About the Orthogonality of derivatives of Orthogonal Polynomials on the unit circle

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Abstract

Given a monic polynomial sequence $\{\phi_n(z)\}$ verifying the following relation:

$$\forall n \geq 1 \qquad (z-\alpha)\frac{\phi_n^{'}(z)}{n} = \phi_n(z) + \beta_n \phi_{n-1}(z)$$

with β_n, $\alpha \in \mathbf{C}$ and $\beta_n \neq 0$, we want to find the linear functional u defined on the vector space of Laurent polynomials and with values in $\mathbf{C}$ with respect to which the sequence $\{\phi_n(z)\}$ is orthogonal.

We also study some questions relative to the semiclassical character of u and the definite positive case.

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1 Introduction

Let $\Sigma = span(z^n)_{n \in \mathbf{Z}}$ be the linear space of Laurent polynomials with complex coefficients, $\Sigma^{'}$ its algebraic dual, $\mathbf{P}$ the linear space of complex polynomials and $\mathbf{P}_n$ the subspace of polynomials whose degrees are less than or equal to n.

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Let $H \subset \Sigma'$ be the subspace of linear functionals which are hermitian. Then, $u \in H$ has associated a Toeplitz matrix $M = (u_{i-j})_{i,j\geq 0}$, with $\forall\, n \in \mathbf{Z}$, $u_n = u(z^n) = \overline{u_{-n}}$.

We say that $u \in H$ is regular (positive definite) if the quadratic form associated to M is regular (positive definite), that is, if for every $n \in \mathbf{N}$ the principal minors $\Delta_n = \det(u_{i-j})_{0\leq i,j\leq n}$ do not vanish (they are positive). Hence, u defines an inner product in $\mathbf{P}$, given by

$$\langle P(z), Q(z)\rangle = u\left(P(z)\overline{Q}(\frac{1}{z})\right).$$

In particular, for a positive definite linear functional it is well known that a finite and positive Borel measure μ on the unit circle $\mathbf{T}$ exists such that

$$\langle u, P(z)\rangle = \frac{1}{2\pi}\int_0^{2\pi} P(e^{i\theta})d\mu(\theta), \quad z = e^{i\theta}.$$

Considering a regular functional $u \in H$, an orthogonalization process yields one and only one **monic orthogonal polynomial sequence related to T** (M.O.P.S.), $\{\phi_n(z)\}_{n\geq 0}$, such that

(i) $\forall n \geq 0$, $\deg \phi_n(z) = n$, .

(ii) $\forall n \geq 0, forallm \geq 0, u\left(\phi_n(z)\overline{\phi_m}(\frac{1}{z})\right) = e_n\delta_{nm} \qquad e_n = \dfrac{\Delta_n}{\Delta_{n-1}} \neq 0,$.

If we define the $*n$-operator by $\phi_n^*(z) = z^n\overline{\phi_n}(\frac{1}{z})$, then the polynomials are related by the following mutually equivalent relations

$$\forall n \geq 1 \qquad \phi_n(z) = z\phi_{n-1}(z) + \phi_n(0)\phi_{n-1}^*(z), \tag{R1}$$

$$\forall n \geq 1 \qquad \phi_n(z) = (1-|\phi_n(0)|^2)z\phi_{n-1}(z) + \phi_n(0)\phi_n^*(z), \tag{R2}$$

$$\forall n \geq 1 \qquad \phi_n^*(z) = \phi_{n-1}^*(z) + \overline{\phi_n(0)}z\phi_{n-1}(z), \tag{R3}$$

$$\forall n \geq 1 \qquad \phi_n^*(z) = (1-|\phi_n(0)|^2)\phi_{n-1}^*(z) + \overline{\phi_n(0)}\phi_n(z) \tag{R4}$$

with $|\phi_n(0)| \neq 1$ ($|\phi_n(0)| < 1$ in the definite positive case).
Besides, the relation $\forall n \geq 1$, $1 - |\phi_n(0)|^2 = \frac{e_n}{e_{n-1}}$ with $e_0 = 1$ allows us to rewrite the recurrence relations (R2) and (R4) as follows

$$\forall n \geq 1 \qquad \phi_n(z) = \frac{e_n}{e_{n-1}}z\phi_{n-1}(z) + \phi_n(0)\phi_n^*(z), \tag{R2}$$

$$\forall n \geq 1 \qquad \phi_n^*(z) = \frac{e_n}{e_{n-1}}\phi_{n-1}^*(z) + \overline{\phi_n(0)}\phi_n(z). \tag{R4}$$

Conversely, for a monic polynomial sequence $\{\phi_n(z)\}_{n\geq 0}$ that verifies the previous relations with $|\phi_n(0)| \neq 1$, there is one and only one $u \in H$ (up to a non-zero real factor) such that

$$u\left(\phi_n(z)z^{-m}\right) = e_n\delta_{nm} \quad 0 \leq m \leq n;\ n \geq 0 \quad \text{with} \quad e_n \neq 0.$$

Another very important result is the extension of Favard's theorem in the regular case: "given a sequence of complex numbers $\{a_n\}_{n\geq 1}$ such that for every n, $|a_n| \neq 1$, there exists one and only one M.O.P.S. $\{\phi_n(z)\}$ related to $\mathbf{T}$ such that $\forall n \geq 1$ $\phi_n(0) = a_n$".

For orthogonal polynomials on the real line a wide literature about differential equations and other differential properties exists.

For orthogonal polynomials on the unit circle very few results are known. One kind of question concerns the study of M.O.P.S. satisfying second order differential equations whose coefficients are polynomials with degree independent of n. The introduction of semiclassical orthogonal polynomials allows to place a theoretical framework for this theory.

Given $u \in H$, we say that u is semiclassical if there exist two polynomials $A(z) \neq 0$ and $B(z)$ such that

$$D\Big(A(z)u\Big) = B(z)u.$$

The derivative of an hermitian linear functional is defined as

$$\forall P \in \Sigma, \langle Du, P(z)\rangle = -i\langle u, zP'(z)\rangle \quad .$$

If $\deg A(z) = p'$ and $\max\{p'-1, \deg\left((p'-1)A(z) + iB(z)\right)\} = q$, we say that u belongs to the class (p', q).

For the M.O.P.S. relative to a semiclassical linear functional appears a difference differential equation called structure relation. An interesting problem is the converse: if a M.O.P.S. on the unit circle satisfies an structure relation, is then the corresponding linear functional, semiclassical?

Considering the structure relation:

$$(z-\alpha)\frac{\phi_n'(z)}{n} = \phi_n(z) + \beta_n\phi_{n-1}(z),$$

in this paper (Section 2) we obtain, by using the hypergeometric series, an explicit representation for the M.O.P.S. solution of the above differential-difference equation. In this way we give an affirmative answer to the problem in a particular situation (see [2], [3], [4], [5], [7], [8], [9] among others.)

2 Representation of the polynomials.

In what follows, unless it is said the contrary, we assume that $\{\phi_n(z)\}$ is a M.O.P.S. on the unit circle with respect to a linear hermitian functional u verifying

$$\forall n \geq 1, (z-\alpha)\frac{\phi_n'(z)}{n} = \phi_n(z) + \beta_n \phi_{n-1}(z) \qquad (*)$$

with $\alpha, \beta_n \in \mathbf{C}$ and $\beta_n \neq 0$.

2.1 Lemma. It follows from $(*)$ that $\phi_n(\alpha) \neq 0$ and $\beta_n = -\dfrac{\phi_n(\alpha)}{\phi_{n-1}(\alpha)}$.

Proof.
Taking $z = \alpha$ and $n = 1, 2, \ldots$ in $(*)$. ■

2.2 Theorem. The sequence $\{\phi_n(z)\}$ admits the explicit representation

$$\forall n \geq 0 \qquad \phi_n(z) = \phi_n(\alpha)\left(\sum_{k=0}^{n} \binom{n}{k} \frac{1}{\phi_k(\alpha)}(z-\alpha)^k\right) \qquad (2.2)$$

Proof Taking derivatives in $(*)$ up to the $(k-1)$-order

$$(z-\alpha)\frac{\phi_n^{(k)}(z)}{n} = \frac{n-(k-1)}{n}\phi_n^{(k-1)}(z) + \beta_n \phi_{n-1}^{(k-1)}(z) \quad 1 \leq k \leq n. \qquad (2.2.1)$$

Hence, putting $z = \alpha$

$$\phi_n^{(k-1)}(\alpha) = -\frac{n}{(n-k+1)}\beta_n \phi_{n-1}^{(k-1)}(\alpha).$$

Substituting the value β_n given in 2.1 and continuing recursively:

$$\phi_n^{(k-1)}(\alpha) = \frac{n!}{(n-k+1)!}\frac{\phi_n(\alpha)}{\phi_{k-1}(\alpha)} \quad 1 \leq k \leq n. \qquad (2.2.2)$$

Finally, the Taylor expansion of $\phi_n(z)$ in $z = \alpha$ besides (2.2.2) yields the statement. ■

2.3 Lemma.

(i) If $\phi_1(0) \neq 0$ then $\forall n \geq 2 \quad \phi_n(0) \neq 0$.

(ii) If $\phi_1(0) = 0$ then $\phi_n(z) = z^n$.

Proof (i) Assume that there exists $n \geq 2$ such that $\phi_n(0) = 0$. By (R1) $\phi_n(z) = z\phi_{n-1}(z)$ and $\beta_n = -\alpha$. Substituting the last expressions in $(*)$ we obtain

$$\phi_n^{'}(z) = n\phi_{n-1}(z) \tag{2.3.1}$$

On the other hand, taking derivatives in the recurrence (R1) and using the previous relation, we get

$$z\phi_{n-1}^{'}(z) = (n-1)\phi_{n-1}(z).$$

Taking $z = 0$ it holds $\phi_{n-1}(0) = 0$ for fixed n, with $n \geq 2$. If $n = 2$ then $\phi_1(0) = 0$ in contradiction with the hypothesis. If $n \geq 2$, continuing recursively, we also reach $\phi_1(0) = 0$.

(ii) We proceed by induction on n.

Since $\phi_1(z) = z$, by (R1) it holds that $\phi_2(z) = z^2 + \phi_2(0)$ and substituting this in $(*)$ for $n = 2$ it follows $\phi_2(0) = 0$.

Let's now assume that $\phi_n(z) = z^n$. Then, $\phi_{n+1}(z) = z^{n+1} + \phi_{n+1}(0)$ and proceeding in the same way as in the case $n = 2$ we obtain $\phi_{n+1}(0) = 0$. ■

• **Remark.** If $\alpha = 0$ necessarily $\phi_1(0) \neq 0$, since $\forall n\ \phi_n(\alpha) \neq 0$. Hence, if $\phi_1(0) = 0$, then $\alpha \neq 0$ and u is the functional associated to the Lebesgue measure which will be represented by $\mathcal{L}_1$.

Throughout the Section 2 we assume that $\phi_1(0) \neq 0$.

2.4 Lemma. The following relations hold:

(i) $\forall n \geq 1 \quad (1 - |\phi_n(0)|^2)z(z-\alpha)\dfrac{\phi_n^{'}(z)}{n} = \left[(1 - |\phi_n(0)|^2)z + \beta_n\right]\phi_n(z) - -\beta_n\phi_n(0)\phi_n^*(z)$.

(ii) $|\alpha| = 1$.

(iii) $\forall n \geq 2 \quad n\phi_n(0)\dfrac{\phi_{n-1}^*(\alpha)}{\phi_{n-1}(\alpha)} - (n-1)\phi_{n-1}(0)\dfrac{\phi_{n-2}^*(\alpha)}{\phi_{n-2}(\alpha)} = \phi_n(0)\overline{\phi_{n-1}(0)}$.

(iv) $\forall n \geq 1 \qquad n\phi_n(0)\dfrac{\phi^*_{n-1}(\alpha)}{\phi_{n-1}(\alpha)} = \sum_{k=1}^{n} \phi_k(0)\overline{\phi_{k-1}(0)}.$

Proof (i) Removing the polynomial $\phi_{n-1}(z)$ in $(*)$ with (R2).
(ii) Applying the $*n$-operator in $(*)$ and substituting the relation $(\phi_n')^*(z) = n\phi_n^*(z) - z(\phi_n^*)'(z)$ ([9]) in the resulting expression we obtain

$$\forall n \geq 1 \qquad (1 - z\overline{\alpha})(\phi_n^*)'(z) = -n\overline{\alpha}\phi_n^*(z) - n\overline{\beta_n}\phi^*_{n-1}(z). \tag{2.4.1}$$

Removing in (2.4.1) $\phi^*_{n-1}(z)$ with (R4)

$$\forall n \geq 1 \qquad (1 - |\phi_n(0)|^2)(1 - z\overline{\alpha})(\phi_n^*)'(z) =$$
$$= (-n)\left[\overline{\alpha}(1 - |\phi_n(0)|^2) + \overline{\beta_n}\right]\phi_n^*(z) + n\bar{\beta}_n\overline{\phi_n(0)}\phi_n(z). \tag{2.4.2}$$

Taking derivatives in (R3), multiplying by $(1 - z\overline{\alpha})$ and substituting (2.4.1) and (2.4.2) it follows

$$-(n+1)\overline{\alpha}(1 - |\phi_n(0)|^2)\phi^*_{n+1}(z) - (n+1)\overline{\beta_{n+1}}(1 - |\phi_n(0)|^2)\phi_n^*(z) =$$
$$= (-n)\left[\overline{\alpha}(1 - |\phi_n(0)|^2) + \overline{\beta_n}\right]\phi_n^*(z) +$$
$$+ \left[n\bar{\beta}_n\overline{\phi_n(0)} + \overline{\phi_{n+1}(0)}(1 - |\phi_n(0)|^2)(1 - z\overline{\alpha})\right]\phi_n(z) +$$
$$+ \overline{\phi_{n+1}(0)}(1 - |\phi_n(0)|^2)z(1 - z\overline{\alpha})\phi_n'(z). \tag{2.4.3}$$

Taking $z = 0$

$$\forall n \geq 0 \qquad n\overline{\beta_n} - (n+1)\overline{\beta_{n+1}} - \overline{\alpha} = \overline{\phi_{n+1}(0)}\phi_n(0). \tag{2.4.4}$$

Next, applying (R3) in (2.4.3) and substituting $\overline{\beta_{n+1}}$ from (2.4.4)

$$\left[n\overline{\beta_n}|\phi_n(0)|^2 + \overline{\phi_{n+1}(0)}\phi_n(0)(1 - |\phi_n(0)|^2)\right]\phi_n^*(z) =$$
$$= \left[n\bar{\beta}_n\overline{\phi_n(0)} + n\overline{\phi_{n+1}(0)}(1 - |\phi_n(0)|^2)\overline{\alpha}z + \overline{\phi_{n+1}(0)}(1 - |\phi_n(0)|^2)\right]\phi_n(z) +$$
$$+ \overline{\phi_{n+1}(0)}(1 - |\phi_n(0)|^2)(1 - z\overline{\alpha})z\phi_n'(z). \tag{2.4.5}$$

Finally, multiplying by $z - \alpha$ in (2.4.5) and removing $\phi_n'(z)$ with 2.4(i)

$$\forall n \geq 1 \qquad \phi_n(0)A_n(z)\phi_n^*(z) = B_n(z)\phi_n(z),$$

with

$$A_n(z) = z\left[n\bar{\beta}_n\overline{\phi_n(0)} + \overline{\phi_{n+1}(0)}(1-|\phi_n(0)|^2) - \overline{\alpha}\overline{\phi_{n+1}(0)}n\beta_n\right] -$$

$$-\alpha\left[n\beta_n\overline{\phi_n(0)} + \overline{\phi_{n+1}(0)}(1-|\phi_n(0)|^2)\right] + \overline{\phi_{n+1}(0)}n\beta_n,$$

and $B_n(z)$ is a polynomial of degree less than or equal to 1. Since $\phi_n(z)$ and $\phi_n^*(z)$ have not common roots ([9]) $A_n(z) = B_n(z) = 0$ if $n \geq 2$. From $A_n(z) = 0$ it follows that $|\alpha|^2 = 1$.

(iii) From (2.4.4), substituting β_n and β_{n+1}:

$$(n+1)\frac{\phi_{n+1}(\alpha)}{\phi_n(\alpha)} - n\frac{\phi_n(\alpha)}{\phi_{n-1}(\alpha)} - \alpha = \phi_{n+1}(0)\overline{\phi_n(0)}.$$

Applying (R1) in $\phi_{n+1}(\alpha)$ and $\phi_n(\alpha)$ we obtain the desired formula.

(iv) Taking $n = 1, 2, \ldots$ in 2.4(iii) and adding up. ∎

2.5 Theorem. *For every $n \geq 1$ it holds that $ne_1 + \phi_1^*(\alpha) \neq 0$.*

Furthermore

$$\forall n \geq 0 \qquad \phi_{n+1}(0) = \frac{(n-1)e_1 + \overline{\phi_1^*(\alpha)}}{ne_1 + \phi_1^*(\alpha)}\alpha\phi_n(0). \tag{2.5}$$

Proof In the first place we prove that $\{\phi_n(z)\}$ admits the following representation in terms of $\{\frac{\phi_n'(z)}{n}\}$:

$$\forall n \geq 1 \quad A_{n+1}\phi_n(z) = (A_{n+1} - \alpha)z\frac{\phi_n'(z)}{n} + \alpha\frac{\phi_{n+1}'(z)}{n+1}. \qquad (2.5.1),$$

where $A_{n+1} = -\frac{\phi_{n+1}(0)}{\phi_n(0)} + \frac{\phi_{n+1}(\alpha)}{\phi_n(\alpha)}$. Writing the relation $(*)$ for $n+1$ and applying (R1)

$$\forall n \geq 0 \quad (z-\alpha)\frac{\phi_{n+1}'(z)}{n+1} = (z + \beta_{n+1})\phi_n(z) + \phi_{n+1}(0)\phi_n^*(z). \tag{2.5.2}$$

Removing $\phi_n^*(z)$ with (2.5.2) and 2.4(i)

$$\left[\beta_n\phi_n(0)\frac{\phi'_{n+1}(z)}{n+1}+\phi_{n+1}(0)(1-|\phi_n(0)|^2)z\frac{\phi'_n(z)}{n}\right](z-\alpha)=$$

$$=\left[\Big(\phi_{n+1}(0)(1-|\phi_n(0)|^2)+\beta_n\phi_n(0)\Big)z+\beta_n\left(\beta_{n+1}\phi_n(0)+\phi_{n+1}(0)\right)\right]\phi_n(z). \tag{2.5.3}$$

Taking $z=\alpha$

$$\alpha\left[\phi_{n+1}(0)(1-|\phi_n(0)|^2)+\beta_n\phi_n(0)\right]=-\beta_n\left[\beta_{n+1}\phi_n(0)+\phi_{n+1}(0)\right]. \tag{2.5.4}$$

Substituting in (2.5.3)

$$\frac{\phi'_{n+1}(z)}{n+1}+\frac{\phi_{n+1}(0)(1-|\phi_n(0)|^2)}{\beta_n\phi_n(0)}z\frac{\phi'_n(z)}{n}=$$

$$=\left[\frac{\phi_{n+1}(0)(1-|\phi_n(0)|^2)+\beta_n\phi_n(0)}{\beta_n\phi_n(0)}\right]\phi_n(z). \tag{2.5.5}$$

By substituting $\beta_{n+1}=-\phi_{n+1}(\alpha)/\phi_n(\alpha)$ in (2.5.4) we obtain that the coefficient of $\phi_n(z)$ in (2.5.5) is $\overline{\alpha}A_{n+1}$.

Taking $z=\alpha$ in (2.5.1) and from (2.2.2) and (R1)

$$\forall n\geq 1 \qquad A_{n+1}=\alpha\frac{\phi_{n+1}(0)\phi_n^*(\alpha)}{\phi_1(0)\phi_n(\alpha)}. \tag{2.5.6}$$

Taking derivatives in (2.5.1), considering in the resulting expression $z=\alpha$ and applying (2.2.2) and (R1)

$$[(n-1)A_{n+1}+\alpha]\frac{\phi_2(0)\phi_1^*(\alpha)}{\phi_1(\alpha)}=n\alpha\frac{\phi_{n+1}(0)\phi_n^*(\alpha)}{\phi_n(\alpha)},$$

substituting (2.5.6)

$$\alpha\frac{\phi_{n+1}(0)\phi_n^*(\alpha)}{\phi_n(\alpha)}[(n-1)\frac{\phi_2(0)\phi_1^*(\alpha)}{\phi_1(\alpha)\phi_1(0)}-n]=-\alpha\frac{\phi_2(0)\phi_1^*(\alpha)}{\phi_1(\alpha)}.$$

Taking into account that $\phi_1^*(\alpha) = e_1 + \overline{\phi_1(0)}\phi_1(\alpha)$, from 2.4(iii) we obtain $\phi_2(0)(e_1+\phi_1^*(\alpha)) = \phi_1(0)\phi_1(\alpha) \neq 0$. Removing $\phi_2(0)$ in the above expression:

$$\forall n \geq 1 \qquad \frac{\phi_{n+1}(0)\phi_n^*(\alpha)}{\phi_n(\alpha)}(ne_1 + \phi_1^*(\alpha)) = \phi_1^*(\alpha)\phi_1(0), \tag{2.5.7}$$

¿From (2.5.7) it holds that $ne_1 + \phi_1^*(\alpha) \neq 0$. ¿From (2.5.7) and (2.5.6) we obtain A_{n+1}. If we insert this result in the value A_{n+1} given in (2.5.1)

$$-\frac{\phi_{n+1}(0)}{\phi_n(0)} + \frac{\phi_{n+1}(\alpha)}{\phi_n(\alpha)} = \alpha\frac{\phi_1^*(\alpha)}{ne_1 + \phi_1^*(\alpha)}.$$

Expanding $\phi_{n+1}(\alpha)$ by (R1) and substituting (2.5.7)

$$\frac{\phi_{n+1}(0)}{\phi_n(0)} = -\frac{\phi_1^*(\alpha)}{ne_1 + \phi_1^*(\alpha)}(\alpha - \phi_1(0)) + \alpha.$$

¿From (R2) after some easy calculations we obtain the desired result. ■

2.6 Corollary.

$$\forall n \geq 1 \qquad |\phi_n(0)|^2 = \frac{|\phi_1(\alpha)|^2|\phi_1(0)|^2}{|(n-1)e_1 + \phi_1^*(\alpha)|^2}. \tag{2.6}$$

Proof By (2.5.7). ■

In what follows, ${}_2F_1(a, b; c; z)$ is the usual notation for the hypergeometric series (see [6] and [10])

$${}_2F_1(a, b; c; z) = 1 + \sum_{\nu=1}^{+\infty} \frac{(a)_\nu(b)_\nu}{(c)_\nu}\frac{z^\nu}{\nu!},$$

where

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = x(x+1)\dots(x+n-1) \quad n = 1, 2\dots.$$

2.7 Theorem If we denote $\gamma = \overline{\phi_1(0)}\phi_1(\alpha)/e_1$. Then $\{\phi_n(z)\}$ admits the following equivalent representations: $\forall n \geq 1$

$$\phi_n(z) = \frac{\Gamma(\overline{\gamma}+n)\Gamma(\gamma+1)}{\Gamma(\overline{\gamma})\Gamma(\gamma+1+n)}\alpha^n \, {}_2F_1(-n, \gamma+1; -\overline{\gamma}-n+1; \overline{\alpha}z). \qquad (2.7)(1)$$

$$\phi_n(z) = \frac{\Gamma(2\Re\gamma+1+n)\Gamma(\gamma+1)}{\Gamma(2\Re\gamma+1)\Gamma(\gamma+1+n)}\alpha^n \, {}_2F_1(-n, \gamma+1; 2\Re\gamma+1; 1-\overline{\alpha}z). \quad (2.7)(2)$$

Proof.
Using (R4), one should observe that the denominator in (2.5) can be written

$$(n+1)e_1 + \phi_1^*(\alpha) - e_1 = (n+1)e_1 + \overline{\phi_1(0)}\phi_1(\alpha) = e_1[(n+1)+\gamma],$$

and using (R2) in the numerator of $\phi_{n+1}(0)$

$$n\alpha e_1 + \phi_1(\alpha) - \alpha e_1 = n\alpha e_1 + \phi_1(0)\phi_1^*(\alpha) = \alpha e_1(n+\overline{\gamma}).$$

Hence

$$\forall n \geq 0 \qquad \phi_{n+1}(0) = \frac{n+\overline{\gamma}}{n+1+\gamma}\alpha\phi_n(0) \qquad (2.7.1)$$

with $\gamma \neq -n$.

On the other hand, from (R1), (R4) and (2.5.7) it holds that $\beta_n = -\frac{2\Re\gamma+n}{\gamma+n}\alpha$, then $\phi_n(\alpha)$ can be written in terms of γ

$$\forall n \geq 1 \quad \phi_n(\alpha) = \frac{(2\Re\gamma+1)(2\Re\gamma+2)\dots(2\Re\gamma+n)}{(\gamma+1)(\gamma+2)\dots(\gamma+n)} \qquad (2.7.2)$$

with $2\Re\gamma \neq -n$, $\forall n \geq 1$.

Putting $z = 0$ in (2.2.1) we obtain the first expression in the statement, so that if $1 \leq k \leq n-1$ then

$$\phi_n^{(k)}(0) = \frac{n!}{(n-k)!}\frac{\overline{\gamma}(\overline{\gamma}+1)\dots(\overline{\gamma}+n-k-1)}{(\gamma+k+1)\dots(\gamma+n)}\alpha^{n-k}.$$

Hence, writing the Taylor expansion of $\phi_n(z)$ in $z = 0$, we obtain

$$\phi_n(z) = \frac{\overline{\gamma}(\overline{\gamma}+1)\dots(\overline{\gamma}+n-1)}{(\gamma+1)(\gamma+2)\dots(\gamma+n)}\alpha^n[1 + \frac{(-n)(\gamma+1)}{1!(-\overline{\gamma}-n+1)}\overline{\alpha}z +$$

$$+\frac{(-n)(-n+1)(\gamma+1)(\gamma+2)}{2!(-\overline{\gamma}-n+1)(-\overline{\gamma}-n+2)}(\overline{\alpha}z)^2+\ldots+$$

$$+\ldots+\frac{(-n)(-n+1)\ldots(-1)(\gamma+1)\ldots(\gamma+n)}{n!(-\overline{\gamma}-n+1)\ldots(-\overline{\gamma}-1)(-\overline{\gamma})}(\overline{\alpha}z)^n].$$

The second expression in the statement is obtained from (2.2) and (2.7.2)

$$\phi_n(z)=\phi_n(\alpha)[1+\frac{(-n)(\gamma+1)}{1!(2\Re\gamma+1)}(1-\overline{\alpha}z)$$

$$+\frac{(-n)(-n+1)(\gamma+1)(\gamma+2)}{2!(2\Re\gamma+1)(2\Re\gamma+2)}(1-\overline{\alpha}z)^2+$$

$$+\ldots+\frac{(-n)(-n+1)\ldots(-1)(\gamma+1)\ldots(\gamma+n)}{n!(2\Re\gamma+1)\ldots(2\Re\gamma+n)}(1-\overline{\alpha}z)^n].\blacksquare$$

Next, we shall discuss the reciprocal of the theorem 2.7.

2.8 Theorem. *Let α,γ be complex numbers, such that $|\alpha|=1$, $\gamma\neq 0$ and $\forall n\in\mathbf{N}$, $2\Re\gamma+n\neq 0$. Then the sequence defined by $\phi_0(z)=1$ and $\forall n\geq 1$ $\phi_n(z)$ given as in (2.7)(1) is a M.O.P.S. related to $\mathbf{T}$ which verifies the relation $(*)$.*

Proof The conditions under the parameters α and γ warrant that $\phi_n(0)\neq 0$ y $|\phi_n(0)|\neq 1$. We obtain the relation $(*)$ for $\phi_{k+1}(z)$ with $k\geq 1$ from the theory of hypergeometric series ([6]), using the following formula:

$$(-1)^n\frac{(a)_n(c-b)_n}{(c)_n}(1-z)^{a-1}\,{}_2F_1(a+n,b;c+n;z)$$
$$=\frac{d^n}{dz^n}[(1-z)^{a+n-1}\,{}_2F_1(a,b:c;z)]$$

where $a=-k-1$, $b=\gamma+1$, $c=-\overline{\gamma}-k$, $n=1$ and

$$\phi_k(\alpha z)=\phi_k(0)\,{}_2F_1(-k,\gamma+1;-\overline{\gamma}-k+1;z).$$

For $\phi_1(z)$ the formula is verified directly. $\blacksquare$

3 The associated functional.

Throughout Section 3 we assume that $u_n = u(z^n)$, with $n \geq 0$, and $u_0 = 1$.

3.1 Corollary. *If $\{\phi_n(z)\}$ is defined by (2.7)(1) then:*

$$\forall n \geq 0 \qquad u_{n+1} = \frac{\overline{\gamma}(\overline{\gamma}-1)(\overline{\gamma}-2)\dots(\overline{\gamma}-n)}{(\gamma+1)(\gamma+2)\dots(\gamma+n+1)}(-\alpha)^{n+1}.$$

Proof We proceed by induction on n.

Due to the orthogonality of $\{\phi_n(z)\}$ with respect to u it holds $u\left(\phi_n(z)\right) = 0$, then $u(z) = u_1 = -\phi_1(0) = -\frac{\overline{\gamma}}{\gamma+1}\alpha$. Hence, the result is true for $n = 1$.

¿From (2.7)(1) it holds $u\left({}_2F_1(-n, \gamma+1; -\overline{\gamma}-n+1; \overline{\alpha}z)\right) = 0$, and after the induction step it holds the following

$$1 + \frac{(-n)(-\overline{\gamma})}{1!(-\overline{\gamma}-n+1)} + \frac{(-n)(-n+1)(-\overline{\gamma})(-\overline{\gamma}+1)}{2!(-\overline{\gamma}-n+1)(-\overline{\gamma}-n+2)} + \dots +$$

$$+ \dots + \frac{(-n)(-n+1)\dots(-2)(-\overline{\gamma})(-\overline{\gamma}+1)(-\overline{\gamma}+n-2)}{(n-1)!(-\overline{\gamma}-n+1)(-\overline{\gamma}-n+2)\dots(-\overline{\gamma}-1)} =$$

$$= -\frac{(\gamma+1)(\gamma+2)\dots(\gamma+n)}{(\overline{\gamma}+n-1)(\overline{\gamma}+n-2)\dots\overline{\gamma}}\overline{\alpha}^n u_n.$$

Using the theory of hypergeometric series, the left-hand side of the above expression can be written as follows

$${}_2F_1(-n, -\overline{\gamma}; -\overline{\gamma}-n+1; 1) - \frac{\overline{\gamma}(\overline{\gamma}-1)\dots(\overline{\gamma}-n+1)}{(-\overline{\gamma}-n+1)(-\overline{\gamma}-n+2)\dots(-\overline{\gamma})}$$

$$= -\frac{(\gamma+1)(\gamma+2)\dots(\gamma+n)}{(\overline{\gamma}+n-1)(\overline{\gamma}+n-2)\dots\overline{\gamma}}\overline{\alpha}^n u_n,$$

and because ${}_2F_1(-n, -\overline{\gamma}; -\overline{\gamma}-n+1; 1) = 0$ ([6] and [10]), we observe that the result holds. ∎

3.2 Theorem. *u is a semiclassical functional that belongs to the class $(1,1)$ if and only if $u = \mathcal{L}_1$ or u is given by 3.1.*

Proof See [9]. ■

3.3 Corollary

$$D[(z-\alpha)u] = \frac{i\phi_1^*(\alpha)}{e_1}(z+\phi_1(0))u,$$

where $\forall P(z) \in \Lambda \qquad Du(P(z)) = -iu(zP'(z)).$

Proof See [9]. ■

3.4 Corollary

$$(z-\alpha)Du = i(\gamma z + \alpha\overline{\gamma})u.$$

Proof Applying in (3.3) the following rule ([9])

$$D(A(z)u) = A(z)Du + izA'(z)u, \quad A(z) \in \Sigma. \blacksquare$$

3.5 Corollary *The sequence $\{\phi_n(z)\}$ verifies the following hypergeometric differential equation*

$$z(\alpha - z)\phi_n^{''}(z) + [(-\gamma + n - 2)z - (\overline{\gamma} + n - 1)\alpha]\,\phi_n^{'}(z) + n(\gamma+1)\phi_n(z) = 0.$$

Proof See [9]. ■

4 The positive definite case.

Since the case $\phi_1(0) = 0$ corresponds to the Lebesgue measure, here we will limit ourselves to the case $\phi_1(0) \neq 0$ or $\gamma \neq 0$.

4.1 Theorem *u is positive defined if and only if $|\phi_1(0)| < 1$ or equivalently $2\Re\gamma > -1$.*

Proof Taking into account that u is positive definite if and only if $\forall n \geq 1\ |\phi_n(0)|^2 < 1$ and 2.6. ■

4.2 Theorem. *We denote:* $G(z) = 1 + 2\sum_{n=1}^{+\infty} \overline{u_n} z^n$. *This function satisfies the following linear differential equation*

$$z(z-\alpha)G'(z) - (\gamma z + \overline{\gamma}\alpha)G(z) = (\gamma z - \overline{\gamma}\alpha).$$

Proof ¿From 3.1 it holds $G(z) = -1 + 2\left({}_2F_1(1, -\gamma; \overline{\gamma}+1; \overline{\alpha}z)\right)$. Using the following relation of hypergeometric series (see [6])

$$(c-a)_n z^{c-a-1}(1-z)^{a+b-c-n}\,{}_2F_1(a-n, b; c; z) =$$

$$= \frac{d^n}{dz^n}[z^{c-a+n-1}(1-z)^{a+b-c}\,{}_2F_1(a, b; c; z)]$$

for $n = 1$ we obtain the equation given in the statement. ■

4.3 Theorem *If* $\alpha = e^{i\psi}$ *and* $2\Re\gamma > -1$, *then the sequence* $\{\phi_n(z)\}$ *is orthogonal with respect to the following weight function*

$$\mu^{'}(\theta) = \sin\left(\frac{\theta-\psi}{2}\right)^{\gamma+\overline{\gamma}} e^{i\frac{\theta}{2}(\gamma-\overline{\gamma})} \qquad 0 \leq \theta \leq 2\pi$$

Proof We prove that

$$\frac{1}{2\pi}\int_0^{2\pi} \phi_n(e^{i\theta})e^{-im\theta}d\mu(\theta) = k_n\delta_{nm}$$

with $0 \leq m \leq n$ and $k_n > 0$.

Let I_{nm} be the above integral, then from 2.7(2)

$$I_{nm} = C_n \int_0^{2\pi} {}_2F_1\left(-n, \gamma+1; 2\Re\gamma+1; -2e^{i\frac{\pi+\theta-\psi}{2}}\sin(\frac{\theta-\psi}{2})\right) e^{-im\theta}d\mu(\theta),$$

with

$$C_n = \frac{1}{2\pi}\frac{\Gamma(2\Re\gamma + n + 1)\Gamma(\gamma + 1)}{\Gamma(2\Re\gamma + 1)\Gamma(\gamma + n + 1)} e^{in\psi}.$$

Expanding the polynomial under the integral in powers of z, the integrals to calculate are as follows:

$$i_{km} = \int_0^{2\pi} \sin\left(\frac{\theta - \psi}{2}\right)^{2\Re\gamma + k} e^{i\theta(i\Im\gamma - m + \frac{k}{2})} d\theta =$$

$$= \int_\psi^{2\pi+\psi} \sin\left(\frac{\theta - \psi}{2}\right)^{2\Re\gamma + k} e^{i\theta(i\Im\gamma - m + \frac{k}{2})} d\theta.$$

with the change of variable $t = \frac{\theta - \psi}{2}$ we obtain

$$i_{km} = 2e^{i\psi(i\Im\gamma - m + \frac{k}{2})} \int_0^\pi (\sin t)^{2\Re\gamma + k} e^{2it(i\Im\gamma - m + \frac{k}{2})} dt$$

whose value is (see [6])

$$i_{km} = 2e^{i(\psi+\pi)[i\Im\gamma - m + \frac{k}{2}]} \frac{\pi}{2^{2\Re\gamma + k}} \frac{\Gamma(2\Re\gamma + k + 1)}{\Gamma(\gamma + k + 1 - m)\Gamma(\overline{\gamma} + 1 + m)},$$

with $0 \leq k \leq n$.

Substituting i_{km} in I_{nm}

$$I_{nm} = \frac{2\pi C_n \Gamma(2\Re\gamma + 1)}{\Gamma(\overline{\gamma} + 1 + m)\Gamma(\gamma + 1 - m)} e^{i(\psi+\pi)[\Im\gamma - m]} \frac{1}{2^{2\Re\gamma}}\, {}_2F_1(-n, \gamma{+}1; \gamma{+}1{-}m; 1)$$

If $m \leq n - 1$

$${}_2F_1(-n, \gamma + 1; \gamma + 1 - m; 1) = \frac{\Gamma(\gamma + 1 - m)\Gamma(n - m)}{\Gamma(\gamma + 1 + n - m)\Gamma(-m)} = 0. \qquad \text{(See [6])}$$

For $m = n$, we use the following relation (see [6]):

$$\frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + 1 - n)} z^{\gamma - n}\, {}_2F_1(-n, \gamma + 1; \gamma + 1 - n; z) = \frac{d^n}{dz^n}\left[z^\gamma (1 - z)^n\right].$$

Taking derivatives in the right-hand side and putting $z = 1$ in the last expression we get

$$\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-n)}\,{}_2F_1(-n,\gamma+1;\gamma+1-n;1) = (-1)^n n!.$$

Substituting the previous in I_{nn} we obtain

$$I_{nn} = \frac{(2\Re\gamma+1)(2\Re\gamma+2)\ldots(2\Re\gamma+n)}{|\gamma+1|^2|\gamma+2|^2\ldots|\gamma+n|^2} I_{00} \neq 0. \blacksquare$$

5 A special case.

If $|\alpha| = 1$, $\gamma \in \mathbf{R} - \{0\}$ and $2\gamma \neq -n$, then

$$\forall n \geq 1 \quad \phi_n(z) = \frac{\gamma}{\gamma+n}\alpha^n\,{}_2F_1(-n,\gamma+1;-\gamma-n+1;\overline{\alpha}z)$$

is a M.O.P.S. related to $\mathbf{T}$.

Taking $\psi_n(z) = \phi_n(\alpha z)$ we obtain $\psi_n(0) = \frac{\gamma}{\gamma+n}\alpha^n$. For $\alpha = \pm 1$ we have the reflection parameters corresponding to the Jacobi weight. $\blacksquare$

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