First of all, here are some new problems.

1. (Suggested by Pat Fitzpatrick). Consider the sequence of digits
   198423768........,
   obtained using the following rule.
   "After 1984 every digit which appears is the final digit of the sum of the previous four digits."

   Does the grouping 1984 appear later in this sequence and, if so, when? What about the grouping 1985?

2. (Due to John Conway and suggested to me by Harold Shapiro.) Imagine playing solitaire on an unlimited board, on which is drawn a horizontal line.

   
   
   
   
   
   
   The aim is to lay out pegs below the line in such a way that, with the usual solitaire moves, a single peg can be manoeuvred as high as possible above the line. The above arrangement of four pegs enables a single peg to reach the second row (above the line).

Find an arrangement which enables a single peg to reach the fourth row and show that there is no arrangement which enables a peg to reach the fifth row.

Now here are the solutions to September's problems.

1. Prove that
   \[ \sum_{k=1}^{n} \cot^2\left(\frac{\pi k}{2n+1}\right) = \frac{n(2n-1)}{3}, \]
   \[ (1) \]
   and deduce that
   \[ \sum_{k=1}^{n} \frac{1}{k^2} = \frac{\pi^2}{6}. \]
   \[ (2) \]

This problem was suggested by Finbarr Holland who proves (1) using the finite Fourier transform. The classical proof (found in turn-of-the-century calculus texts) goes as follows.

By De Moivre's theorem,

\[ \sin(2n+1)\theta = \text{Im}[(\cos\theta + i\sin\theta)^{2n+1}] = \sin^{2n+1}\theta \text{Im}[(\cot\theta + i)^{2n+1}] = \sin^{2n+1}\theta \text{P(}\cot^2\theta), \]

where \( \text{P} \) is the polynomial

\[ \text{P}(x) = \left(\frac{2^{n+1}}{3}\right)x^n - \left(\frac{2^{n+1}}{3}\right)x^{n-1} + \ldots + (-1)^n. \]

For \( \theta = k\pi/(2n+1), \) \( k = 1,2,\ldots,n, \) we have \( \sin(2n+1)\theta = \sin k\pi = 0 \) and \( \sin\theta < 0, \) so \( \text{P}(x) = 0 \) when \( x = \cot^2(k\pi/(2n+1)), \)

\( k = 1,2,\ldots,n. \) These \( n \) numbers are the roots of \( \text{P} \) and so their sum is

\[ \frac{(2n+1)}{3} = \frac{n(2n-1)}{3}, \]

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which proves (1).

To deduce (2) from (1) rewrite the inequality
\[ \sin \theta < \theta < \tan \theta, \quad 0 < \theta < \pi/2, \]
as
\[ \cot^2 \theta < \frac{1}{\theta^2} < 1 + \cot^2 \theta, \quad 0 < \theta < \pi/2, \]
to obtain
\[ \sum_{k=1}^{n} \cot^2 \left( \frac{k}{2n+1} \right) < \left( \frac{2n+1}{\pi} \right)^2 \sum_{k=1}^{n} \frac{1}{k^2} < \sum_{k=1}^{n} \left( 1 + \cot^2 \left( \frac{k\pi}{2n+1} \right) \right). \]
Using (1) and letting \( n \to \infty \) gives (2).

It is remarkable that this proof of (2) remained unnoticed until 1953 when it was published by R.M. Yaglom and I.M. Yaglom. It has been rediscovered several times since (Monthly, 80 (1973) 424-425).

Before leaving this problem I can't resist including a related 'proof without words'.

2. Prove that, if \( a_0, a_1 \) are not both zero then the sequence
\[ a_{n+2} = |a_{n+1}| - a_n, \quad n = 0, 1, 2, \ldots \]
has period 9.
which takes \((a_n, a_{n+1})\) to \((a_{n+1}, a_{n+2})\) can be decomposed as

\[ t = q_{\pi/4} s q_{\pi/2}, \]

where \(q_{\theta}\) denotes reflection in the line making a positive angle \(\theta\) with the \(x\)-axis, and

\[ s : (x, y) \mapsto (x + |y|, y) \]

is a "piecewise shear". Thus

\[ t = q_{\pi/4} (s r_{\pi/2}) q_{\pi/4}, \]

where \(r_{\pi/2} = q_{\pi/2} q_{\pi/4}\) is a positive rotation through \(\pi/2\), so that \(t\) is conjugate to \(w = s r_{\pi/2}\). The effect of \(u\) on a triangle \(\Delta_s\) is illustrated below. Here \(\Delta_n = (u^n(\Delta_s), n = 0, 1, \ldots, 8).\)

\[ \begin{array}{c}
\text{(1,0)} \quad \text{(1,1)} \\
\text{(0,1)} \quad \text{(1,1)} \\
\text{(0,0)} \quad \text{(0,1)} \\
\text{(0,0)} \quad \text{(1,0)} \\
\text{(0,0)} \quad \text{(0,0)} \\
\end{array} \]

To find other sequences having a similar property to (5) Mike Crampin (Open University) has considered mappings obtained by gluing together a pair of shears each of which fixes the \(x\)-axis. In this way he finds, for example, that if \(a_0, a_1\) are not both \(0\) then the sequence

\[ a_{n+2} = \frac{1}{2} (a_{n+1} + |a_{n+1}|) - a_n, \quad n = 0, 1, 2, \ldots, \]

has period 5.

With regard to (b) the main question seems to be "Is \(\{a_n\}\) bounded?" Since \(\text{Im}[a_{n+2}] = \text{Im}[a_n]\), this question reduces to "Is \(\text{Im}[a_n]\) bounded?" Dav Ratanov has shown that it is enough to investigate the two cases:

\[ (i) \quad a_0 = -x + i\epsilon, \quad a_1 = 1 + i\eta, \quad 0 \leq x \leq 1, \quad \epsilon > 0, \quad \eta > 0 \text{ arbitrarily small, and} \]

\[ (ii) \quad a_0 = -x + i\epsilon, \quad a_1 = 1 + i\eta, \]

where \(|\epsilon| < \max(|\epsilon|, \eta)|).\]

Taking the special case \(a_0 = a \in \mathbb{R}\) and \(a_1 = i\), the points \((\text{Re}[a_n], \text{Re}[a_{n+1}]), n = 0, 1, 2, \ldots,\) in \(\mathbb{R}^2\) form an orbit of the (area-preserving) mapping

\[ (u, v) = (\sqrt{x^2 + 1} - x, |u| - y). \]

These orbits are plotted below for \(a = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \). As \(a \to \infty\) the orbits seem more and more to resemble the boundary of the previous figure!

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