions ℓ for which there exists $\widetilde{q}\in\left]0,q\right[$ such that $\ell(x)=O\left(\|x\|^{\widetilde{q}}\right)$, and by $\widetilde{\mathcal{G}}$ the set $\bigcup_{q>0}\mathcal{G}_q$.

In this framework we define the notion of *asymptotically of minimal risk* estimator, which can be considered as an optimal asymptotic property in the class of regular estimators.

Definition 1. *Let ℓ be a function belonging to $\mathcal{G}$. We say that δ_n, a regular estimator of the parameter on a model $\mathcal{M}_n$, is ℓ-asymptotically of minimal risk if for any regular estimator $\widetilde{\delta}_n$ of the parameter on $\mathcal{M}_n$*

$$\limsup_{n\to\infty}\mathbf{E}_\theta\ell\left(\sqrt{n}\left(\delta_n-\theta\right)\right)\leq\liminf_{n\to\infty}\mathbf{E}_\theta\ell\left(\sqrt{n}\left(\widetilde{\delta}_n-\theta\right)\right),\quad\forall\theta\in\Theta.$$

It can be noticed that if δ_n is a ℓ-asymptotically of minimal risk estimator then $\mathbf{E}_\theta\ell\left(\sqrt{n}\left(\delta_n-\theta\right)\right)$ admits a limit when n tends to the infinity. Furthermore, if an estimator δ_n is ℓ-asymptotically of minimal risk for all $\ell\in\mathcal{G}_q$, then all the moments of $\sqrt{n}\left(\delta_n-\theta\right)$ whose order is lower than q converge to the moments of his asymptotic distribution.

The following result characterizes the ℓ-asymptotically of minimal risk estimators in the regular class.

Proposition 2. *Let $\mathcal{M}$ be a DQM model such that I is nonsingular, δ_n a regular estimator on $\mathcal{M}_n$ and ℓ belonging to $\mathcal{G}$.*

Then, for all $\theta\in\Theta$

$$\liminf_{n\to\infty}\mathbf{E}_\theta\ell\left(\sqrt{n}\left(\delta_n-\theta\right)\right)\geq\mathbf{E}\ell\left(Z_\theta\right),\tag{2.4}$$

[1]nonnegative measurable function w_n, defined on Θ^2 such that for all $\theta\in\Theta$, $w_n(\theta,\theta)=0$.

where the distribution of Z_θ *is* $\mathcal{N}\left(0, I^{-1}(\theta)\right)$.

If for all $\theta \in \Theta$

$$\liminf_{n\to\infty} \mathbf{E}_\theta \ell\left(\sqrt{n}\left(\delta_n - \theta\right)\right) = \mathbf{E}\ell\left(Z_\theta\right), \tag{2.5}$$

then δ_n *is asymptotically efficient and* ℓ*-asymptotically of minimal risk.*

Moreover, any estimator which is ℓ*-asymptotically of minimal risk satisfies* (2.5).

Proof. See [10] □

The *asymptotic minimaxity* is another way, including the notion of loss function, to characterize the asymptotic optimality of an estimator.

Definition 2. *Let* ℓ *be a function belonging to* $\mathcal{G}$. *We say that* δ_n, *an estimator of the parameter on a model* $\mathcal{M}_n$, *is* ℓ*-asymptotically minimax if for any estimator* $\widetilde{\delta}_n$ *of the parameter on* $\mathcal{M}_n$

$$\begin{aligned} &\lim_{\tau\to+\infty} \limsup_{n\to+\infty} \sup_{u\in\mathcal{B}\left(\theta,\frac{\tau}{\sqrt{n}}\right)} \mathbf{E}_u \ell\left(\sqrt{n}\left(\delta_n - u\right)\right) \\ \leq &\lim_{\tau\to+\infty} \liminf_{n\to+\infty} \sup_{u\in\mathcal{B}\left(\theta,\frac{\tau}{\sqrt{n}}\right)} \mathbf{E}_u \ell\left(\sqrt{n}\left(\widetilde{\delta}_n - u\right)\right), \forall\theta \in \Theta. \end{aligned}$$

Note that the asymptotic minimaxity is an optimal property in the class of all the estimators and not only in the class of regular ones.

By Proposition 1, if one defines some assumptions under which an estimator satisfies (2.2), the latter is, in particular, regular. To show that it is ℓ-asymptotically of minimal risk, it hence sufficies to verify that it satisfies (2.5). This is the approach that we follow to define a class of models on which the Bayes estimator possesses this property. Actually it is ℓ-asymptotically minimax under the same assumptions. It is mainly due to the fact that there exists a version of Proposition 2, in the framework of the asymptotic minimaxity, inspired by the Hájek's theorem (see [10, Theorem 2.6]).

3. Preamble

First, we set up some assumptions allowing us to the asymptotic efficiency and the regularity of the Bayes estimator, denoted by $\widehat{\theta}_n$. Recall that this estimator is a version of the conditional expectation

$\mathbf{E}\left(\Delta\right|\left(X_1,\cdots,X_n\right))$. Thus, using the notations of Sec. 2, we can write $\mathbf{P}_\theta^n$-almost everywhere

$$\sqrt{n}\left(\widehat{\theta}_n-\theta\right)-I^{-1}\left(\theta\right)S_{n,\theta}^{k_n}=\frac{\int\limits_{\sqrt{n}(\Theta-\theta)}\left(t-I^{-1}\left(\theta\right)S_{n,\theta}^{k_n}\right)\Psi_{\theta,t}\pi\left(\theta_{n,t}\right)dt}{\int\limits_{\sqrt{n}(\Theta-\theta)}\Psi_{\theta,t}\pi\left(\theta_{n,t}\right)dt}. \tag{3.1}$$

To show that $\widehat{\theta}_n$ satisfies assertion (2.2), it is therefore sufficient to state that in the right side the numerator is $o_{\mathbf{P}_\theta^n}(1)$, and the inverse of the denominator is $O_{\mathbf{P}_\theta^n}(1)$. Recall first the following result that presents the narrow link that exists between the Lipschitz property, according to $L^2\left(\mathbf{P}_\theta^n\right)$-norm, of the local likelihood ratio's square root, and the fact that the Fisher information is bounded.

Lemma 1. *Let $\mathcal{M}$ be a DQM model and $\theta\in\Theta$. If I is bounded in a convex neighbourhood $\widetilde{\Theta}$ of θ, included in Θ, then the map $t\mapsto\sqrt{\Psi_{\theta,t}}$ is Lipschitz on $\sqrt{n}\left(\widetilde{\Theta}-\theta\right)$ according to $L^2\left(\mathbf{P}_\theta^n\right)$-norm, with the Lipschitz constant $\frac{1}{2}\sup_{u\in\widetilde{\Theta}}\sqrt{\|I\left(u\right)\|}$.*

Proof. It suffices to note that for all $\theta\in\Theta$ and $(t_1,t_2)\in\sqrt{n}\left(\Theta-\theta\right)$, $\left\|\sqrt{\Psi_{\theta,t_1}}-\sqrt{\Psi_{\theta,t_2}}\right\|_{L^2\left(\mathbf{P}_\theta^n\right)}\leq\sqrt{n}d\left(\theta_{n,t_1},\theta_{n,t_2}\right)$, and to use (1.2). □

As we have already written, the main problems occur when the Fisher information of the model is not globally bounded. On the other hand, I is generally locally bounded. This leads us to propose the following general framework:

- **(A1)** The model $\mathcal{M}$ is DQM and the Fisher information I is locally bounded.
- **(A2)** The Fisher information is nonsingular.

The previous assumption is actually only useful to study the asymptotic properties of the estimator and particularly the asymptotic efficiency.

The following proposition precises the behavior of the denominator and of a part of the numerator.

Proposition 3. *Assume that* (A1) *and* (A2) *hold.*
Then, for all $\theta\in\Theta$

$$\left(\int\limits_{\sqrt{n}(\Theta-\theta)}\Psi_{\theta,t}\pi\left(\theta_{n,t}\right)dt\right)^{-1}=O_{\mathbf{P}_\theta^n}(1) \tag{3.2}$$

and there exists an increasing sequence of nonnegative real numbers $(\alpha_n)_n$ *tending to the infinity such that*

$$\int\limits_{\mathcal{B}(0,\alpha_n)\cap\sqrt{n}(\Theta-\theta)} \left(t - I^{-1}(\theta)\, S_{n,\theta}^{k_n}\right) \Psi_{\theta,t}\pi(\theta_{n,t})\, dt = o_{\mathbf{P}_\theta^n}(1)\,. \tag{3.3}$$

We now have to study the numerator of the right side of (3.1), when the integral is calculed on the exterior of a ball $\mathcal{B}(0,\alpha_n)$. For this, we take of interest the asymptotic behavior of the more general expression $\mathbf{E}_\theta \sqrt{\int\limits_{\mathcal{B}(0,\alpha_n)^c\cap\sqrt{n}(\Theta-\theta)} \|t\|^p\, \Psi_{\theta,t}\pi(\theta_{n,t})\, dt}$.

4. Technical results and assumptions

Let Θ be a Borel subset of $\mathbb{R}^r$, $\mathcal{C}(a,\gamma) = a + [0,\gamma[^r$ the semi-open cube of $\mathbb{R}^r$ with the vertex a and the length's edge γ. If $m \in \mathbb{N}^*$, it's therefore possible to partition the cube $\mathcal{C}(a,\gamma)$ in m^r small semi-open cubes with length's edge $\frac{\gamma}{m}$ in order to have $\mathcal{C}(a,\gamma) = \bigcup_{i\in\Gamma_m} \mathcal{C}\left(a_i, \frac{\gamma}{m}\right)$ where $\Gamma_m = \{1,\ldots,m^r\}$ and $a_1 = a$. Denote by $i_m(\theta)$ the indice of the cube in the partition that contains θ, that is $\theta \in \mathcal{C}\left(a_{i_m(\theta)}, \frac{\gamma}{m}\right)$. Define χ_m, a map from Θ to Θ such that for all $\theta\in\Theta$, $\chi_m(\theta) \in \mathcal{C}\left(a_{i_m(\theta)}, \frac{\gamma}{m}\right)$, and for each $i \in \Gamma_m$, χ_m is constant on $\mathcal{C}\left(a_i, \frac{\gamma}{m}\right)\cap\Theta$. This constant is denoted by $\overline{\chi}_{i,m}$.

Proposition 4. *Let* $p \geq 1$ *and* φ *be a nonnegative and measurable function defined on* $\Theta \times \mathfrak{X}$*, such that*

- *there exists* $t \geq 1$ *and* $A > 0$ *such that*

$$\sup_{\theta\in\mathcal{C}(a,\gamma)\cap\Theta} \|\varphi_\theta\|_{L^t(\mathbf{P})} \leq A; \tag{4.1}$$

- *there exists* $\alpha \in]0,1]$ *and* $K > 0$ *such that for all* $m \in \mathbb{N}^*$

$$\int\limits_{\mathcal{C}(a,\gamma)\cap\Theta} \left\|\varphi_\theta - \varphi_{\chi_m(\theta)}\right\|_{L^p(\mathbf{P})} d\nu(\theta) \leq \frac{K}{m^\alpha}. \tag{4.2}$$

Then, for all $s \in \left[1, 1 + t\left(1 - \frac{1}{p}\right)\right]$*, there exists* $\omega \in]0,1[$ *and* $C > 0$ *such that*

$$\mathbf{E}\left(\int\limits_{\mathcal{C}(a,\gamma)\cap\Theta} \varphi_\theta^s d\nu(\theta)\right)^{\frac{1}{s}} \leq C\left(\sup_{\theta\in\mathcal{C}(a,\gamma)\cap\Theta} \|\varphi_\theta\|_{L^1(\mathbf{P})}\right)^\omega.$$

Remark 1. If the map $\theta \mapsto \varphi_\theta$ is Lipschitz on $\mathcal{C}(a,\gamma) \cap \Theta$ according to $L^p(\mathbf{P})$-norm, with the Lipschitz constant $\widetilde{K}$, then it satisfies assertion (4.2) with $\alpha = 1$ and $K = \widetilde{K}\gamma\nu(\mathcal{C}(a,\gamma)\cap\Theta)$. If we add assertion (4.1), then by construction, $C = \overline{C}\nu(\mathcal{C}(a,\gamma)\cap\Theta)^{\frac{1}{s}} \leq \overline{C}\gamma^{\frac{r}{s}} \sup\limits_{u \in \mathcal{C}(a,\gamma)\cap\Theta} \pi^{\frac{1}{s}}(u)$ where $\overline{C}$ is a constant which depends only on A and $\widetilde{K}$.

To simplify the notation, the sets of the form $E \cap \Theta$ (where E is a subset of $\mathbb{R}^r$) are now denoted by E.

Proposition 2 plays a central role in the study of the expression $\mathbf{E}_\theta \sqrt{\int_{\mathcal{B}(0,\alpha_n)^c} \|t\|^p \,\Psi_{\theta,t}\pi(\theta_{n,t})\,dt}$. The idea consists in partitioning the set $\mathcal{B}(0,\alpha_n)^c$ in cubes with length's edge 1, and then, in defining the assumptions under which this proposition can be applied with for φ, the square root of the local likelihood ratio and for s, p and t the number 2. Let us precise that by the definition of the map $t \mapsto \Psi_{\theta,t}$, for all $t \in \sqrt{n}(\Theta - \theta)$, $\left\|\sqrt{\Psi_{\theta,t}}\right\|_{L^2\left(\mathbf{P}_\theta^n\right)} \leq 1$. Assertion (4.1) is therefore satisfied with $A = 1$. The difficulty is in fact concentrated in the verification of assertion (4.2). Actually, if I is globally bounded on Θ, then by Lemma 1 and Remark 1, the latter is fulfilled. On the other hand, if I is not globally bounded on Θ but only locally bounded (like for Cauchy or Weibull distributions), Proposition 4 cannot be applied, except in a neighbourhood of any $\theta \in \Theta$, thanks to the Lemma 1. We therefore propose to impose an additional assumption on the boundary of Θ.

(B1) The space Θ is convex and there exists k elements $\theta_1, \ldots, \theta_k$ belonging to $\partial\Theta$, the boundary of Θ, and k convex neighbourhoods $\mathcal{V}_{\theta_1}, \cdots, \mathcal{V}_{\theta_k}$, respectively, of $\theta_1, \cdots, \theta_k$ such that $\partial\Theta \subset \cup_{i=1}^k \mathcal{V}_{\theta_i}$. Moreover, for each $i = 1, \cdots, k$ there exists a r-uplet $\varepsilon_i = \left(\varepsilon_i^{(1)}, \cdots, \varepsilon_i^{(r)}\right)$ of elements belonging to $[0,2[$ such that $\sum_{j=1}^r \left(\varepsilon_i^{(j)} - 1\right)^+ < 1$ and there exists a positive continue function $\widetilde{C}_{\theta_i}$ defined on $\mathcal{V}_{\theta_i} \cap \overline{\Theta}$ such that for all $u \in \mathcal{V}_{\theta_i} \cap \Theta$

$$\|I(u)\| \leq \widetilde{C}_{\theta_i}(u)\,|u - \theta_i|^{-2\varepsilon_i} = \widetilde{C}_{\theta_i}(u) \prod_{j=1}^r \left|u^{(j)} - \theta_i^{(j)}\right|^{-2\varepsilon_i^{(j)}}.$$

It is now possible to treat the part of the integral corresponding to a set of the form $\mathcal{B}(0,\alpha)^c \cap \mathcal{B}(0, A\sqrt{n})$.

Proposition 5. *Assume that* (A1) *and* (B1) *hold.*

Then, for all $\theta \in \Theta$, $A > 0$, $p \geq 0$, *and* $s > 0$

$$\sup_n \mathbf{E}_\theta \sqrt{\int_{\mathcal{B}(0,\alpha)^c \cap \mathcal{B}(0,A\sqrt{n})} \|t\|^p \, \Psi_{\theta,t} \pi(\theta_{n,t})\, dt} = O\left(\alpha^{-s}\right).$$

As a consequence, for all sequence of nonnegative real numbers $(\alpha_n)_n$ tending to the infinity, we have

$$\lim_{n\to\infty} \mathbf{E}_\theta \sqrt{\int_{\mathcal{B}(0,\alpha_n)^c \cap \mathcal{B}(0,A\sqrt{n})} \|t\|^p \, \Psi_{\theta,t} \pi(\theta_{n,t})\, dt} = 0. \tag{4.3}$$

Eventually, there remains the part of the integral corresponding to the set $\mathcal{B}(0, A\sqrt{n})^c$ to be studied. It is obviously useful only if Θ is not bounded. In this case we impose additional assumptions on the asymptotic behavior of the functions $\widetilde{C}_{\theta_i}$, of the Fisher information and of the density π.

- **(B2)** There exists $\widetilde{p}_0 \geq 0$ and $\widetilde{p}_i \geq 0$, $i = 1, \cdots, k$ such that for all $u \in \left(\cup_{i=1}^k \mathcal{V}_{\theta_i}\right)^c$, $\|I(u)\| = O\left(\|u\|^{\widetilde{p}_0}\right)$ and for all $u \in \mathcal{V}_{\theta_i}$, $\widetilde{C}_{\theta_i}(u) = O\left(\|u\|^{\widetilde{p}_i}\right)$.
- **(B3)** For all $\theta \in \Theta$, $\liminf_{\|u\|\to\infty} d(\theta, \theta + u) > 0$.
- **(B4)** There exists $m > 2r + z + 1$ such that $\pi(u) = O\left(\|u\|^{-m}\right)$ where $z = \max_{i=0,\cdots,k}\left(\frac{\widetilde{p}_i(1-\omega_i)}{2}\right)$, $\omega_i = \frac{\alpha_i}{r+\alpha_i}$, $\alpha_i = 1 - \sum_{j=1}^r \left(\varepsilon_i^{(j)} - 1\right)^+$, $i = 1, \cdots, k$ and $\alpha_0 = 1$.

Proposition 6. *Assume that* (A1), (B1), (B2), (B3), *and* (B4) *hold.*

Then, for all $\theta \in \Theta$ *there exists* $A > 0$ *such that for all* $s \geq 0$, *and* $p < m - 2r - z$, $\sup_n \sqrt{n}^s \mathbf{E}_\theta \sqrt{\int_{\mathcal{B}(0,A\sqrt{n})^c} \|t\|^p \, \Psi_{\theta,t} \pi(\theta_{n,t})\, dt}$ *is finite.*

Actually, a stronger result is stated, that is

$$\lim_{A\to\infty} \sup_n \sqrt{n}^s \mathbf{E}_\theta \sqrt{\int_{\mathcal{B}(0,A\sqrt{n})^c} \|t\|^p \, \Psi_{\theta,t} \pi(\theta_{n,t})\, dt} = 0.$$

In the framework of this paper, the previous proposition must notably be used with $p = 1$. That is why we impose in assumption (B4), m to be strictly above $2r + z + 1$, and not only strictly above $2r + z$.

5. Main results

To show that an asymptotically efficient and regular estimator δ_n is ℓ-asymptotically of minimal risk for any ℓ belonging to $\mathcal{G}_q$, it sufficies to verify that the sequence $\left(\mathbf{E}_\theta \left\|\sqrt{n}\left(\delta_n - \theta\right)\right\|^q\right)_n$ is bounded for n sufficiently large. It is a consequence (see [1, Chapter II, Theorem 22]). Similar arguments are used to prove that δ_n is ℓ-asymptotically minimax.

In fact, when Θ is bounded, the assumptions we propose to state that the Bayes estimator is best regular are sufficient to state that it's ℓ-asymptotically of minimal risk and ℓ-asymptotically minimax.

Theorem 1. *Assume that Θ is bounded and that* (A1), (A2), *and* (B1) *hold.*

Then, the Bayes estimator on $\mathcal{M}_n$ relative to the quadratic loss function exists, is asymptotically efficient, regular, ℓ-asymptotically of minimal risk and ℓ-asymptotically minimax, for any ℓ belonging to $\widetilde{\mathcal{G}}$.

If Θ is not bounded, we must add assumptions (B2), (B3)m and (B4).

Theorem 2. *Assume that* (A1), (A2), (B1), (B2), (B3), *and* (B4) *hold.*

Then, the Bayes estimator on $\mathcal{M}_n$ relative to the quadratic loss function exists, is asymptotically efficient and regular.

Moreover, if the assumption (B4) *holds with $m > 3\left(r+1\right) + z$ then this estimator is ℓ-asymptotically of minimal risk and ℓ-asymptotically minimax, for any ℓ belonging to $\mathcal{G}_{\frac{m-3(r+1)-z}{r+3}}$.*

6. Applications

Example 1 (Bounded case). Let f_1 and f_2 be two densities with respect to a σ-finite measure μ such that $\mu\left(f_1 \neq f_2\right) > 0$. Suppose $X_1, \cdots, X_n$ are a sample from a mixture distribution of density $\theta f_1 + \left(1-\theta\right) f_2$ with respect to μ. The parameter we want to estimate is $\theta \in \left]0,1\right[$.

It is not difficult to show that the model is DQM and that for all $\theta \in \left]0,1\right[$, $I\left(\theta\right) \leq \frac{1}{\theta^2} \vee \frac{1}{(1-\theta)^2}$. Thus, I is locally bounded on $\left]0,1\right[$. Moreover assumption (B1) is fullfilled with for instance, $\theta_1 = 0$, $\theta_2 = 1$, $\mathcal{V}_{\theta_1} = \left]-\alpha_0, \alpha_0\right[$ (where $\alpha_0 \in \left]0, \frac{1}{2}\right[$), $\mathcal{V}_{\theta_2} = \left]\beta_0, 1+\beta_0\right[$ (where $\beta_0 \in \left]\frac{1}{2}, 1\right[$), $\varepsilon_1 = \varepsilon_2 = 1$ and $\widetilde{C}_{\theta_1} = \widetilde{C}_{\theta_2} \equiv 1$.

Let ν be a probability measure on $]0,1[$ with a continuous and positive density with respect to the restriction of the Lebesgue measure on $]0,1[$. By Theorem 1, the Bayes estimator relative to the quadratic loss function exists, is asymptotically efficient, regular, ℓ-asymptotically of minimal risk and ℓ-asymptotically minimax for any ℓ belonging to $\widetilde{\mathcal{G}}$. Note that it can be shown (see [10]) that the maximum likelihood estimator possesses the same properties.

Example 2 (Unbounded case). Suppose $X_1, \cdots, X_n$ are a sample from the Cauchy distribution with a scale and location parameter $(\alpha, \beta) \in \mathbb{R}_+^* \times \mathbb{R}$. The density with respect to the Lebesgue measure on $\mathbb{R}$ is

$$f_{(\alpha,\beta)}(x) = \frac{\alpha}{\pi\left(\alpha^2 + (x-\beta)^2\right)}.$$

It is not difficult to show that the model is DQM and that for all $(\alpha, \beta) \in \mathbb{R}_+^* \times \mathbb{R}$, $I(\alpha, \beta) = \frac{1}{2\alpha^2}\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Thus, I is locally bounded (because continuous) on $\mathbb{R}_+^* \times \mathbb{R}$ and the assumption (B1) is fullfilled with $\theta_1 = (0,0)$, $\mathcal{V}_{\theta_1} =]-\alpha_0, \alpha_0[\times \mathbb{R}$ (where $\alpha_0 \in \mathbb{R}_+^*$), $\varepsilon = (1,0)$ and for all $u \in [0, \alpha_0[\times \mathbb{R}$, $\widetilde{C}_{\theta_1}(u) = 1$. Hence, $\widetilde{p}_1 = 0$ is convenient and since $\|I(\alpha,\beta)\| \leq \frac{1}{2\alpha_0^2}$ for all $(\alpha, \beta) \in]\alpha_0, \infty[\times \mathbb{R}$, we can choose $\widetilde{p}_0 = 0$ and $z = 0$.

Note that (B3) is equivalent to

$$\limsup_{\|u\| \to \infty} \mathbf{E}_\theta \sqrt{\frac{f_u}{f_\theta}(X_1)} < 1, \quad \forall \theta \in \Theta. \tag{6.1}$$

Even if we use a change of variable, assume that $\theta = (1,0)$.

Letting $\phi(\alpha, \beta) = \mathbf{E}_{(1,0)} \sqrt{\frac{f_{(\alpha,\beta)}}{f_{(1,0)}}(X_1)}$ on $\mathbb{R}_+^* \times \mathbb{R}$, we extend this function by continuity on $\mathbb{R}_+ \times \mathbb{R}$ with $\phi(0, \beta) = 0$. Note that $\phi(\alpha, \beta) \leq 1$ and that $\phi(\alpha, \beta) = 1$ iff $(\alpha, \beta) = (1,0)$. If we note $\varphi(\alpha, \beta) = \left(\frac{1}{\alpha}, -\frac{\beta}{\alpha}\right)$, it can easily be checked that $\phi = \phi \circ \varphi$. Furthermore, the function $\phi(\alpha, \cdot)$ is even. Hence it can be supposed that $\beta \geq 0$.

Consider the case where $\beta < \alpha$. Then we have for all $a > 1$

$$\begin{aligned}
\sup_{(\alpha,\beta) \in]a,\infty[\times [0,\infty[, \beta < \alpha} \phi(\alpha, \beta) &= \sup_{(\alpha,\beta) \in \varphi(]a,\infty[\times [0,\infty[, \beta < \alpha)} \phi(\alpha, \beta) \\
&\leq \sup_{(\alpha,\beta) \in \left]0, \frac{1}{a}\right] \times [0,1]} \phi(\alpha, \beta) \\
&< 1 \text{ using the continuity of } \phi
\end{aligned}$$

hence $\limsup_{\|(\alpha,\beta)\|\to\infty,\beta<\alpha} \phi(\alpha,\beta) < 1.$

In the case where $\beta \geq \alpha$, we have

$$\begin{aligned}\phi(\alpha,\beta) &\leq \frac{1}{\pi}\int \frac{1}{\sqrt{2}\sqrt{|x-\beta|}\sqrt{1+x^2}}dx \\ &\leq \frac{1}{\pi}\left(\frac{2}{\sqrt{3}}\int_{\left[0,\frac{1}{2}\right]} \frac{\sqrt{\beta}}{\sqrt{1+\beta^2x^2}}dx + \int_{\left[-\frac{1}{2},\frac{1}{2}\right]^c} \frac{1}{\sqrt{2}\sqrt{|x-1|}\sqrt{\beta}\,|x|}dx\right).\end{aligned}$$

The first integral in the right member of the inequality is an $O\left(\frac{\ln(\beta)}{\sqrt{\beta}}\right)$, hence it tends to zero when β tends to the infinity. The conclusion is the same for the second integral which is an $O\left(\frac{1}{\sqrt{\beta}}\right)$ hence $\limsup_{\|(\alpha,\beta)\|\to\infty,\beta\geq\alpha} \phi(\alpha,\beta) = 0.$

Finally, assertion (6.1) is checked.

Choosing for the prior distribution ν on Θ, any distribution with a continuous and positive density satisfying (B4) the Bayes estimator relative to the quadratic loss function exists, is asymptotically efficient and regular. Moreover, if (B4) is fullfilled with $m > 9$ then this estimator is ℓ-asymptotically of minimal risk and ℓ-asymptotically minimax for any ℓ belonging to $\mathcal{G}_{\frac{m-9}{5}}$. Actually, for this particular model, it can be proved that the assumptions imposed on the asymptotic comportment of π are not essential, and hence this estimator possesses the two precedent properties for any ℓ belonging to $\widetilde{\mathcal{G}}$. Once again, it can be shown that the maximum likelihood estimator possesses the same properties with also ℓ belonging to $\widetilde{\mathcal{G}}$ (see [ ]10).

The approach used to treat the model associated to the Cauchy distribution can be easily extended to a large class of distributions with scale and location parameter. Indeed, most of the assumptions we impose are on the Fisher information which does not depend on the location parameter. Actually, the assumption that really imposes constraints on the density of the studied distribution is (B3).

Example 3 (Unbounded case). Suppose $X_1, \cdots, X_n$ are a sample from the Weibull with parameter $(\alpha,\lambda) \in \Theta = \mathbb{R}_+^* \times \mathbb{R}_+^*$. The density with respect to the Lebesgue measure on $\mathbb{R}_+^*$ is

$$f_{(\alpha,\lambda)}(x) = \alpha\lambda x^{\alpha-1}\exp\left(-\lambda x^{\alpha}\right).$$

First, it is not difficult to show that the model is DQM.

As can easily be checked, there exists three constants D_1, D_2 and D_3 such that for all $(\alpha,\lambda)\in\Theta$, $\|I(\alpha,\lambda)\|\le$ trace $(I(\alpha,\lambda))\le\frac{1}{\alpha^2}D_1+\frac{|\log\lambda|}{\alpha^2}D_2+\frac{1}{\lambda^2}D_3$.

Given $a>1$, $\widetilde{a}<1$ we can choose the three neighbourhoods $\mathcal{V}_{\theta_1}=\left]-a,a\right[\times\left]a,\infty\right[$, $\mathcal{V}_{\theta_2}=\left]a,\infty\right[\times\left]-\widetilde{a},\widetilde{a}\right[$ and $\mathcal{V}_{\theta_3}=\left]-a,a\right[\times\left]-a,a\right[$ to recover the boundary of Θ. Denote $E_i=\mathcal{V}_{\theta_i}\cap\Theta$ $(i=1,2,3)$ and $E_0=\left[a,\infty\right[\times\left[\widetilde{a},\infty\right[$.

We now show that using the neighbourhoods defined before, assumption (B1) is fullfilled.

- Let $\theta_1=(0,\lambda_0)$ belonging to $\{0\}\times\left]a,\infty\right[$. It is clear that for all $(\alpha,\lambda)\in E_1$, $\|I(\alpha,\lambda)\|\le\widetilde{C}_{\theta_1}(\alpha,\lambda)\frac{1}{\alpha^2}$ where $\widetilde{C}_{\theta_1}(\alpha,\lambda)=O(|\log\lambda|)$.

We can therefore take $\varepsilon_1=(1,0)$ and any positive real number for $\widetilde{p}_1$.

- Let $\theta_2=(\alpha_0,0)$ belonging to $\left]a,\infty\right[\times\{0\}$. It is clear that for all $(\alpha,\lambda)\in E_2$, $\|I(\alpha,\lambda)\|\le\widetilde{C}_{\theta_2}(\alpha,\lambda)\frac{1}{\lambda^2}$ where $\widetilde{C}_{\theta_2}=O(1)$.

We can therefore take $\varepsilon_2=(0,1)$ and $\widetilde{p}_2=0$.

- Let $\theta_3=(0,0)$. It is clear that for all $(\alpha,\lambda)\in E_3$, $\|I(\alpha,\lambda)\|\le\widetilde{C}_{\theta_3}(\alpha,\lambda)\frac{1}{\alpha^2\lambda^2}$ where $\widetilde{C}_{\theta_3}=O(1)$.

We can therefore take $\varepsilon_3=(1,1)$ and $\widetilde{p}_3=0$.

For all $(\alpha,\lambda)\in E_0$, $\|I(\alpha,\lambda)\|\le a_1+a_2|\log\lambda|$. Thus, any positive real number is convenient for $\widetilde{p}_0$ and finally, any positive real number is convenient for z.

In order to prove (B3) it suffices to verify that there exists $a>1$ such that

$$\sup_{u\in\mathcal{B}(0,a)^c}\mathbf{E}_\theta\sqrt{\frac{f_{\theta+u}}{f_\theta}(X_1)}<1$$

Even if we use a change of variable, assume that $(\alpha,\lambda)=(1,1)$ and let us prove that $\sup_{(\alpha,\lambda)\in\mathcal{B}(0,a)^c}\mathbf{E}_{(1,1)}\sqrt{\frac{f_{(\alpha,\lambda)}}{f_{(1,1)}}(X_1)}<1$ where a is fixed in the interval $\left]1,\infty\right[$.

Letting $\phi(\alpha,\lambda)=\mathbf{E}_{(1,1)}\sqrt{\frac{f_{(\alpha,\lambda)}}{f_{(1,1)}}(X_1)}$ on $\mathbb{R}_+^*\times\mathbb{R}_+^*$, we extend this function by continuity on $\mathbb{R}_+\times\mathbb{R}_+$ with $\phi(0,\lambda)=0$ and $\phi(\alpha,0)=0$. Note that $\phi(\alpha,\lambda)\le 1$ and $\phi(\alpha,\lambda)=1$ iff $(\alpha,\lambda)=(1,1)$. Furthermore, $\sup_{\mathcal{B}(0,a)^c}\phi(\alpha,\lambda)$ equals to the max of the sup on each of the three sets E_0, E_1 and E_2 defined before.

If we note $\varphi(\alpha,\lambda) = \left(\alpha^{-1}, \lambda^{-\frac{1}{\alpha}}\right)$ it can easily be checked that $\phi = \phi \circ \varphi$. Thus

$$\begin{aligned}\sup_{(\alpha,\lambda)\in E_0} \phi(\alpha,\lambda) &= \sup_{(\alpha,\lambda)\in\varphi(E_0)} \phi(\alpha,\lambda)\\ &\leq \sup_{(\alpha,\lambda)\in\left[0,\frac{1}{a}\right]\times\left[0,\tilde{a}^{-\frac{1}{a}}\right]} \phi(\alpha,\lambda)\\ &< 1 \quad \text{using the continuity of} \quad \phi\end{aligned}$$

and likewise

$$\sup_{(\alpha,\lambda)\in E_2} \phi(\alpha,\lambda) \leq \sup_{(\alpha,\lambda)\in\left[0,\frac{1}{a}\right]\times]1,\infty[} \phi(\alpha,\lambda)$$

$$= \max\left(\sup_{(\alpha,\lambda)\in\left[0,\frac{1}{a}\right]\times[1,a[} \phi(\alpha,\lambda), \sup_{(\alpha,\lambda)\in\left[0,\frac{1}{a}\right]\times[a,\infty[} \phi(\alpha,\lambda)\right)$$

$$\leq \max\left(\sup_{(\alpha,\lambda)\in\left[0,\frac{1}{a}\right]\times[1,a[} \phi(\alpha,\lambda), \sup_{(\alpha,\lambda)\in E_1} \phi(\alpha,\lambda)\right).$$

We now have to show that $\sup_{(\alpha,\lambda)\in E_1} \phi(\alpha,\lambda) < 1$. Regarding the configuration of the set E_1, if $\sup_{(\alpha,\lambda)\in E_1} \phi(\alpha,\lambda) = 1$, then using the continuity of ϕ, there exists a sequence $(\alpha_n,\lambda_n)_n$ of elements belonging to E_1, of limit (α_0,∞) (where $\alpha_0 \in [0,a]$) such that $\sup_{(\alpha,\lambda)\in E_1} \phi(\alpha,\lambda) = \lim_{n\to\infty} \phi(\alpha_n,\lambda_n)$.

Note that

$$\phi(\alpha,\lambda) \leq \sqrt{\alpha} \sup_{u\in\mathbb{R}_+} u \exp\left(-\frac{u^2}{2}\right) \int_0^\infty x^{-\frac{1}{2}} \exp\left(-\frac{x}{2}\right) dx.$$

This implies in particulary that $\lim_{\alpha\to 0^+} \phi(\alpha,\lambda) = 0$ uniformly with respect to λ. Therefore, α_0 is obviously different from zero. Thus, using once again the transformation φ, $\sup_{(\alpha,\lambda)\in E_1} \phi(\alpha,\lambda)$ is reached for an element belonging to the interior of E_1. We conclude thanks to the continuity of ϕ.

The conclusion is then strictly the same as the one in the precedent example.

It can be shown that the maximum likelihood estimator is asymptotically efficient and regular. On the other hand, the regularity's assumptions proposed by Ibragimov and Hasminskii (see [4]) or Lhéritier ([10])

do not allow us to affirm that it is ℓ-asymptotically of minimal risk, nor ℓ-asymptotically minimax.

7. Proofs

In this section, we present the proofs of the results stated in the previous sections.

Proof of Proposition 3. Fix $\theta \in \Theta$ and $\varepsilon > 0$. We suppose n sufficiently large for the ball $\mathcal{B}(0,\varepsilon)$ to be included in $\sqrt{n}(\Theta - \theta)$ and we note $Y_n = \int\limits_{\mathcal{B}(0,\varepsilon)} \Psi_{\theta,t}\pi(\theta_{n,t})\,dt$.

In order to show (3.2), it sufficies to prove that $\frac{1}{Y_n} = O_{\mathbf{P}_\theta^n}(1)$, that is

$$\lim_{\beta\to 0}\limsup_{n\to\infty}\mathbf{P}_\theta^n(\{Y_n \le \beta\}) = 0.$$

First note that if the model is DQM, then for all $t \in \mathbb{R}^r$, $\mathbf{E}_\theta \Psi_{\theta,t} = 1 - \varepsilon_n(t)$ where $\varepsilon_n(t) \in [0,1[$ and $\lim_{n\to\infty}\varepsilon_n(t) = 0$. Thus, denoting $\beta_n(\varepsilon) = \frac{1}{2}\int\limits_{\mathcal{B}(0,\varepsilon)} \mathbf{E}_\theta\Psi_{\theta,t}\pi(\theta_{n,t})\,dt$ we have $\beta_n(\varepsilon) = \frac{1}{2}\int\limits_{\mathcal{B}(0,\varepsilon)}\pi(\theta_{n,t})\,dt - \widetilde{\varepsilon}_n$ where $\widetilde{\varepsilon}_n = \int\limits_{\mathcal{B}(0,\varepsilon)}\pi(\theta_{n,t})\varepsilon_n(t)\,dt$. Let us precise that $\widetilde{\varepsilon}_n$ is positive and that $\limsup_{n\to\infty}\widetilde{\varepsilon}_n = 0$ because π is locally bounded. This implies that $\liminf_{n\to\infty}\beta_n(\varepsilon) = \frac{1}{2}\pi(\theta)\varepsilon^r$ and that for n sufficiently large

$$\mathbf{P}_\theta^n\left(\left\{Y_n \le \frac{1}{2}\liminf_{n\to\infty}\beta_n(\varepsilon)\right\}\right) \le \mathbf{P}_\theta^n(\{Y_n \le \beta_n(\varepsilon)\}).$$

It can then be noticed that by the Cauchy–Schwarz's inequality and Lemma 1, for all $t \in \mathcal{B}(0,\varepsilon)$, $\mathbf{E}_\theta|\Psi_{\theta,t} - 1| \le 2\sqrt{n}d(\theta,\theta_{n,t}) \le \sup_{u\in\mathcal{B}\left(\theta,\frac{\varepsilon}{\sqrt{n}}\right)}\sqrt{\|I(u)\|}\,\|\varepsilon\|$. Thus, using the Chebychev inequality,

$$\mathbf{P}_\theta^n(\{Y_n \le \beta_n(\varepsilon)\}) \le \mathbf{P}_\theta^n\left(\left|\int\limits_{\mathcal{B}(0,\varepsilon)}(\Psi_{\theta,t} - 1)\pi(\theta_{n,t})\,dt\right| \ge \right.$$
$$\left. \ge \frac{1}{2}\int\limits_{\mathcal{B}(0,\varepsilon)}\pi(\theta_{n,t})\,dt + \varepsilon_n\right)$$

$$\leq \frac{\mathbf{E}_\theta \left| \int\limits_{\mathcal{B}(0,\varepsilon)} (\Psi_{\theta,t} - 1)\, \pi(\theta_{n,t})\, dt \right|}{\frac{1}{2} \int\limits_{\mathcal{B}(0,\varepsilon)} \pi(\theta_{n,t})\, dt + \varepsilon_n}$$

$$\leq 2\varepsilon \sup_{u \in \mathcal{B}\left(\theta, \frac{\varepsilon}{\sqrt{n}}\right)} \sqrt{\|I(u)\|}$$

$$\leq 2\varepsilon \sup_{u \in \mathcal{B}(\theta,\varepsilon)} \sqrt{\|I(u)\|}.$$

Assertion (3.2) is then stated when ε tends to zero.

For the study of (3.3), the radius α is fixed in $\mathbb{R}_+^*$ and we suppose n sufficiently large for the ball $\mathcal{B}(0,\alpha)$ to be included in $\sqrt{n}(\Theta - \theta)$.

Let us decompose $\Psi_{\theta,t}$ in the form

$$\Psi_{\theta,t} = (\Psi_{\theta,t} - \Phi_{k_n,\theta,t}) + \Phi_{k_n,\theta,t}.$$

- Considering the first term, for all $\alpha > 0$

$$\int\limits_{\mathcal{B}(0,\alpha)} \left\| \left(t - I^{-1}(\theta)\, S_{n,\theta}^{k_n}\right) (\Psi_{\theta,t} - \Phi_{k_n,\theta,t}) \right\| \pi(\theta_{n,t})\, dt$$

$$\leq \sup_{t \in \mathcal{B}(0,\alpha)} \pi(\theta_{n,t}) \left(\alpha + \left\| I^{-1}(\theta)\, S_{n,\theta}^{k_n} \right\|\right) \int\limits_{\mathcal{B}(0,\alpha)} |\Psi_{\theta,t} - \Phi_{k_n,\theta,t}|\, dt = o_{\mathbf{P}_\theta^n}(1).$$

Indeed, on the one hand, $I^{-1}(\theta)\, S_{n,\theta}^{k_n} = O_{\mathbf{P}_\theta^n}(1)$ (because it converges in $\mathbf{P}_\theta^n$-distribution) and, on the other hand,

$$\mathbf{E}_\theta\, |\Psi_{\theta,t} - \Phi_{k_n,\theta,t}| \leq \mathbf{E}_\theta\, |\Psi_{\theta,t} - C_{k_n,\theta,t}\Phi_{k_n,\theta,t}| + \left| 1 - \frac{1}{C_{k_n,\theta,t}} \right|$$

thus using (2.3), and noticing that $t \mapsto \frac{1}{C_{k_n,\theta,t}}$ is bounded on every compact and that $\lim\limits_{n\to\infty} C_{k_n,\theta,t} = 1$, by the Lebesgue theorem, for all $t \in \mathbb{R}^r$

$$\int\limits_{\mathcal{B}(0,\alpha)} |\Psi_{\theta,t} - \Phi_{k_n,\theta,t}|\, dt \overset{L^1(\mathbf{P}_\theta^n)}{\longrightarrow} 0.$$

Therefore, for all $\alpha > 0$, the diagonal method allows us to affirm that there exists an increasing sequence of positive real numbers $(\alpha_n)_n$ tending to the infinity, along which the convergence occurs.

• Considering the second term, let us note that $\pi(u) = 0$ if $u \notin \Theta$. Thus, for all $\alpha > 0$

$$\int_{\mathcal{B}(0,\alpha_n)} \left(t - I^{-1}(\theta) S_{n,\theta}^{k_n}\right) \Phi_{k_n,\theta,t} \pi(\theta_{n,t})\, dt$$

$$= \exp\left(\frac{1}{2}\left\langle S_{n,\theta}^{k_n}, S_{n,\theta}^{k_n}\right\rangle_{I^{-1}(\theta)}\right) \int_{\mathcal{B}\left(-I^{-1}(\theta)S_{n,\theta}^{k_n},\alpha_n\right)}$$

$$\times t \exp\left(-\frac{1}{2}\langle t,t\rangle_{I(\theta)}\right) \pi\left(\theta_{n,t+S_{n,\theta}^{k_n}}\right) dt.$$

Note that $\exp\left(\frac{1}{2}\left\langle S_{n,\theta}^{k_n}, S_{n,\theta}^{k_n}\right\rangle_{I^{-1}(\theta)}\right) = O_{\mathbf{P}_\theta^n}(1)$ (because $S_{n,\theta}^{k_n}$ converges in $\mathbf{P}_\theta^n$-distribution) and passing to the limit in the integral (that is possible because π is continuous and bounded), we have $\int_{\mathbb{R}^r} t\exp\left(-\frac{1}{2}\langle t,t\rangle_{I(\theta)}\right) dt = 0$.

Finally, assertion (3.3) is stated. □

Proof of Proposition 4. Suppose s to be fixed in the interval $\left[1, 1+\frac{t}{q}\right]$ where $q = \frac{1}{1-\frac{1}{p}}$.

We first show that

$$\int_{\mathcal{C}(a,\gamma)\cap\Theta} \left\|\varphi_\theta^s - \varphi_{\chi_m(\theta)}^s\right\|_{L^1(\mathbf{P})} d\nu(\theta) \le 2sA^{s-1}\frac{K}{m^\alpha}. \tag{7.1}$$

Using the mean value theorem and the Hölder's inequality, for all $(\theta_1,\theta_2) \in (\mathcal{C}(a,\gamma)\cap\Theta)^2$

$$\begin{aligned}
\mathbf{E}\left|\varphi_{\theta_1}^s - \varphi_{\theta_2}^s\right| &\le s\mathbf{E}\left\{\max\left(\varphi_{\theta_1}^{s-1},\varphi_{\theta_2}^{s-1}\right)\left|\varphi_{\theta_1}-\varphi_{\theta_2}\right|\right\}\\
&\le s\left\{\mathbf{E}\varphi_{\theta_1}^{(s-1)q} + \mathbf{E}\varphi_{\theta_2}^{(s-1)q}\right\}^{\frac{1}{q}} \left\|\varphi_{\theta_1}-\varphi_{\theta_2}\right\|_{L^p(\mathbf{P})}\\
&\le s\left\{\left\|\varphi_{\theta_1}\right\|_{L^{(s-1)q}(\mathbf{P})}^{s-1} + \left\|\varphi_{\theta_2}\right\|_{L^{(s-1)q}(\mathbf{P})}^{s-1}\right\}\left\|\varphi_{\theta_1}-\varphi_{\theta_2}\right\|_{L^p(\mathbf{P})}\\
&\le 2sA^{s-1}\left\|\varphi_{\theta_1}-\varphi_{\theta_2}\right\|_{L^p(\mathbf{P})}.
\end{aligned}$$

By (4.1) (because $(s-1)q \le t$) and the fact that $b \mapsto \left(\mathbf{E}|X|^b\right)^{\frac{1}{b}}$ increases on $\mathbb{R}_+^*$ when X is a random variable.

It then suffices to integrate the inequality with respect to ν on $\mathcal{C}(a,\gamma)\cap\Theta$, replacing θ_2 by $\chi_m(\theta)$ and then to use (4.2).

Furthermore, it can be written $\mathbf{P}$-almost everywhere on $\mathfrak{X}$ that

$$\int\limits_{\mathcal{C}(a,\gamma)\cap\Theta} \varphi_\theta^s d\nu(\theta) \leq \int\limits_{\mathcal{C}(a,\gamma)\cap\Theta} \left|\varphi_\theta^s - \varphi_{\chi_m(\theta)}^s\right| d\nu(\theta) + \int\limits_{\mathcal{C}(a,\gamma)\cap\Theta} \varphi_{\chi_m(\theta)}^s d\nu(\theta)$$

hence by the inequality

$$\left(\sum_{i=1}^k x_i\right)^{\frac{1}{s}} \leq \sum_{i=1}^k x_i^{\frac{1}{s}}, \quad \forall k\in\mathbb{N}^*, \quad \forall x_i\geq 0, \quad \forall s\geq 1$$

and then by the Jensen's inequality, we have

$$\mathbf{E}\left(\int\limits_{\mathcal{C}(a,\gamma)\cap\Theta} \varphi_\theta^s d\nu(\theta)\right)^{\frac{1}{s}} \leq \left(\mathbf{E}\int\limits_{\mathcal{C}(a,\gamma)\cap\Theta} \left|\varphi_\theta^s - \varphi_{\chi_m(\theta)}^s\right| d\nu(\theta)\right)^{\frac{1}{s}} + \sum_{i\in\Gamma_m} \mathbf{E}\varphi_{\overline{\chi}_{i,m}}\nu\left(\mathcal{C}\left(a_i,\frac{\gamma}{m}\right)\cap\Theta\right)^{\frac{1}{s}}.$$

The first term in the right member of the previous inequality is treated by (7.1).

Concerning the second, since the maximum of $\sum\limits_{i\in\Gamma_m} x_i^{\frac{1}{s}}$ on $\sum\limits_{i\in\Gamma_m} x_i = 1$ is $(\operatorname{Card}\Gamma_m)^{1-\frac{1}{s}} = m^{\frac{r(s-1)}{s}}$, we have

$$\sum_{i\in\Gamma_m} \mathbf{E}\varphi_{\overline{\chi}_{i,m}}\nu\left(\mathcal{C}\left(a_i,\frac{\gamma}{m}\right)\cap\Theta\right)^{\frac{1}{s}} \leq m^{\frac{r(s-1)}{s}}\nu\left(\mathcal{C}(a,\gamma)\cap\Theta\right)^{\frac{1}{s}} \sup_{\theta\in\mathcal{C}(a,\gamma)\cap\Theta} \mathbf{E}\varphi_\theta.$$

Thus,

$$\mathbf{E}\left(\int\limits_{\mathcal{C}(a,\gamma)\cap\Theta} \varphi_\theta^s d\nu(\theta)\right)^{\frac{1}{s}} \leq \inf_{m\in\mathbb{N}^*} \psi(m),$$

where $\psi : x \mapsto D_s x^{-\frac{\alpha}{s}} + Bx^{\frac{r(s-1)}{s}}$ with $D_s = \left(2sA^{s-1}K\right)^{\frac{1}{s}}$ and $B = \nu(\mathcal{C}(a,\gamma)\cap\Theta)^{\frac{1}{s}} \sup\limits_{\theta\in\mathcal{C}(a,\gamma)\cap\Theta} \mathbf{E}\varphi_\theta$.

Note that ψ is a decreasing function on the interval $]0,\widetilde{x}[$ and an increasing one on the interval $]\widetilde{x},\infty[$, where $\widetilde{x} = \left(\frac{\alpha D_s}{r(s-1)B}\right)^{\frac{s}{\alpha+r(s-1)}}$.

- If $r(s-1)B < \alpha D_s$, then $\widetilde{x} > 1$ and

$$\inf_{m\in\mathbb{N}^*} \psi(m) = \min\left(\psi(\lfloor\widetilde{x}\rfloor), \psi(\lfloor\widetilde{x}\rfloor+1)\right) \leq \psi(\lfloor\widetilde{x}\rfloor+1).$$

But $\widetilde{x} < \lfloor\widetilde{x}\rfloor + 1 \leq 2\widetilde{x}$, hence

$$\inf_{m\in\mathbb{N}^*} \psi(m) \leq {D_s}^{1-\omega} F_\alpha B^\omega.$$

where $\omega = \frac{\alpha}{\alpha+r(s-1)}$ and $F_\alpha = \left(\frac{r(s-1)}{\alpha}\right)^\omega \left(2^{-\frac{\alpha}{s}} + 2^{\frac{r(s-1)}{s}}\frac{\alpha}{r(s-1)}\right)$.

- If $\alpha D_s \leq r(s-1)B$, then $\widetilde{x} \leq 1$ and hence

$$\begin{aligned}\inf_{m\in\mathbb{N}^*} \psi(m) &= \psi(1)\\ &< \frac{1}{\omega}B\\ &\leq \frac{1}{\omega}A^{1-\omega}\nu(\mathcal{C}(a,\gamma)\cap\Theta)^{\frac{1-\omega}{s}} B^\omega \quad \text{using} \quad (4.1).\end{aligned}$$

Finally, we can pose

$$C = \max\left(\left(\frac{2sA^{s-1}K}{\nu(\mathcal{C}(a,\gamma)\cap\Theta)}\right)^{\frac{1-\omega}{s}} F_\alpha, \frac{1}{\omega}A^{1-\omega}\right)\nu(\mathcal{C}(a,\gamma)\cap\Theta)^{\frac{1}{s}}. \quad (7.2)$$

It is important to note that the map

$$(x,y) \mapsto \max\left(\left(\frac{2sA^{s-1}x}{y}\right)^{\frac{1-\omega}{s}} F_\alpha, \frac{1}{\omega}A^{1-\omega}\right) y^{\frac{1}{s}}$$

defined on $\mathbb{R}_+^* \times \mathbb{R}_+^*$, is increasing in y when x is fixed and increasing in x when y is fixed. $\square$

In this paper, Proposition 4 is applied with $A = 1$, $s = 2$, $p = 2$, and $\varphi_{\cdot} = \sqrt{\psi_{\theta,\cdot}}$. If I is bounded in a convex neighbourhood $\widetilde{\Theta}$ of θ, included in Θ, using Lemma 1, Remark 1, and (7.20), we have

$$C \le \overline{C}_0 \gamma^{\frac{r}{2}} \sqrt{\sup_{t \in \mathcal{C}(a,\gamma) \cap \sqrt{n}(\widetilde{\Theta} - \theta)} \pi(\theta_{n,t})}, \tag{7.3}$$

where $\overline{C}_0 = \max\left(\left(2\gamma \sup_{u \in \widetilde{\Theta}} \sqrt{\|I(u)\|}\right)^{\frac{1-\omega}{2}} F_\alpha, \frac{1}{\omega}\right)$.

Before stating Propositions 5 and 6, let us show the two following technical lemmas.

Lemma 2. *Assume that* (A1) *holds.*

Then for all $\theta \in \Theta$, for all θ_i belonging to $\partial\Theta$ and satisfying the assumption (B1), *there exists $\alpha_i \in]0,1]$ such that for all $\gamma > 0$, $n \in \mathbb{N}^*$ and $a \in \mathbb{R}^r$, there exists $K_{i,n} > 0$ such that for all $m \in \mathbb{N}^*$*

$$\int_{\mathcal{C}(a,\gamma) \cap \mathcal{U}_{n,\theta_i}} \left\| \sqrt{\Psi_{\theta,t}} - \sqrt{\Psi_{\theta,\chi_m(t)}} \right\|_{L^2(\mathbf{P}_\theta^n)} \pi(\theta_{n,t})\, dt \le \frac{K_{i,n}}{m^{\alpha_i}}, \tag{7.4}$$

where $\mathcal{U}_{n,\theta_i} = \sqrt{n}(\mathcal{V}_{\theta_i} - \theta)$.

Proof. Let θ_i belonging to $\partial\Theta$, satisfying assumption (B1). There exists a positive function $\widetilde{C}_{\theta_i}$ defined on $\mathcal{V}_{\theta_i} \cap \overline{\Theta}$ such that for all $u \in \mathcal{V}_{\theta_i} \cap \Theta$, $\|I(u)\| \le \widetilde{C}_{\theta_i}(u)\, |u - \theta_i|^{-2\varepsilon_i}$.

By Lemma 1, for all $t \in \sqrt{n}(\Theta - \theta)$

$$\left\| \sqrt{\Psi_{\theta,t}} - \sqrt{\Psi_{\theta,\chi_m(t)}} \right\|_{L^2(\mathbf{P}_\theta^n)} \le \frac{1}{2} \sup_{v \in [t,\chi_m(t)]} \sqrt{\|I(\theta_{n,v})\|}\, \|t - \chi_m(t)\|.$$

Moreover, by hypothesis, for all $t \in \mathcal{U}_{n,\theta_i}$

$$\sqrt{\|I(\theta_{n,t})\|} \le \sqrt{n}^{\sum_{j=1}^r \varepsilon_i^{(j)}} \sqrt{\widetilde{C}_{\theta_i}(\theta_{n,t})}\, \left|t - \sqrt{n}(\theta_i - \theta)\right|^{-\varepsilon_i}.$$

Let fix $m \in \mathbb{N}^*$ and note for every cube $\mathcal{C}\left(a_l, \frac{\gamma}{m}\right)$ belonging to the partition of $\mathcal{C}(a,\gamma)$

$$J_l = \left\{ j \in \{1, \cdots, r\} : \left[a_l^{(j)}, a_l^{(j)} + \frac{\gamma}{m}\right] \right.$$
$$\left. \cap \left[\sqrt{n}\left(\theta_i^{(j)} - \theta^{(j)}\right) - \frac{\gamma}{m}, \sqrt{n}\left(\theta_i^{(j)} - \theta^{(j)}\right) + \frac{\gamma}{m}\right] \ne \emptyset \right\}$$

and $O_m = \{l \in \Gamma_m : J_l \neq \emptyset\}$. If the set J_l is not empty then it can be written in the form

$$J_l = \left\{ j \in \{1, \cdots, r\} : a_l^{(j)} \in \left[\sqrt{n}\left(\theta_i^{(j)} - \theta^{(j)}\right) - \frac{2\gamma}{m}, \sqrt{n}\left(\theta_i^{(j)} - \theta^{(j)}\right) + \frac{\gamma}{m} \right] \right\}.$$

Hence the cardinal of O_m is lower than $3rm^{r-1}$. Using on the set O_m the fact that $\left\| \sqrt{\Psi_{\theta,t}} - \sqrt{\Psi_{\theta,\chi_m(t)}} \right\|_{L^2\left(\mathbf{P}_\theta^n\right)}$ is bounded above by $\sqrt{2}$, we have

$$\int_{\mathcal{C}(a,\gamma)\cap\mathcal{U}_{n,\theta_i}} \left\| \sqrt{\Psi_{\theta,t}} - \sqrt{\Psi_{\theta,\chi_m(t)}} \right\|_{L^2\left(\mathbf{P}_\theta^n\right)} \pi\left(\theta_{n,t}\right) dt$$

$$\leq \sup_{t\in\mathcal{C}(a,\gamma)} \pi\left(\theta_{n,t}\right) \left(3r\sqrt{2}\frac{\gamma}{m}\gamma^{r-1} + \frac{1}{2}\sqrt{n}^{\sum_{j=1}^r \varepsilon_i^{(j)}} \right.$$

$$\left. \times \sqrt{\sup_{t\in\mathcal{C}(a,\gamma)} \widetilde{C}_{\theta_i}\left(\theta_{n,t}\right) \frac{\gamma}{m}} \int_{\bigcup_{l\in O_m^c} \mathcal{C}\left(a_l,\frac{\gamma}{m}\right)} \sup_{v\in[u,\chi_m(u)]} \left| v - \sqrt{n}\left(\theta_i - \theta\right) \right|^{-\varepsilon_i} du \right).$$

Note that if we note p_j (respectively, q_j) the smallest (respectively, largest) natural number such that $a^{(j)} + p_j \frac{\gamma}{m} \geq \sqrt{n}\left(\theta_i^{(j)} - \theta^{(j)}\right) + \frac{\gamma}{m}$ respectively, $a^{(j)} + q_j \frac{\gamma}{m} \leq \sqrt{n}\left(\theta_i^{(j)} - \theta^{(j)}\right) - \frac{\gamma}{m}\Big)$, then

$$\bigcup_{l\in O_m^c} \mathcal{C}\left(a_l, \frac{\gamma}{m}\right) = \prod_{j=1}^r \left(\left[a^{(j)} + p_j\frac{\gamma}{m}, a^{(j)} + \gamma \right] \cup \left[a^{(j)}, a^{(j)} + q_j\frac{\gamma}{m} \right] \right).$$

Thus,

$$\int_{\bigcup_{l\in O_m^c} \mathcal{C}\left(a_l,\frac{\gamma}{m}\right)} \sup_{v\in[u,\chi_m(u)]} \left| v - \sqrt{n}\left(\theta_i - \theta\right) \right|^{-\varepsilon_i} du$$

$$\leq \prod_{j=1}^r \left(\int_{\left[a^{(j)}+p_j\frac{\gamma}{m}, a^{(j)}+\gamma\right]} \sup_{v_j\in\left[u_j,(\chi_m(u))_j\right]} \left(v_j - \sqrt{n}\left(\theta_i^{(j)} - \theta^{(j)}\right) \right)^{-\varepsilon_i^{(j)}} du_j \right.$$

$$\left. + \int_{\left[a^{(j)}, a^{(j)}+q_j\frac{\gamma}{m}\right]} \sup_{v_j\in\left[u_j,(\chi_m(u))_j\right]} \left(\sqrt{n}\left(\theta_i^{(j)} - \theta^{(j)}\right)^{-\varepsilon_i^{(j)}} - v_j \right)^{-\varepsilon_i^{(j)}} du_j \right)$$

$$= \prod_{j=1}^r \left(I_{1,j} + I_{2,j}\right).$$

Using the decrease of the real valued function $u \mapsto u^{-\alpha}$ on $\mathbb{R}_+^*$ $(\alpha \geq 0)$

$$\begin{aligned}
I_{1,j} &\leq \frac{\gamma}{m} \sum_{l=p_j}^{m} \left(a^{(j)} + l\frac{\gamma}{m} - \sqrt{n}\left(\theta_i^{(j)} - \theta^{(j)}\right)\right)^{-\varepsilon_i^{(j)}} \\
&\leq \int\limits_{\left[a^{(j)}+p_j\frac{\gamma}{m}, a^{(j)}+\gamma\right]} \left(u_j - \sqrt{n}\left(\theta_i^{(j)} - \theta^{(j)}\right)\right)^{-\varepsilon_i^{(j)}} du_j \\
&\quad + \frac{\gamma}{m}\left(a^{(j)} + p_j\frac{\gamma}{m} - \sqrt{n}\left(\theta_i^{(j)} - \theta^{(j)}\right)\right)^{-\varepsilon_i^{(j)}} \\
&\leq \underbrace{\int\limits_{\left[a^{(j)}+p_j\frac{\gamma}{m}-\sqrt{n}\left(\theta_i^{(j)}-\theta^{(j)}\right), a^{(j)}+\gamma-\sqrt{n}\left(\theta_i^{(j)}-\theta^{(j)}\right)\right]} u_j^{-\varepsilon_i^{(j)}} du_j}_{L_j} + \left(\frac{\gamma}{m}\right)^{1-\varepsilon_i^{(j)}}.
\end{aligned}$$

But the function $x \mapsto \int\limits_x^{x+a} u^{-\alpha} du$ decreases on $\mathbb{R}_+$ (if $a \geq 0$ and $\alpha \geq 0$) thus three cases appear:

- if $\varepsilon_i^{(j)} \in [0, 1[$, $\quad L_j \leq \displaystyle\int\limits_{\left[0,\left(1-\frac{p_j}{m}\right)\gamma\right]} u_j^{-\varepsilon_i^{(j)}} du_j \leq \frac{\gamma^{1-\varepsilon_i^{(j)}}}{1-\varepsilon_i^{(j)}}$,

- if $\varepsilon_i^{(j)} = 1$, $\quad L_j \leq \displaystyle\int\limits_{\left[\frac{\gamma}{m},\left(1-\frac{p_j-1}{m}\right)\gamma\right]} u_j^{-1} du_j = \log(m - p_j + 1) \leq \log m$,

- if $\varepsilon_i^{(j)} > 1$, $\quad L_j \leq \displaystyle\int\limits_{\left[\frac{\gamma}{m},\left(1-\frac{p_j-1}{m}\right)\gamma\right]} u_j^{-\varepsilon_i^{(j)}} du_j \leq \left(\frac{m}{\gamma}\right)^{\varepsilon_i^{(j)}-1} \frac{1}{\varepsilon_i^{(j)}-1}$.

The treatment of the integral $I_{2,j}$ is similar.

Therefore there exists $\widehat{C}_{\theta_i,j} > 0$, which depends only on $\varepsilon_i^{(j)}$ and γ such that

$$I_{1,j} + I_{2,j} \leq \widehat{C}_{\theta_i,j} m^{\sigma_j\left(\varepsilon_i^{(j)}\right)},$$

where, on the one hand, $\sigma_j(x) = (x-1)^+$ if $x \neq 1$ and $\sigma_j(1) = \xi_j$ (chosen such that $\sum_{j=1}^r \sigma_j\left(\varepsilon_i^{(j)}\right) < 1$), and on the other hand $\widehat{C}_{\theta_i,j} = \frac{\gamma^{1-\varepsilon_i^{(j)}}}{\left|1-\varepsilon_i^{(j)}\right|}$ if $\varepsilon_i^{(j)} \neq 1$ and $\widehat{C}_{\theta_i,j} = \frac{1}{\xi_j}$ otherwise. Thus, noting $\widehat{C}_{\theta_i} = \frac{1}{2}\prod_{j=1}^r \widehat{C}_{\theta_i,j}$ and using the fact that $\frac{1}{m} \leq \frac{1}{m^{1-\sum_{j=1}^r\left(\varepsilon_i^{(j)}-1\right)^+}}$, we have

$$\int_{\mathcal{C}(a,\gamma)\cap\mathcal{U}_{n,\theta_i}} \left\|\sqrt{\Psi_{\theta,t}} - \sqrt{\Psi_{\theta,\chi_m(t)}}\right\|_{L^2\left(\mathbf{P}_\theta^n\right)} \pi(\theta_{n,t})\,dt$$

$$\leq \left(3r\sqrt{2}\gamma^r + \gamma\sqrt{n}^{\sum_{j=1}^r \varepsilon_i^{(j)}}\sqrt{\sup_{t\in\mathcal{C}(a,\gamma)} \widetilde{C}_{\theta_i}(\theta_{n,t})\widehat{C}_{\theta_i}}\right)$$

$$\times \sup_{t\in\mathcal{C}(a,\gamma)} \pi(\theta_{n,t}) \frac{1}{m^{1-\sum_{j=1}^r\left(\varepsilon_i^{(j)}-1\right)^+}}$$

Inequality (7.4) is hence true with $\alpha_i = 1 - \sum_{j=1}^r\left(\varepsilon_i^{(j)}-1\right)^+$ and

$$K_{i,n} = \left(3r\sqrt{2} + \gamma^{1-r}\sqrt{n}^{\sum_{j=1}^r \varepsilon_i^{(j)}}\sqrt{\sup_{t\in\mathcal{C}(a,\gamma)} \widetilde{C}_{\theta_i}(\theta_{n,t})\widehat{C}_{\theta_i}}\right)\gamma^r$$

$$\times \sup_{t\in\mathcal{C}(a,\gamma)} \pi(\theta_{n,t}).$$

□

Under the assumptions of the the previous lemma, inequality (4.2) is fullfilled in $\mathcal{C}(a,\gamma)\cap\sqrt{n}\left(\mathcal{V}_{\theta_i}-\theta\right)$ with $\varphi_\cdot = \sqrt{\psi_{\theta,\cdot}}$. Therefore, Proposition 4 can be applied with $A = 1$, $s = 2$, $p = 2$, and using (7.2), we have

$$C \leq \overline{C}_{i,n}\gamma^{\frac{r}{2}}\sqrt{\sup_{t\in\mathcal{C}(a,\gamma)} \pi(\theta_{n,t})}, \tag{7.5}$$

where

$$\overline{C}_{i,n} = \max\left(\left(4\left(3r\sqrt{2} + \gamma^{1-r}\sqrt{n}^{\sum_{j=1}^r \varepsilon_i^{(j)}}\sqrt{\sup_{t\in\mathcal{C}(a,\gamma)} \widetilde{C}_{\theta_i}(\theta_{n,t})\widehat{C}_{\theta_i}}\right)\right)^{\frac{1-\omega_i}{2}} F_{\alpha_i}, \frac{1}{\omega_i}\right)$$

and $\omega_i = \frac{\alpha_i}{\alpha_i + r}$.

Lemma 3. *Assume that* (A1) *and* (B1) *hold.*

Then for all $\theta \in \Theta$, *there exists* $\omega \in]0,1[$ *such that for all* $\gamma > 0$, $n \in \mathbb{N}^*$, *and* $a \in \mathbb{R}^r$ *there exists* $\overline{C}_n > 0$ *such that*

$$\mathbf{E}_\theta \sqrt{\int_{\mathcal{C}(a,\gamma)} \Psi_{\theta,t}\, \pi(\theta_{n,t})\, dt} \leq \overline{C}_n \sqrt{\sup_{t\in\mathcal{C}(a,\gamma)} \pi(\theta_{n,t})} \left(\sup_{t\in\mathcal{C}(a,\gamma)} \mathbf{E}_\theta \sqrt{\Psi_{\theta,t}} \right)^{\omega}.$$

Proof. Fix $\theta \in \Theta$, $n \in \mathbb{N}^*$, $(a,\gamma) \in \mathbb{R}^r \times \mathbb{R}_+^*$.

By hypothesis, the Fisher information of the model $\mathcal{M}$ is locally bounded, hence bounded on every compact included in $\sqrt{n}\,(\Theta - \theta)$ and in particulary on $\mathcal{C}(a,\gamma) \cap \sqrt{n}\,(\Theta-\theta)$. Therefore, using (7.3), there exists $\overline{C}_0 > 0$ such that

$$\mathbf{E}_\theta \sqrt{\int_{\mathcal{C}(a,\gamma)\cap\sqrt{n}\left(\left(\cup_{i=1}^k \mathcal{V}_{\theta_i}\right)^c - \theta\right)} \Psi_{\theta,t}\, \pi(\theta_{n,t})\, dt}$$

$$\leq \overline{C}_0 \gamma^{\frac{r}{2}} \sqrt{\sup_{t\in\mathcal{C}(a,\gamma)} \pi(\theta_{n,t})} \left(\sup_{t\in\mathcal{C}(a,\gamma)} \mathbf{E}_\theta \sqrt{\Psi_{\theta,t}} \right)^{\omega_0},$$

where $\omega_0 = \frac{1}{1+r}$.

Moreover, by Lemma 3, assumption (4.2) is fullfilled in each of the k neighbourhoods $\mathcal{C}(a,\gamma) \cap \sqrt{n}\,(\mathcal{V}_{\theta_i} - \theta)$, hence Proposition 4 can be applied. Using inequality (7.5) and noting $\widetilde{\omega} = \min_{i=1,\cdots,k} \omega_i$, there exists a constant $\overline{C}_{i,n}$ such that

$$\mathbf{E}_\theta \sqrt{\int_{\mathcal{C}(a,\gamma)\cap\bigcup_{i=1}^k \mathcal{U}_{n,\theta_i,\tau}} \Psi_{\theta,t}\, \pi(\theta_{n,t})\, dt}$$

$$\leq \left(\sum_{i=1}^k \overline{C}_{i,n} \right) \gamma^{\frac{r}{2}} \sqrt{\sup_{t\in\mathcal{C}(a,\gamma)} \pi(\theta_{n,t})} \left(\sup_{t\in\mathcal{C}(a,\gamma)} \mathbf{E}_\theta \sqrt{\Psi_{\theta,t}} \right)^{\widetilde{\omega}}.$$

We note that $\overline{C}_n = \gamma^{\frac{r}{2}} \sum_{i=0}^k \overline{C}_{i,n}$ and $\omega = \min(\omega_0, \widetilde{\omega})$ to conclude. □

Remark 2. It is important to note that if $\mathcal{C}(a,\gamma) \subset \mathcal{B}(0, \tau\sqrt{n})$ then by construction, the constant $\overline{C}_n$ can be bounded above by $\sqrt{n}^{\eta} C_\tau$ where $\eta = \max_{i=1,\cdots,k} \left(\frac{1-\omega_i}{2} \sum_{j=1}^r \varepsilon_i^{(j)} \right)$ and C_τ is a constant that does not depend on the vertex a of the cube, nor on n, but on τ. It suffices to note that $\overline{C}_0$ does not depend on n (see (7.3)), that $n^{-\frac{\eta}{2}} \leq 1$ and to use (7.5).

Recall the following inequality, which is of interest only when $\mathcal{K}$ is bounded.

Lemma 4. *Let $\mathcal{M}$ be a model, $\mathcal{K}$ be a part of Θ and fix $\theta \in \Theta$.*

Then, for all $s \geq 0$,

$$\sup_{t\in\mathcal{B}(0,s)^c\cap\sqrt{n}(\mathcal{K}-\theta)} \mathbf{E}_\theta \sqrt{\Psi_{\theta,t}} \leq \exp\left(-\frac{1}{2}\rho_{\theta,\mathcal{K}}^2 s^2\right),$$

where $\rho_{\theta,\mathcal{K}} = \inf\limits_{u\in\mathcal{K}-\theta} \frac{d(\theta,\theta+u)}{\|u\|}$.

Proof. See Lemma 5.3 in [4, Chapter I]. □

It is not difficult to see that if $\mathcal{M}$ is DQM with I nonsingular and locally bounded, and $\mathcal{K}$ a convex bounded subset of $\mathbb{R}^r$ then $\rho_{\theta,\mathcal{K}}$ is positive.

Proof of Proposition 5. Let partition the exterior of the ball of center 0 and radius α with crowns $\mathcal{B}(0,l)^c \cap \mathcal{B}(0,l+1)$ and denote by $(\mathcal{C}(a_i,1))_{i\in I_l}$ a partition, with cubes, of $\mathcal{B}(0,l)^c \cap \mathcal{B}(0,l+1)$. Since I is locally bounded, there exists $\varepsilon_\theta > 0$ such that I is bounded on $\mathcal{B}(\theta,\varepsilon_\theta)$. Therefore, using once again (7.3) and then Lemma 4, there exists $\overline{C}_{\varepsilon_\theta} > 0$ such that

$$\begin{aligned}
&\mathbf{E}_\theta \sqrt{\int_{\mathcal{B}(0,\alpha)^c\cap\mathcal{B}\left(0,\varepsilon_\theta\sqrt{n}\right)} \|t\|^p\, \Psi_{\theta,t}\pi(\theta_{n,t})\, dt} \\
&\leq \sum_{\alpha\leq l<\varepsilon_\theta\sqrt{n}} (l+1)^{\frac{p}{2}} \sum_{i\in I_l} \mathbf{E}_\theta \sqrt{\int_{\mathcal{C}(a_i,1)\cap\mathcal{B}\left(0,\varepsilon_\theta\sqrt{n}\right)} \Psi_{\theta,t}\pi(\theta_{n,t})\, dt} \\
&\leq r\overline{C}_{\varepsilon_\theta} \sqrt{\sup_{u\in\mathcal{B}(\theta,\varepsilon_\theta)} \pi(u)} \sum_{l\geq\alpha} (l+1)^{\frac{p}{2}+r-1} \exp\left(-\frac{1}{2}\rho_{\theta,\mathcal{B}(\theta,\varepsilon_\theta)}^2 \omega l^2\right).
\end{aligned}$$

Using successively Lemmas 3 and 4, Remark 2, and the fact that $\sqrt{n}$ is bounded above by $\frac{l+1}{\varepsilon_\theta}$, there exists $C_A > 0$ such that

$$\mathbf{E}_\theta \sqrt{\int_{\mathcal{B}\left(0,\alpha\vee\varepsilon_\theta\sqrt{n}\right)^c\cap\mathcal{B}\left(0,A\sqrt{n}\right)} \|t\|^p\, \Psi_{\theta,t}\pi(\theta_{n,t})\, dt}$$

$$\leq \sum_{\alpha \vee \varepsilon_\theta \sqrt{n} \leq l < A\sqrt{n}} (l+1)^{\frac{p}{2}} \sum_{i \in I_l} \mathbf{E}_\theta \sqrt{\int_{\mathcal{C}(a_i,1)\cap \mathcal{B}\left(0,A\sqrt{n}\right)} \Psi_{\theta,t}\pi\left(\theta_{n,t}\right) dt}$$

$$\leq rC_A \sqrt{\sup_{u\in\mathcal{B}(\theta,A)} \pi\left(u\right)} \sum_{l\geq\alpha} (l+1)^{\eta+\frac{p}{2}+r-1} \exp\left(-\frac{1}{2}\rho^2_{\theta,\mathcal{B}(\theta,A)}\omega l^2\right).$$

It suffices then to remark that for all $s \geq 0$, $a > 0$, and $b > 0$

$$\sum_{l\geq\alpha} (l+1)^a \exp\left(-bl^2\right) = O\left(\alpha^{-s}\right).$$

□

Proof of Proposition 6. Let $A > 2\left\|\theta\right\|$. Studying with attention the proof of Lemma 3 and using Remark 2, it can be noticed that if $\mathcal{C}\left(a,\gamma\right) \subset \mathcal{B}\left(0,\tau\sqrt{n}\right)$, the constant $\overline{C}_n$ is bounded above by $\sqrt{n}^{\eta} C_\tau$ where $C_\tau = O\left(\tau^{\frac{z}{2}}\right)$. Let use again the partitions with crowns, and then cubes, defined in the proof of Proposition 5. Then there exists, thanks to the Lemma 3 and assumption (B4), a constant C such that

$$\sqrt{n}^s \mathbf{E}_\theta \sqrt{\int_{\mathcal{B}\left(0,A\sqrt{n}\right)^c} \left\|t\right\|^p \Psi_{\theta,t}\pi\left(\theta_{n,t}\right) dt}$$

$$\leq 2^{\frac{m}{2}} rC\sqrt{n}^{s+\eta-\frac{z-m}{2}} \Big(\sup_{t\in\mathcal{B}\left(0,A\sqrt{n}\right)^c} \mathbf{E}_\theta \sqrt{\Psi_{\theta,t}}\Big)^{\omega} \sum_{l\geq A} \left(\frac{l}{l+1}\right)^{-\frac{m}{2}} (l+1)^{r-1+\frac{z-m}{2}+\frac{p}{2}}.$$

But $\sup_{t\in\mathcal{B}\left(0,A\sqrt{n}\right)^c} \mathbf{E}_\theta \sqrt{\Psi_{\theta,t}} = \left(\sup_{v\in\mathcal{B}(0,A)^c} \mathbf{E}_\theta \sqrt{\frac{f_{\theta+v}}{f_\theta}\left(X_1\right)}\right)^n$. Note that $\limsup_{\|v\|\to\infty} \mathbf{E}_\theta \sqrt{\frac{f_{\theta+v}}{f_\theta}\left(X_1\right)} < 1$, by the equality

$$\mathbf{E}_\theta \sqrt{\frac{f_{\theta+v}}{f_\theta}\left(X_1\right)} = 1 - \frac{1}{2} d^2\left(\theta,\theta+v\right)$$

and the assumption (B3). Therefore, there exists $A > 2\left\|\theta\right\|$ such that for all $s \geq 0$, $\left(\sqrt{n}^s \left(\sup_{v\in\mathcal{B}(0,A)^c} \mathbf{E}_\theta \sqrt{\frac{f_{\theta+v}}{f_\theta}\left(X_1\right)}\right)^{n\omega}\right)_n$ is a bounded sequence.
□

The proof of Theorem 1 is actually contained in the more general one of Theorem 2, which takes into account the unbounded part of Θ, which imposes assumptions (B2), (B3), and (B4).

Proof of Theorem 2. First we show that the Bayes estimator is asymptotically efficient and regular.

In the right side of inequality (3.1), let separate the numerator in two parts, the one treating of the interior of a ball $\mathcal{B}(0,\alpha)$, and the other of its exterior. By Proposition 3, there exists an increasing sequence of nonnegative real numbers $(\alpha_n)_n$ tending to the infinity such that

$$\frac{\int\limits_{\mathcal{B}(0,\alpha_n)} \left(t - I^{-1}(\theta)\, S_{n,\theta}^{k_n}\right) \Psi_{\theta,t}\pi(\theta_{n,t})\, dt}{\int\limits_{\sqrt{n}(\Theta-\theta)} \Psi_{\theta,t}\pi(\theta_{n,t})\, dt} = o_{\mathbf{P}_\theta^n}(1).$$

Furthermore,

$$\begin{aligned} &\left\| \int\limits_{\mathcal{B}(0,\alpha_n)^c} \left(t - I^{-1}(\theta)\, S_{n,\theta}^{k_n}\right) \Psi_{\theta,t}\pi(\theta_{n,t})\, dt \right\| \\ &\leq \int\limits_{\mathcal{B}(0,\alpha_n)^c} \left(\|t\| + \left\| I^{-1}(\theta)\, S_{n,\theta}^{k_n} \right\|\right) \Psi_{\theta,t}\pi(\theta_{n,t})\, dt. \end{aligned}$$

Let now separate $\mathcal{B}(0,\alpha_n)^c$ in a bounded part $\mathcal{B}(0,\alpha_n)^c \cap \mathcal{B}(0, A\sqrt{n})$ and a nonbounded part $\mathcal{B}(0,\alpha_n \vee A\sqrt{n})^c$, where A is a positive real number. Note that $I^{-1}(\theta)\, S_{n,\theta}^{k_n}$ converges in $\mathbf{P}_\theta^n$-distribution to $\mathcal{N}\left(0, I^{-1}(\theta)\right)$, it sufficies then to use, with $p=1$ and $p=0$, assertion (4.3) on $\mathcal{B}(0,\alpha_n)^c \cap \mathcal{B}(0, A\sqrt{n})$ and then Proposition 6 on $\mathcal{B}(0,\alpha_n \vee A\sqrt{n})^c$.

Let q be a positive number. Since $\widehat{\theta}_n = \mathbf{E}(\Delta | (X_1,\cdots,X_n))$ $\mathbf{P}_\theta^n$-almost everywhere, by the Jensen's inequality and passing to the expectation, we have

$$\mathbf{E}_\theta \left\| \sqrt{n}\left(\widehat{\theta}_n - \theta\right) \right\|^q \leq \sum_{l\geq 0} (l+1)^q\, \mathbf{E}_\theta \frac{\int\limits_{\mathcal{B}(0,l)^c \cap \mathcal{B}(0,l+1)} \Psi_{\theta,t}\pi(\theta_{n,t})\, dt}{\int\limits_{\sqrt{n}(\Theta-\theta)} \Psi_{\theta,t}\pi(\theta_{n,t})\, dt}.$$

Let $\xi > 0$ such that $\mathcal{B}(\theta,\xi) \subset \Theta$ and $\sup_{u\in\mathcal{B}(\theta,\xi)} \sqrt{\|I(u)\|}$ is finite.

We can apply the following inequality

$$\frac{x}{z} \leq 1_{\{y\leq b\}} + 1_{\{x\geq a\}} + \frac{a}{b}, \quad \forall (a,b) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \text{ if } 0 < x \leq y \leq z$$

to $x = \int\limits_{\mathcal{B}(0,l)^c \cap \mathcal{B}(0,l+1)} \Psi_{\theta,t}\pi(\theta_{n,t})\,dt$, $z = \int\limits_{\sqrt{n}(\Theta-\theta)} \Psi_{\theta,t}\pi(\theta_{n,t})\,dt$, $y = \int\limits_{\mathcal{B}(0,\xi\sqrt{n})} \Psi_{\theta,t}\pi(\theta_{n,t})\,dt$, and $a = J_{n,l}^{\zeta}$, where

$$J_{n,l} = \mathbf{E}_\theta \sqrt{\int\limits_{\mathcal{B}(0,l)^c \cap \mathcal{B}(0,l+1)} \Psi_{\theta,t}\pi(\theta_{n,t})\,dt}$$

and $b = K_{n,l}$. Two positive constants $K_{n,l}$ and ζ will be chosen later. Note that if we suppose that $\frac{K_{n,l}}{\inf_{u\in\mathcal{B}(\theta,\xi)}\pi(u)} < \xi_{n,l}^r \leq (\xi\sqrt{n})^r$ where $\xi_{n,l}$ will be chosen later, we have

$$\begin{aligned}
&\mathbf{P}_\theta^n\left(\left\{\int\limits_{\mathcal{B}(0,\xi\sqrt{n})} \Psi_{\theta,t}\pi(\theta_{n,t})\,dt \leq K_{n,l}\right\}\right)\\
\leq&\mathbf{P}_\theta^n\left(\left\{\int\limits_{\mathcal{B}(0,\xi\sqrt{n})} \Psi_{\theta,t}dt \leq \frac{K_{n,l}}{\inf_{u\in\mathcal{B}(\theta,\xi)}\pi(u)}\right\}\right)\\
\leq&\mathbf{P}_\theta^n\left(\left\{\int\limits_{\mathcal{B}(0,\xi_{n,l})} \Psi_{\theta,t}dt \leq \frac{K_{n,l}}{\inf_{u\in\mathcal{B}(\theta,\xi)}\pi(u)}\right\}\right)\\
\leq&\mathbf{P}_\theta^n\left(\left\{\int\limits_{\mathcal{B}(0,\xi_{n,l})} (\Psi_{\theta,t}-1)\,dt \leq \frac{K_{n,l}}{\inf_{u\in\mathcal{B}(\theta,\xi)}\pi(u)} - \xi_{n,l}^r\right\}\right).
\end{aligned}$$

By the Chebychev's inequality, we have

$$\mathbf{P}_\theta^n\left(\left\{\int\limits_{\mathcal{B}(0,\xi\sqrt{n})} \Psi_{\theta,t}\pi(\theta_{n,t})\,dt \leq K_{n,l}\right\}\right) \leq \frac{\mathbf{E}_\theta \int\limits_{\mathcal{B}(0,\xi_{n,l})} |\Psi_{\theta,t}-1|\,dt}{\xi_{n,l}^r - \frac{K_{n,l}}{\inf_{u\in\mathcal{B}(\theta,\xi)}\pi(u)}}.$$

Since

$$\begin{aligned}
\mathbf{E}_\theta \int\limits_{\mathcal{B}(0,\xi_{n,l})} |\Psi_{\theta,t}-1|\,dt &\leq 2\sqrt{n} \int\limits_{\mathcal{B}(0,\xi_{n,l})} d(\theta,\theta_{n,t})\,dt\\
&\leq \xi_{n,l}^{r+1} \sup_{u\in\mathcal{B}(\theta,\xi)} \sqrt{\|I(u)\|},
\end{aligned}$$

we can then choose $K_{n,l} = \frac{1}{2}\xi_{n,l}^r \inf_{u\in\mathcal{B}(\theta,\xi)} \pi(u)$ and thus

$$\mathbf{P}_\theta^n \left(\left\{ \int\limits_{\mathcal{B}\left(0,\xi\sqrt{n}\right)} \Psi_{\theta,t}\pi(\theta_{n,t})\,dt \leq K_{n,l} \right\} \right) \leq 2\xi_{n,l} \sup_{u\in\mathcal{B}(\theta,\xi)} \sqrt{\|I(u)\|}.$$

It allows us to state the following inequality

$$\begin{aligned} &\mathbf{E}_\theta \left\| \sqrt{n}\left(\widehat{\theta}_n - \theta\right) \right\|^q \\ &\leq \sum_{l\geq 0} (l+1)^q \left(2\xi_{n,l} \sup_{u\in\mathcal{B}(\theta,\xi)} \sqrt{\|I(u)\|} + J_{n,l}^{1-\frac{\zeta}{2}} \right. \\ &\left. + \left(\frac{1}{2} \inf_{u\in\mathcal{B}(\theta,\xi)} \pi(u) \right)^{-1} \frac{J_{n,l}^{\zeta}}{\xi_{n,l}^r} \right). \end{aligned}$$

Choosing $\zeta = \frac{2(r+1)}{r+3}$ and $\xi_{n,l} = J_{n,l}^{\frac{2}{r+3}}$, there exists $C > 0$ such that

$$\mathbf{E}_\theta \left\| \sqrt{n}\left(\widehat{\theta}_n - \theta\right) \right\|^q \leq C \left(J_{n,0}^{\frac{2}{r+3}} + \sum_{l\geq 1} (l+1)^q J_{n,l}^{\frac{2}{r+3}} \right).$$

First we note that by the Jensen inequality and since $\left\| \sqrt{\Psi_{\theta,t}} \right\|_{L^2\left(\mathbf{P}_\theta^n\right)} \leq 1$, we have

$$J_{n,0} \leq \sqrt{\int\limits_{\mathcal{B}(0,1)} \pi(\theta_{n,t})\,dt}.$$

Then the continuity of π implies that $J_{n,0}$ is uniformly bounded in n.

Second, for all $p \geq 0$ and $l \in \mathbb{N}^*$

$$J_{n,l} \leq l^{-\frac{p}{2}} \mathbf{E}_\theta \sqrt{\int\limits_{\mathcal{B}(0,l)^c\cap\mathcal{B}(0,l+1)} \|t\|^p\, \Psi_{\theta,t}\pi(\theta_{n,t})\,dt}.$$

Let A a positive number defined in Proposition 6.

If $l \in \mathbb{N}^* \cap [1, A\sqrt{n}[$ then, using Proposition 5, there exists a positive constant $\widetilde{C}$, independant of n such that

$$\begin{aligned} J_{n,l} &\leq l^{-\frac{p}{2}} \mathbf{E}_\theta \sqrt{\int\limits_{\mathcal{B}(0,1)^c\cap\mathcal{B}\left(0,(A+1)\sqrt{n}\right)} \|t\|^p\, \Psi_{\theta,t}\pi(\theta_{n,t})\,dt} \\ &\leq \widetilde{C} l^{-\frac{p}{2}} \end{aligned}$$

If $l \geq A\sqrt{n}$ then, using Proposition 6 for all $p < m - z - 2r$, there exists a positive constant $\widehat{C}$ independent of n such that

$$\begin{aligned} J_{n,l} &\leq l^{-\frac{p}{2}} \mathbf{E}_\theta \sqrt{\int\limits_{\mathcal{B}\left(0, A\sqrt{n}\right)^c} \|t\|^p \, \Psi_{\theta,t} \, \pi\left(\theta_{n,t}\right) dt} \\ &\leq \widehat{C} l^{-\frac{p}{2}}. \end{aligned}$$

Therefore, for all $l \geq 1$, $J_{n,l} \leq \max\left(\widetilde{C}, \widehat{C}\right) l^{-\frac{p}{2}}$ and thus $\sum\limits_{l>0} (l+1)^q \, J_{n,l}^{\frac{2}{r+3}}$ is bounded above by $\max\left(\widetilde{C}, \widehat{C}\right) \sum_{l>0} (l+1)^{q - \frac{p}{r+3}}$ which converges if $(q+1)(r+3) < p < m - z - 2r$.

Since $\widehat{\theta}_n$ is asymptotically efficient, it suffices to apply Theorem 2.22 of Chapter II in [1], to prove that $\widehat{\theta}_n$ is ℓ-asymptotically of minimal risk.

Studying with attention the definition of the constants, it can be note that they are locally bounded in θ. To establish that $\widehat{\theta}_n$ is ℓ-asymptotically minimax, a "uniform" version of Lemma 4 is necessary. Most precisely, it can be shown (see [10]) that for all $s \geq 0$ and $\tau > 0$, there exists $\rho_{\theta,\mathcal{K},\tau} > 0$ and $C_{\theta,\mathcal{K},\tau} > 0$ such that for all $n \in \mathbb{N}^*$

$$\sup_{u \in \mathcal{B}(\theta,\tau)} \sup_{t \in \mathcal{B}(0,s)^c \cap \sqrt{n}(\mathcal{K}-u)} \mathbf{E}_u \sqrt{\Psi_{u,t}} \leq C_{\theta,\mathcal{K},\tau} \exp\left(-\frac{1}{2}\rho^2_{\theta,\mathcal{K},\tau} s^2\right).$$

Furthermore, it should be noticed that the assumption (B3) implies the fact that there exists $\tau > 0$ and $A > 0$ such that

$$\inf_{u \in \mathcal{B}(\theta,\tau)} \inf_{v \in \mathcal{B}(0,A)^c} d\left(u, u+v\right) > 0.$$

□

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