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# LARGE DEVIATIONS FOR SAMPLE PATHS OF GAUSSIAN PROCESSES QUADRATIC VARIATIONS

ABSTRACT. We show a functional large deviations principle for the family of random functions $\left\{ V_n(x) = \sum_{k=1}^{[nx]} (Z_{k/n} - Z_{k-1/n})^2,\ x \in [0,1] \right\}$, where $\{Z_t,\ t \in [0,1]\}$ is a real valued centered Gaussian process.

## 1. INTRODUCTION

Quadratic variations of Gaussian processes were first studied by Lévy [4] who showed that for a standard Wiener process $Z$ on $[0,1]$, almost surely, as $n \to +\infty$,

$$\sum_{k=1}^{2^n} [Z_{k/2^n} - Z_{k-1/2^n}]^2 \longrightarrow 1.$$

Further, Baxter [3] showed that for a Gaussian nondifferentiable process $Z$, some smoothness assumptions on the mean value and on the covariance function outside the diagonal ensure the almost sure convergence of

$$V_n = \sum_{k=1}^{n} (Z_{k/n} - Z_{k-1/n})^2. \tag{1}$$

Later on, Gladyshev [2] improved this result. An important generalization of these variations was considered by Guyon and Léon [17]. They studied so-called $H$-variations, which generalize the quadratic variations: for $Z$ a Gaussian centered stationary nondifferentiable process, they introduced

$$V_{H,n} = \sum_{k=1}^{n} H\Big( \frac{Z_{k/n} - Z_{k-1/n}}{\sqrt{2(r(0) - r(1/n))}} \Big),$$

where $H$ is a sufficiently smooth real function, and the covariance function $r$ of the process $Z$ has form

$$r(t) = 1 - t^{\beta} L(t), \quad 0 < \beta < 2,$$

where $L$ is a slowly varying function at 0. Next, Istas and Lang [16] introduced general quadratic forms to estimate the Hölder index of a process. The discretization in this case is no longer regular, and $Z$ is taken at values $Z_{j\Delta(n)}$, where $j = 1, \cdots, n$ and $(\Delta(n), n) \to (0, +\infty)$.

Perrin [5] studied the random process of quadratic variations corresponding to the linear interpolation of the points $(p/n; V_n(p/n))$, where

$$V_n\left(\frac{p}{n}\right) = \sum_{k=1}^{p} (Z_{k/n} - Z_{k-1/n})^2 \tag{2}$$

and $Z$ is a Gaussian process with sufficient smoothness of covariance function. He showed the asymptotic normality of this process. He applied this result to study the deformation of a Gaussian stationary process by a function $\Phi$: the estimator $\widehat{\Phi}$ of $\Phi$ is built from the quadratic form (2)).

The aim of this paper is to evaluate the associated large deviations, and to derive a LDP for $\widehat{\Phi}$.

Section 2 is devoted to the presentation of the notation and of the main results of the paper. Section 3 is devoted to the proof of these results. Section 4 gathers the proofs of technical lemmas.

## 2. Main results

### 2.1. Framework and notation

Let $Z = \{Z_t, t \in [0,1]\}$ be a real valued centered Gaussian process with covariance function $r : (x,y) \in [0,1]^2 \to r(x,y) \in \mathbb{R}^+$ satisfying:

**(A1)**: $r(x,y)$ is continuous on $[0,1]^2$. It has second derivatives uniformly bounded for $x \neq y$.

We define

$$D^-(x) = \lim_{y \to x, y < x} \frac{\partial}{\partial y} r(x,y), \quad x \in ]0,1]$$

$$D^+(x) = \lim_{y \to x, x < y} \frac{\partial}{\partial y} r(x,y), \quad x \in [0,1[$$

and the singularity function $\alpha$ of $Z$ by

$$\alpha(x) = D^-(x) - D^+(x), \quad x \in ]0,1[. \tag{3}$$

By assumptions on $r$, $\alpha$ is uniformly continuous on $[0,1]$. We consider quadratic variations of $Z$ built on a regular subdivision of $[0,1]$: for $n \in \mathbb{N}$, $n \geq 1$, define for $k = 1, \cdots, n$

$$\Delta Z_k = Z_{k/n} - Z_{k-1/n},$$

and the random function

$$V_n(x) = \sum_{k=1}^{[nx]} (\Delta Z_k)^2, \quad x \in [0,1], \tag{4}$$

where $[x]$ denotes the largest integer smaller than or equal to $x$. For $[nx] = 0$, we assume the sum to be null. The function $V_n$ is càdlàg on $[0,1]$. We define a continuous version of $V_n$ which is the linear interpolation of the points $(p/n, V_n(p/n))$ with $p \in \{1, \cdots, n\}$:

$$\begin{cases} v_n(x) = V_n(x) + (nx - [nx])(\Delta Z_{[nx]+1})^2, & x \in [0,1[ \\ v_n(1) = V_n(1). \end{cases}$$

Assume

**(A2)**: $\alpha$ has bounded first derivative on $[0,1]$.

We know from Baxter [3] that $V_n$ is a strongly consistent estimator of $\int_0^{\cdot} \alpha(u)du$. Perrin showed the corresponding convergence in distribution:

**Theorem 2.1** (Perrin). *Under assumptions* **(A1)**–**(A2)**, *the process*

$$\left\{ \sqrt{n}(v_n(x) - \int_0^x \alpha(u)du), \;\; x \in [0,1] \right\}$$

*converges in distribution in* $\mathcal{C}[0,1]$ *– the space of continuous functions on* $[0,1]$, *as* $n$ *tends to* $\infty$, *towards*

$$\left\{ \int_0^x \sqrt{2}\alpha(u)dW_u, \;\; x \in [0,1] \right\},$$

*where* $W$ *is a standard Wiener process.*

The aim of this paper is to derive a large deviation principle for the random function $V_n$ (or, equally, $v_n$). For that purpose, we consider the distribution derivative $\nu_n$ of $V_n$:

$$\nu_n = \sum_{k=1}^{n} (\Delta Z_k)^2 \delta_{k/n}. \tag{5}$$

Analogous random measures have been studied by Dembo and Zeitouni [10], Gamboa and Gassiat[12], Gamboa, Rouault, and Zani[13]. In Zani

[15], one can find a version of the present result: large deviations for $\sum_{k=1}^{[nt]} X_k^2$ when $(X_n)_n$ is a stationary Gaussian seqvence with positive spectral density on the torus. When the $X_k^2$ are i.i.d., it is the famous Mogulskii theorem (see [6] and [7]). In [1], Najim studied LDP for general measures $\sum_{k=1}^{n} Z_i \delta_{x_i^n}$ where $(Z_i)$ are i.i.d and the empirical measure based on the $(x_i^n)$ converges weakly to a probability measure $R$.

## 2.2. Large deviations

The Gärtner–Ellis theorem (see [11]), in its abstract version (Baldi), however not valid in this case, will be the main thread of our work. The principal tool is the normalized cumulant generating function (log-Laplace transform).

For any $h \in \mathcal{C}([0,1])$, define

$$\Lambda(h) = \begin{cases} -\dfrac{1}{2} \displaystyle\int_{[0,1]} \log(1 - 2\alpha(t)h(t))\, dt & \text{if} \quad h \in \mathcal{D}_\alpha \\ +\infty \quad \text{otherwise,} \end{cases}$$

where

$$\mathcal{D}_\alpha = \{h \in \mathcal{C}([0,1]) : \quad \text{for any} \quad t \in [0,1], \quad \alpha(t)h(t) < 1/2\}.$$

Let $\mathcal{M}([0,1])$ be the set of positive bounded measures on $[0,1]$, endowed with the weak topology. Define $\Lambda^*$ the Legendre dual of $\Lambda$ for any $\mu$ in $\mathcal{M}([0,1])$ by

$$\Lambda^*(\mu) = \sup_{h \in \mathcal{C}([0,1])} \left\{ \int_{[0,1]} h\, d\mu - \Lambda(h) \right\}. \tag{6}$$

From Rockafellar ([14, Theorem 5]), we can give an explicit expression of $\Lambda^*$. Set for any $\tau > 0$, $\gamma(\tau) = \frac{1}{2}(\tau - 1 - \log \tau)$.

**Proposition 2.2.** *Let $\mu \in \mathcal{M}([0,1])$ having with respect to the Lebesgue measure $\lambda$ the decomposition $\mu = l\lambda + \mu^\perp$, where $l \in \mathcal{C}([0,1])$ and $\mu^\perp$ is the singular part. Then whenever $\gamma(\frac{l}{\alpha}) \in L_1([0,1])$ and $\alpha^{-1} \in L_1(d\mu^\perp)$,*

$$\Lambda^*(\mu) = \int_{[0,1]} \gamma\Big(\frac{l(t)}{\alpha(t)}\Big)\, dt + \int_{[0,1]} \frac{d\mu^\perp(t)}{2\alpha(t)},$$

*and otherwise, $\Lambda^*(\mu) = +\infty$.*

We make an additional assumption on $\alpha$:

**(A3)**: $\alpha$ has a countable number $\{\alpha_k\}_{k\geq 0}$ of zeros in $[0,1]$.

We can now present the main result of this paper.

**Theorem 2.3.** *Under assumptions* **(A1)**–**(A3)**, *the family* $\{\nu_n\}$ *satisfies a* LDP *in* $\mathcal{M}([0,1])$ *endowed with the weak topology, with speed* $n$ *and good rate function* $\Lambda^*$.

We can give a functional version of this result. Let $D([0,1],\mathbb{R})$ be the space of càdlàg real functions on $[0,1]$ and let $bv([0,1],\mathbb{R}) \subset D([0,1],\mathbb{R})$ be the space of functions with bounded variation. We can identify $bv([0,1],\mathbb{R})$ and $\mathcal{M}([0,1])$ as follows: to $\varphi$ in $bv([0,1],\mathbb{R})$ corresponds $d\varphi$ in $\mathcal{M}([0,1])$ characterized by $d\varphi([0,t]) = \varphi(t)$. Endow $bv([0,1],\mathbb{R})$ with the weak-$*$ topology denoted by $\sigma$. Now, let us define the rate function associated to $V_n$. Let $\varphi$ be in $bv([0,1],\mathbb{R})$ and $d\varphi$ the associated measure in $\mathcal{M}([0,1])$. Set

$$J(\varphi) = \int\limits_{[0,1]} \gamma\left(\frac{\dot{\varphi}(t)}{\alpha(t)}\right)\, dt + \int\limits_{[0,1]} \frac{d\varphi^{\perp}(t)}{2\alpha(t)}.$$

where $\dot{\varphi}(t)dt$ and $d\varphi^{\perp}$ are respectively, the absolute continuous part and the singular part of the measure $d\varphi$ associated to $\varphi$. The following theorem is the transcription of Theorem 2.3 in the space $bv([0,1],\mathbb{R})$ (see de Acosta [19]).

**Theorem 2.4.** *The family of random functions* $\{V_n(t),\, t \in [0,1]\}$ *satisfies a* LDP *on the space* $(bv([0,1]),\sigma)$, *with speed* $n$ *and rate function* $J$.

**Corollary 2.5.** *Almost surely,* $\nu_n$ *converges weakly towards* $\nu$.

### 2.3. Application to time deformation

Let $Z$ be our previous Gaussian process satisfying **(A1)** to **(A3)**, and suppose it has form

$$Z_t = \delta(\Phi(t)) \quad t \in [0,1], \tag{7}$$

where $\delta$ is a stationary process with known covariance $R$ and $\Phi$ is the deformation function satisfying

**(B)**: $\Phi$ is bijective and has uniformly bounded second derivatives in $[0,1]$, as well as its inverse.

Expression (7) is equivalent to the following one:

$$r(x,y) = R(\Phi(y) - \Phi(x)).$$

The singularity function $\alpha$ (see [5] for more details) satisfies

$$\Phi(x) = \frac{\int_0^x \alpha(u)du}{\int_0^1 \alpha(u)du},$$

hence an estimator of $\Phi$ is given by

$$\widehat{\Phi}_n(x) = \frac{v_n(x)}{v_n(1)}.$$

Perrin showed that $||\widehat{\Phi}_n - \Phi||_\infty \to 0$ as $n \to \infty$. We give a large deviation result on $\widehat{\Phi}_n$.

**Theorem 2.6.** *Under assumptions* **(A1)** *to* **(A3)** *and* **(B)**, *the family* $\{\widehat{\Phi}_n\}_n$ *satifies a large deviations principle in* $(bv([0,1],\sigma)$ *with rate function*

$$H(\psi) := \inf\left\{J(\phi); \phi \in bv([0,1]); \psi(\cdot) = \frac{\phi(\cdot)}{\phi(1)}\right\}.$$

## Proof of Theorem 2.3

### 3.1. Convergence of $\nu_n$

We first focuse on the asymptotic properties of the measure $\nu_n$.

**Proposition 3.1.** *Let* $h \in \mathcal{C}([0,1])$, *then* $(\nu_n(h))_n$ *converges in probability to* $\nu(h)$, *where the measure* $\nu$ *is defined, for any* $h$ *in* $\mathcal{C}[0,1]$, *by*

$$\nu(h) = \int_{[0,1]} h(t)\alpha(t)\, dt. \tag{8}$$

**Proof of Proposition 3.1.** We reduce the quadratic form:

$$\nu_n(h) = \sum_{k=1}^{n} (\Delta Z_k)^2 h\left(\frac{k}{n}\right) = \sum_{k=1}^{n} \lambda_{k,n}\chi_{k,n}, \tag{9}$$

where $\{\chi_{k,n}\}_{k=1,\cdots,n}$ are $\chi^2(1)$–distributed, i.i.d. random variables, and $\{\lambda_{k,n}\}_{k=1,\cdots,n}$ are the eigenvalues of the matrix $\Gamma_n^{1/2} D_n(h)\Gamma_n^{1/2}$. The matrix $\Gamma_n = L_n \Sigma_n^t L_n$ is the covariance matrix of the vector $(\Delta Z_1, \cdots, \Delta Z_n)$ and $\Sigma_n$ the covariance matrix of the Gaussian vector

$(Z_{0/n}, Z_{1/n}, \cdots, Z_1)$, $L_n$ is the matrix with $n$ rows and $n+1$ columns, defined as follows

$$L_n = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & -1 & 1 \end{pmatrix}$$

and $D_n(h)$ is the $n \times n$ diagonal diag $\{h(i/n),\ i = 1, \cdots, n\}$. The two following lemmas, whose proofs are postponed to the Appendix (see Sec. 4), detail the asymptotic properties of $(\lambda_{k,n})$.

**Lemma 3.2.** *The family $\{n\lambda_{k,n}\}$ is uniformly bounded in $(k, n)$.*

**Lemma 3.3.** *For any $p \in \mathbb{N}$, $p \geq 1$,*

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=1}^{n} (n\lambda_{k,n})^p = \int_{[0,1]} (\alpha(t)h(t))^p \, dt.$$

Applying Lemma 3.3,

$$\lim_{n\to+\infty} E(\nu_n(h)) = \int_{[0,1]} \alpha(t)h(t) \, dt = \nu(h),$$

and

$$\lim_{n\to+\infty} n\mathrm{Var}\ (\nu_n(h)) = 2 \int_{[0,1]} (\alpha(t)h(t))^2 \, dt,$$

which ends the proof of Proposition 3.1.

**Proof of Theorem 2.3**

A result analogous to Theorem 2.3 was established in another framework in [12] with both time-invariant discretization and random weights (see also [10] and [9]).

The proof is based on the ideas of Baldi's theorem (see [11, Theorem 4.5.20]). We introduce the normalized cumulant generating function of $(\nu_n)$: any $h \in \mathcal{C}([0,1])$, $\Lambda_n(h) := \frac{1}{n} \log E(\exp\{n\nu_n(h)\})$. This function $\Lambda_n$ will not be defined everywhere but a careful study of the exposed points shows that it does not matter.

From expression (9), we have easily, for any $h \in \mathcal{C}([0,1])$

$$\Lambda_n(h) = \begin{cases} -\dfrac{1}{2n} \displaystyle\sum_{k=1}^{n} \log(1 - 2n\lambda_{k,n}) & \text{if for any} \quad k, \quad n\lambda_{k,n} < 1/2 \\ +\infty & \text{otherwise.} \end{cases}$$

From Lemma 3.3, we can determine the limit of $\Lambda_n$ in two cases:

- if for any $t \in [0,1]$ $\alpha(t)h(t) < 1/2$, then

$$\lim_{n\to+\infty} \Lambda_n(h) = -\frac{1}{2}\int_{[0,1]} \log\left(1 - 2\alpha(t)h(t)\right) dt = \Lambda(h).$$

- if there exists $t \in [0,1]$ $\alpha(t)h(t) > 1/2$ then for $n$ large enough, $\Lambda_n(h) = +\infty$ and

$$\lim_{n\to+\infty} \Lambda_n(h) = +\infty = \Lambda(h).$$

Whenever $h$ does not belong to any of these two categories, we do not know what happens for the asymptotic behaviour of $\Lambda_n$. These two cases do not cover the whole set $\mathcal{C}([0,1])$. Nevertheless, this will be sufficient for the LDP, since they contain a dense subset of exposing hyperplanes of $\Lambda^*$. Recall that $\mu$ in $\mathcal{M}([0,1])$ is an exposed point of $\Lambda^*$ if there exists a function $h$ of $\mathcal{C}([0,1])$ such that for any $\zeta$ in $\mathcal{M}([0,1])$, $\zeta \neq \mu$,

$$\int h\, d\mu - \Lambda^*(\mu) > \int h\, d\zeta - \Lambda^*(\zeta). \tag{10}$$

The function $h$ is called the exposing hyperplane associated to $\mu$. Let us now prove the upper and lower bound of the Theorem 2.3.

**Upper bound**

From Theorem 4.5.3 b) of [11], and the following lemma, which proof is postponed to the Appendix, the upper bound holds for compact sets.

**Lemma 3.4.** *For any $\delta > 0$ and $\mu$ in $\mathcal{M}([0,1])$, there exists $h_\delta$ in $\mathcal{C}([0,1])$ such that:*

$$\text{for any} \quad t \in [0,1], \quad \alpha(t)h_\delta(t) < 1/2$$

$$\int_{[0,1]} h_\delta(t)\mu(dt) - \Lambda(h_\delta) \geq \Lambda^*_\delta(\mu), \tag{11}$$

*where*

$$\Lambda^*_\delta(\mu) = \min\left\{\Lambda^*(\mu) - \delta, \frac{1}{\delta}\right\}.$$

**Exponential tightness**

Note that for a real number $a$,

$$\Big\{ \sup_{\|h\|_\infty \leq 1} \nu_n(h) \geq a\} \subset \{\nu_n(1) \geq a \Big\}.$$

Denote by $A = \operatorname*{esssup}_{t \in [0,1]}\{\alpha(t)\}$. For any $\theta < 1/2A$,

$$\limsup_n \frac{1}{n} \log P(\nu_n(1) \geq a) \leq -\theta a - \frac{1}{2} \int_{[0,1]} \log\left(1 - 2\theta\alpha(t)\right) dt$$

and

$$\lim_{a \to +\infty} \limsup_n \frac{1}{n} \log P(\nu_n(1) \geq a) = -\infty.$$

Hence the sequence $(\nu_n)$ is exponentially tight, and the upper bound holds for any closed set of $\mathcal{M}([0,1])$.

**Lower bound**

We study the set of exposed points of $\Lambda^*$. Define $\mathcal{F}$ a subset of $\mathcal{M}([0,1])$ such that $\mu \in \mathcal{F}$ if

- $\mu = l\lambda$ with $l \in \mathcal{C}[0,1]$, $l(t) > 0$ on $[0,1]$.
- For any $\alpha_k$, a zero of $\alpha$, $\lim_{t \to \alpha_k} \frac{l(t) - \alpha(t)}{\alpha(t)}$ is finite.

In the two following lemmas, we describe $\mathcal{F}$ as a dense subset of the exposed points of $\Lambda^*$.

**Lemma 3.5.** *Let $\mu = l\lambda$ be in $\mathcal{F}$. Set*

$$f(t) = \frac{1}{2\alpha(t)}\Big(1 - \frac{\alpha(t)}{l(t)}\Big), \quad t \in [0,1].$$

*Then $\mu$ is an exposed point of $\Lambda^*$ with exposing hyperplane $f$.*

**Lemma 3.6.** *Let $\mu$ be in $\mathcal{M}([0,1])$ such that $\Lambda^*(\mu) < +\infty$. There exists a sequence of measures in $(\mu_n)$ in $\mathcal{G}$ such that $(\mu_n)$ converges weakly to $\mu$ and $\lim_{n \to +\infty} \Lambda^*(\mu_n) = \Lambda^*(\mu)$.*

**Proof of Corollary 2.5.** We apply Theorem 2.3 along with Borel–Cantelli lemma.

## 4. Appendix

**Proof of Lemma 3.2**

In order to prove this lemma, we bound the norm of the covariance matrix $n\Gamma_n^{1/2} D_n(h)\Gamma_n^{1/2}$. Let us consider the following norm on operators: if $A$ is a linear operator on a Hilbert space, set

$$||A|| = \sup_{||x||=1} ||Ax||,$$

where $\langle\, ,\, \rangle$ is the inner product and $||x|| = \langle x, x\rangle$. When $A$ is hermitian,

$$||A|| \;=\; \sup_{||x||=1} |\langle x, Ax\rangle|. \tag{12}$$

This norm $||.||$ is unitarily invariant, hence submultiplicative (see [18, Proposition IV 2.4]). Consequently, when $A$ is normal,

$$||A|| = \max_j\{|\lambda_j|\}, \tag{13}$$

where $\{\lambda_i\}$ is the set of the eigenvalues of $A$. Now, from Proposition V 1.8 and Theorem X 1.1 of [18] and Lemma 4.1 of [5],

$$||(n\Gamma_n)^{1/2} D_n(h)(n\Gamma_n)^{1/2}|| \;\le\; ||n\Gamma_n||\,||D_n(h)|| \le ||\alpha||_\infty ||h||_\infty, \tag{14}$$

where $||.||_\infty$ is the supremum norm. □

**4.2. Proof of Lemma 3.3**

Since the eigenvalues are bounded, it is enough to find the limit, for any $p \geq 1$, of $\operatorname{tr}(n\Gamma_n D_n(h))^p/n$, where $\operatorname{tr} A$ denotes the trace of the matrix $A$. From [5], we have the following approximations

$$\Big|(n\Gamma_n)_{k,k} - \alpha\Big(\frac{k}{n}\Big)\Big| \le \frac{C_0}{n}, \quad \Big|(n\Gamma_n)_{j,k}\Big| \le \frac{C_0}{n} \quad \text{if} \quad j \neq k.$$

We detail the computation for $p = 3$, it is easy to see that the other cases follow similarly

$$\frac{1}{n}\,(n\Gamma_n D_n(h))^3 = \frac{1}{n}\sum_{j=1}^{n} [(n\Gamma_n)_{j,j}]^3\, h\Big(\frac{j}{n}\Big)^3$$

$$+\frac{3}{n}\sum_{j=1}^{n}\ \sum_{k\neq j}(n\Gamma_n)_{j,j}(n\Gamma_n)_{j,k}(n\Gamma_n)_{k,j}h\Big(\frac{j}{n}\Big)^2 h\Big(\frac{k}{n}\Big)$$

$$+\frac{1}{n}\sum_{j=1}^{n}\ \sum_{k\neq l\neq j}(n\Gamma_n)_{j,k}(n\Gamma_n)_{k,l}(n\Gamma_n)_{l,j}h\Big(\frac{j}{n}\Big)h\Big(\frac{k}{n}\Big)h\Big(\frac{l}{n}\Big).$$

By an argument of Riemann sums, the first term of the RHS above converges to

$$\int_{[0,1]} (\alpha(t)h(t))^3 \, dt.$$

It is easy to see that the other terms are negligible.

**Proof of Lemma 3.4**

For any $\delta > 0$, from the definition of $\Lambda^*$, there exists $h_\delta$ in $\mathcal{C}([0,1])$ such that (11) is statisfied. In case $th_\delta(t) \leq 1/2$, we may add some $\delta_0 < 0$ so that (11) holds with another $\delta$.

**Proof of Lemma 3.5**

Since $\gamma$ is strictly convex, for any $x$, $y \geq 0$ such that $x \neq y$,

$$\gamma(x) - \gamma(y) < (x - y)\gamma'(x) \tag{15}$$

and

$$\gamma'(x) = \frac{1}{2}\Big(1 - \frac{1}{x}\Big).$$

Integrating (15) with $\mu = l\lambda$ and $\zeta \neq \mu$, we obtain (10).

**Proof of Lemma 3.6**

A version of this lemma can be found in [8] and [13].

**Step 1.** Suppose $\mu = l\lambda + \mu^\perp$, and $\Lambda^*(\mu) < \infty$. Assume that the function $l \in \mathcal{C}[0,1]$, is nonnegative and the singular part $\mu^\perp$ is such that $1/\alpha(t)$ in $L^1(\mu^\perp)$. Since $\lambda$ has full support on $[0,1]$, there exists a sequence $(h_n)$ of positive functions in $\mathcal{C}([0,1])$ such that $h_n \, d\lambda \Rightarrow \mu^\perp/\alpha(t)$. From the lower semi-continuity of $\Lambda^*$,

$$\liminf_{n\to+\infty} \Lambda^*((l + \alpha(t)h_n)\lambda) \geq \Lambda^*(\mu).$$

Furthermore, one can see that for $\tau > 0$ and $\tau' \geq 0$,

$$\gamma(\tau + \tau') \leq \gamma(\tau) + \frac{\tau'}{2}. \tag{16}$$

Integrating this inequality,

$$\Lambda^*((l + \alpha(t)h_n)\lambda) \leq \Lambda^*(l\lambda) + \frac{1}{2}\int h_n(t)dt.$$

Hence,

$$\liminf_{n\to+\infty} \Lambda^*((l + \alpha(t)h_n)\lambda) \geq \Lambda^*(\mu),$$

which yields the limit.

**Step 2.** Suppose $\mu = l\lambda$ with $l$ and $\gamma(l/\alpha(t))$ in $L^1([0,1])$.

Fix $\varepsilon > 0$. We approximate $l$ by $l_4$ in $\mathcal{F}$ by the following construction:

(1) Let $l_1$ be the function defined by $l_1(x) = 1$ if $x \in [\alpha_k - \beta_k, \alpha_k + \beta_k]$, for any zero $\alpha_k$ of $\alpha$, and $l_1 = l$ elsewhere. Furthermore, $(\beta_k)$ are chosen so that $|\Lambda^*(l_1) - \Lambda^*(l)| \leq \varepsilon$ and $||l_1 - l||_1 \leq \varepsilon$.

(2) Set $l_2 = l_1 \mathbb{1}_{l_1 > \eta} + \eta \mathbb{1}_{l_1 \leq \eta}$, where $\eta$ is chosen such that $|\Lambda^*(l_2) - \Lambda^*(l_1)| \leq \varepsilon$ and $||l_2 - l_1||_1 \leq \varepsilon$. Therefore $l_2 \geq \eta$.

(3) Set $l_3 = 1$ on any $[\alpha_k - \beta_k, \alpha_k + \beta_k]$ and $l_3$ is positive continuous elsewhere, such that $l_3 \geq \eta/2$ and $||l_3 - l_2||_1 \leq \varepsilon$.

(4) Let $l_4$ be in $\mathcal{C}([0,M])$, defined by

$$l_4(x) = \begin{cases} 1 \text{ if for any } k \geq 0, x \in [\alpha_k - \beta_k, \alpha_k + \beta_k] \ \text{ linear on} \\ [\alpha_k - \beta_k - \delta_k, \alpha_k - \beta_k] \cup [\alpha_k + \beta_k, \alpha_k + \beta_k + \delta_k] \\ l_3(x) \text{ elsewhere} \end{cases}$$

with $(\delta_k)$ chosen such that $|\Lambda^*(l_4) - \Lambda^*(l_3)| \leq \varepsilon$ and $||l_4 - l_3||_1 \leq \varepsilon$.

Therefore $l_4 \in \mathcal{F}$ and $|\Lambda^*(l_4) - \Lambda^*(l)| \leq 4\varepsilon$ and $||l_4 - l||_1 \leq 4\varepsilon$.

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