SPECTRAL PROJECTIONS

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1. If $T$ is a bounded linear operator on a complex Banach space $X$, and if $0 \notin \sigma(T)$ is not an accumulation point of the spectrum $sp(T)$, then the formula

$$ I - P = \frac{1}{2\pi i} \oint_{\gamma} (zI - T)^{-1} \, dz, \quad (1.1) $$

in which integration is conducted around a contour which winds once positively around the point 0 and winds zero times around every point of $sp(T) \setminus \{0\}$, defines a projection $P = P^2$ which is bounded and linear on $X$, and satisfies three conditions:

$$ TP = PT \quad (1.2) $$

there are bounded linear $U$ and $V$ on $X$ for which $UT = PTV$; (1.3)

$$ \|T^n(I - P)\|^{1/n} \to 0 \text{ as } n \to \infty. \quad (1.4) $$

This note is in response to a feeling that while it may be tolerable to use heavyindustry like the Cauchy integrals of (1.1) to construct a projection like $P$, it ought to be possible to define one in a much more elementary context. We claim in fact that the three conditions (1.2) - (1.4) determine $P$ uniquely, and then force

$$ T^*T = TT^* \implies PT^* = T^*P \quad (1.5) $$

which we can use to show that operators $T$ for which $P$ exists are stable under certain multiplications and additions.

2. Formally,

**DEFINITION 1:** The bounded linear operator $T$ on $X$ is called quasi-polar if there exists a projection $P$ satisfying the conditions (1.2), (1.3) and (1.4).
satisfying (2.2) and (2.3) also commutes with every \( T' \) commuting with \( T \). We shall write

\[
S = T^X.
\]

If in particular (1.4) can be sharpened to

\[
T^n(1 - P) = 0 \text{ for some } n \in \mathbb{N}
\]

then we shall call the operator \( T \) polar, and refer to \( S = T^X \) as the Brozis Inverse of \( T \) ([2], 5.1).

3. Without any contour integrals it is clear that invertibles, quasinilpotents and idempotents all satisfy the conditions of Definition 1: the projection \( P \) is either \( I \), \( 0 \) or the operator \( T \) itself. Another familiar example is an operator \( T \) "of finite ascent and descent", in the sense that

\[
\text{cl}(T^X) = T^X = T^k T^X \quad \text{for some } k \in \mathbb{N}
\]

here \( T^X \) is the range and \( T^k \) the null space of the projection \( P \). If \( 0 \) is not an accumulation point of the spectrum of \( T \) then the usual contour integration theory still tells us that \( T \) is almost invertible, but we also know something new: the projection \( P \) given by the formula (1.1) is the only one around. Conversely, and without contour integration, the condition that \( 0 \) is at worst an isolated point of spectrum is necessary.

**THEOREM 2:** If \( T \) is quasi-polar and if \( T' \) commutes with \( T \) then:

\[
\begin{align*}
T + T' &\quad \text{is invertible if } T' \text{ and } T + T' \text{ are invertible}; \\
T T' &\quad \text{is quasi-polar if } T' \text{ is quasinilpotent}; \\
T T' &\quad \text{is quasi-polar if } T' \text{ is quasi-polar.}
\end{align*}
\]

**PROOF:** If \( T' \) commutes with \( T \) then by Theorem 1 it also commutes with \( P \) and therefore leaves the range and the null space of \( P \) invariant. To derive (3.2) we observe that the restriction of \( T + T' \) to \( P(X) \) is inverted by \((1 + T' X)^{-1} T^X\), while the restriction of \( T + T' \) to \( P^{-1} \bar{0} \) is the sum of an invertible operator and a quasinilpotent which commute with one another, therefore again invertible. To derive (3.3) we observe that the restriction of \( T + T' \) to \( P(X) \) is the commuting sum of an invertible and a quasinilpotent, therefore invertible, while the restriction of \( T + T' \) to \( P^{-1} \bar{0} \) is the sum of two commuting quasinilpotents, therefore quasinilpotent. To derive (3.4) we consider the product of the projections \( P \) and \( P' \) associated with \( T \) and \( T' \), which by Theorem 1 commute with \( T \), \( T' \) and one another: the restriction of \( T T' \) to the range of \( P P' \) is the product of two invertibles and therefore invertible, while the restriction of \( T T' \) to the null space of \( P P' \) is the sum of three commuting quasinilpotents and therefore quasinilpotent.

4. Sufficient for (3.2) is that \( T' \) is invertible with

\[
||T^X|| \leq ||T'|| < 1.
\]

Specialising to the case in which

\[
T' = \lambda I,
\]

for sufficiently small \( \lambda \neq 0 \) in \( \mathbb{I} \), shows that \( 0 \) cannot be an accumulation point of the spectrum of a quasi-polar: thus the contour integral (1.1) can always be used to give \( I - P \) ([4], Prop. 50.1). The converse of (3.4) is liable to fail: for example

\[
T = 0 \implies T T' = T T' = T T' \text{ quasi-polar}
\]

without restriction on \( T' \). For Fredholm operators however the converse of (3.4) does hold:

**THEOREM 3:** If \( T \) and \( T' \) are arbitrary then

\[
T \text{ Browder } \implies T \text{ quasi-polar Fredholm}
\]

and

\[
T T' \text{ quasi-polar Fredholm } \implies T, T' \text{ Browder}
\]
PROOF: If we write
\[ \phi(t) = BtA = BL(X,Y) + BL(X,X)/KL(X,X) = B \] (4.6)
for the "Calkin map" which quotients out the ideal KL(X,X)
of compact operators then it is Atkinson's theorem ([2], Thm 3.2.8) that
\[ T \text{ Fredholm } \Leftrightarrow \phi(T) \in B^{-1} \text{ invertible.} \] (4.7)
If in particular
\[ T = S + K \text{ with } S \in A^{-1}, \phi(K) = 0 \text{ and } SK = KS \] (4.8)
we shall call T a Browder operator. One more preliminary:
if K \in KL(X,Y) is compact then I + K has closed range and finite ascent and descent in the sense of (3.1) ([2], Thm 1.4.5; [4], Thm 40.1): thus
\[ \phi(K) = 0 \Leftrightarrow I + K \text{ quasi-polar.} \] (4.9)
Now if T = S + K is Browder than S^{-1}T = I + S^{-1}K is quasi-polar,and hence by (3.4) so is T = S(S^{-1}T). Conversely, without using (4.9), suppose T^* = T^*T = T^*T = T^*T = quasi-polar, with
\[ P^* = (P^*)^2 \text{ the projection of definition 1. Then also (in an obvious sense) } \phi(T^*) \in B \text{ is quasi-polar, with projection} \]
\[ \phi(P^*) \in B. \] If also T^* is Fredholm, so that \phi(T^*) \in B^{-1} is invertible, then by the uniqueness component (2.5) of Theorem 1 we have
\[ \phi(P^*) = \phi(I) \in B. \] (4.10)
Now
\[ S^* = T^*P^* + (I - P^*), K^* = (T^*T - I)(I - P^*) \] (4.11)
gives a Browder decomposition for T^*. By the doubly commuting component (2.6) of Theorem 1 both T and T' commute with P^*; now
\[ (TP^* + I - P^*)(T'P^* + I - P^*) = S^* = (T'P^* + I - P^*)(T'P^* + I - P^*), \] (4.12)
so that S = T'P^* + I - P^* and S' = T'P^* + I - P^* are also invertible.

Also K = (T - I)(I - P^*) and K' = (T' - I)(I - P^*) are both compact:
thus T = S + K and T' = S' + K' are both Browder.

Theorem 3 was very nearly proved in [3] (Theorem 1, Theorem 2), using the contour integral (1.1); (4.5) is however slightly stronger than (2.8) of [3]. As in [3] the whole theory is valid for arbitrary Banach algebras A and B, or indeed general rings, provided we are content with "polar" rather than "quasi-polar" elements. It seems to be quite a delicate problem to decide what the "quasinilpotent" elements of a general ring should be.

References

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