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ALGEBRAIC VERSUS TOPOLOGIC ANOMALIES

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ABSTRACT. Within the frame of a Group Approach to Quantization anomalies arise in a quite natural way. We present in this talk an analysis of the basic obstructions that can be found when we try to translate symmetries of the Newton equations to the Quantum Theory. They fall into two classes: *algebraic* and *topologic* according to the local or global character of the obstruction. We present here one explicit example of each.

1. The concept of anomaly

Roughly speaking *anomalies* are *obstructions* to the *quantum realization* of *classical symmetries* and they fall into two classes according to the local or global character of the obstruction. On the one hand, there are generators in the symmetry group that do not preserve any distinction between the basic $\hat{q}$ and $\hat{p}$ quantum operators, and this pathology shows up at the Lie algebra level. We shall call this obstruction *algebraic anomaly.* On the other hand, we can find symmetry generators, the local action of which (in the sense of local chart) is well-behaved, whereas their global action does not preserve the Hilbert space of the theory. We shall call this type of obstruction *topologic anomaly* and those generators, *bad operators.*

Algebraic anomalies firstly appeared in Quantum Field Theory when trying to quantize a current algebra made of fermionic matter [1]. Let G be a group of internal symmetries of a classical relativistic theory, generated by T_a such that

$$[T_a, T_b] = C_{ab}^{c} T_c \tag{1}$$

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The classical Lagrangian theory provides Noether currents $j_a \equiv *i_{T_a}\Theta$ ($j^\mu_a = \frac{\partial\mathcal{L}}{\partial\psi_\mu}T_a$) so that the equation $\partial_\mu j^\mu_a = 0$ implies the conservation of the charges $Q_a \equiv \int_\Sigma d\sigma_\mu j^\mu_a$, which close the Lie algebra $\mathcal{G}$ of G under the Poisson bracket:

$$\{Q_a, Q_b\} = C^c_{ab} Q_c \ . \tag{2}$$

Occasionally, we can formally apply the Poisson bracket to j^μ_a itself, even though it is not a Noether invariant, and find a closed current algebra:

$$\{j^0_a(\vec{x}), j^i_b(\vec{y})\} = C^c_{ab}\delta(\vec{x}-\vec{y})j^i_a, \ \Sigma \equiv \{x^0 = cte\} \ , \tag{3}$$

thus interpreting this algebra as a classical symmetry.

The problem then arises of up to what extent the algebra (3) can be quantized *in terms of the basic quantum* (fermionic) *operators* of the theory. One usually finds, in fact, *extra (Schwinger) terms* on the right hand side of the quantum version of (3), which are referred to as *anomalous terms.*

A more precise example is that of *conformal symmetry* in *string theory* (see, e.g. [2]). Here the classical conformal symmetry is unambiguously defined. The classical generators L_m are written in terms of the classical normal modes α_m (the *basic* variables) of the string:

$$L_m = \sum_n \alpha_n \alpha_{m-n} \ . \tag{4}$$

L_m are true Noether invariants satisfying the Poisson algebra:

$$\{L_m, L_n\} = (m-n)L_{m+n} \ . \tag{5}$$

However, the quantization procedure allows $\hat{L}_n$ to be written in terms of the *basic operators* $\hat{\alpha}_m$ *only* for a particular value of the deformation parameter of the conformal algebra:

$$[\hat{L}_m, \hat{L}_n] = (m-n)\hat{L}_{(m+n)} + \frac{(d=26)}{12} n^3 \delta_{m,n} \hat{1} \ . \tag{6}$$

For different values of d, $\hat{L}$ cannot be written in terms of the $\hat{\alpha}$'s and, therefore, $\hat{L}$ *must also be considered as basic operators* of the theory.

Topologic anomalies have been characterized by the failure of the Ehrenfest theorem in quantum systems formulated on a configuration space with non-trivial topology [3, 4]:

$$\frac{d}{dt} < \hat{A} >= \frac{i}{\hbar} < [\hat{H}, \hat{A}] > +anomaly \ . \tag{7}$$

This is a *global* problem *without* classical counterpart provided that Classical Mechanics is formulated by *initial value problems*, and it is often related to the lack of hermiticity of quantum operators.

Topologic anomalies also appear along with *non-equivalent quantizations* of the starting classical system, the phase space of which has *non-trivial homotopy group* π^1 [5] (see also [6]).

2. Anomalies in a Group Approach to Quantization

Anomalies are more accurately formulated in a framework where symmetry and quantization are strongly related issues. We shall discuss them in an already formulated Group Approach to Quantization [7, 8].

On a Lie group we have two types of *mutually commuting* operators: Left- and Right-invariant vector fields

- Right-invariant fields are *pre-quantum operators* providing a *reducible* representation
- Left-invariant fields are used to *reduce* the representation in a compatible way

In addition, the quantization group $\tilde{G}$ is supposed to wear a principal bundle structure with fibre a group T containing $U(1)$:

$$\tilde{G}, T \to \tilde{G}/T \; ; \tag{8}$$

the subgroup $U(1)$ realizes the ordinary phase invariance in Quantum Mechanics and establishes the basic canonically conjugate (symplectic) variables: those producing a $U(1)$-term on the r.h.s. of their commutator.

For the sake of simplicity let us assume that $\tilde{G}$ is a $U(1)$-central extension with Lie algebra co-cycle

$$\Sigma : \mathcal{G} \times \mathcal{G} \to R \; . \tag{9}$$

The *kernel of* Σ, $\mathcal{G}_\Sigma$, is made of *non-symplectic generators.*

We start from $U(1)$-functions on $\tilde{G}$ ($\Xi\psi = i\psi$, $\Xi \in u(1)$) on which we impose the *polarization conditions*

$$\tilde{X}^L \psi = 0 \;\; \forall \tilde{X}^L \in \mathcal{P} \;\; , \tag{10}$$

where $\mathcal{P}$, the polarization, is a maximal left subalgebra *containing* a proper subalgebra $\mathcal{A} \subset \mathcal{G}_\Sigma$ and *excluding* Ξ. It must be stressed that a *full* polarization, i.e. containing the whole $\mathcal{G}_\Sigma$ *might not* exist.

On polarized $U(1)$-functions the right-invariant vector fields act properly and *irreducibly* defining the operators of the *true quantization.* Those operators whose left counterpart are in $\mathcal{A} \subset \mathcal{G}_\Sigma$ can be *written* as a *function* of the *basic ones.*

If the subgroup T is bigger than $U(1)$, all generators of T different from Ξ provide *constraints.* In that case, *not all* right-invariant vector fields are *good* operators since constraints are imposed on the right. If H is the space of T-*functions*, i.e. functions satisfying:

$$\tilde{X}^R_a \psi = D^{(\epsilon)}(T_a)\psi \;\; , \tag{11}$$

where D is a particular representation of T, the index ϵ of which characterizes the quantization, good operators must preserve this space. The subgroup G_{good} satisfying either of the following conditions involving the group commutator [8, 4]:

$$Ad(\tilde{G})\,[T, G_{good}] \subset G_{\mathcal{P}} \;\; or \;\; [G_{good}, T] \subset ker D(T) \tag{12}$$

is made of good operators.

Algebraic anomalies

They appear when a *full* polarization *does not exist* and only a polarization containing a proper subalgebra $\mathcal{A}$ of $\mathcal{G}_\Sigma$ can be found [9].

A non-full polarization causes those operators in $\mathcal{G}_\Sigma \backslash \mathcal{A}$, the anomalous operators, *not to be expressible in terms of the basic ones.* In other words, anomalous operators behave as *if they were basic*, thus giving a *deformation term* on the r.h.s. of (some of) their commutators. Furthermore, the fact that we do not impose on the wave functions as many polarization conditions as it should in order to reduce the quantum representation (half of the generators in $\mathcal{G}_\Sigma \backslash \mathcal{A}$ are absent from $\mathcal{P}$) and $\mathcal{P}$ is nevertheless maximal, creates some sort of paradox. The solution to the polarization equations results, however, in a rather natural situation: the quantum representation is *reducible but not completely reducible.* Now, some kind of mechanism of restriction to the invariant subspace is in order. We then resort to the concept of *higher-order polarization.*

The definition of a higher-order polarization $\mathcal{P}^{HO}$ is analogous to that of first-order one except for the fact that its elements are allowed to belong to the enveloping algebra $U\mathcal{G}^L$. However, the existence of a (full) higher-order polarization is only guaranteed for particular values of the deformation parameters, *the quantum values of the anomaly.* The solutions to the equations associated with $\mathcal{P}^{HO}$ constitute the invariant subspace in the quantum representation, which appears for those particular values of the anomaly.

Topologic anomalies

They appear when the configuration space of the physical system is of the form $R^n/\check{T}$ where $\check{T}$ is a (probably discrete) "surgery" group. The quantization of this system via our group-theoretical method is achieved by considering as quantization group that of the corresponding unconstrained system (with configuration space R^n) but with fibre a group T such that $\check{T} \equiv T/U(1)$.

As mentioned above, when the structure group is bigger than $U(1)$, not all the operators in the theory are good, i.e. not all the operators preserve the subspace of T-functions. Those operators destroying the T-function property of wave functions are in general called *bad operators* and in the present case are identified with *topologic anomalies.* Among them some can be found which destroy the T-function property in a rather benign way (they are not so bad): they just change the label associated with the T-property. To be more precise, if we denote by $\mathcal{H}^{(\epsilon)}$ the particular subspace of the whole set of T-functions, H, carrying the irreducible representation $D^{(\epsilon)}$ of T, bad operators do not preserve in general the specific properties associated with the ϵ-property but some of them can occasionally transform wave functions $\psi^\epsilon \in \mathcal{H}^{(\epsilon)}$ into $\psi^{\epsilon'} \in \mathcal{H}^{(\epsilon')}$. In this way, these not so bad operators generate *quantization-changing transformations*, because the index ϵ parameterizes non-equivalent quantizations.

3. The simplest, yet relevant example of an algebraic anomaly

The simplest Lie group wearing an algebraically anomalous structure is the Schrödinger group in $1+1$ dimensions, the symmetry group of the quadratic potential in one dimension [10]:

$$Ax^2 + Bx + C \tag{13}$$

The classical (extended) Schrödinger Lie algebra is realized immediately as the Poisson algebra generated by either

$$\{1, p, x, p^2, x^2, p\cdot x\} \quad or \quad \{1, p, x, p^2 + x^2, x^2, x\cdot p\}\ , \tag{14}$$

where we have distinguished between two different choices of Hamiltonian in the same algebra. Each choice corresponds to a different class of irreducible representations: in the first case the entire group is represented on the carrier space for the irreducible representations of the Galilei subgroup, and in the second, the entire group is represented on the carrier space for the irreducible representations of the Newton subgroup (the harmonic oscillator symmetry).

For the sake of concreteness we shall limit ourselves to the second case and adopt the more appropriate notation $x \equiv \frac{1}{\sqrt{C+C^*}}$, $p \equiv \frac{-im\omega}{\sqrt{2m\omega}}(C-C^*)$, associated with the harmonic oscillator. We then write for the Schrödinger Poisson subalgebra the generators:

$$\{1, C, C^*, CC^*, \frac{1}{2}C^2, \frac{1}{2}C^{*2}\} \equiv \{1, C, C^*, h, z, z^*\}\ \ , \tag{15}$$

where the quadratic functions generating the $sL(2,R)$-subalgebra that replaces the time of the Newton group, has been denote by a single character to mean that they really correspond to linear elements in the abstract Lie algebra. The Schrödinger group can also be seen, in this way, as the classical symmetry of the two-photon laser system with a Hamiltonian of the form:

$$H = CC^* + \alpha C^2 + \alpha^* C^{*2} + \beta C + \beta^* C^* + \gamma\ \ . \tag{16}$$

The classical commutators between the Lie algebra generators are easily derived by computing the Poisson bracket, giving rise to:

$$
\begin{aligned}
\{h, C^*\} &= C^* & \{C^*, C\} &= 1 \\
\{h, C\} &= -C & \{z^*, z\} &= -\frac{1}{2}h \\
\{h, z^*\} &= 2z^* & \{C^*, z^*\} &= 0 \\
\{h, z\} &= -2z & \{C, z\} &= 0
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
\{C, z^*\} &= \frac{1}{\sqrt{2}}C^* \\
\{C^*, z\} &= \frac{1}{\sqrt{2}}C \quad .
\end{aligned} \tag{18}
$$

The last two commutators reveal a non-diagonalizable action of the $sL(2,R)$ subalgebra, which generates the kernel of the co-cycle, on the rest. This action is precisely responsible for an algebraic anomaly. In fact, there is no full polarization containing the entire $sL(2,R)$ subalgebra. At most, a non-full polarization can be found:

$$
\mathcal{P} = < \tilde{X}^L_C, \tilde{X}^L_h, \tilde{X}^L_z > \quad . \tag{19}
$$

Quantizing with the non-full polarization (19) results in a breakdown of the naively expected correspondence between $\hat{z}, \hat{z}^\dagger$ operators and the basic ones:

$$
\begin{aligned}
\hat{z} &\neq \frac{1}{2}\hat{C}^2 \\
\hat{z}^\dagger &\neq \frac{1}{2}\hat{C}^{\dagger^2} \quad .
\end{aligned} \tag{20}
$$

Unlike in the classical case, the operators $(\hat{h}), \hat{z}, \hat{z}^\dagger$ behave independently of $\hat{C}, \hat{C}^\dagger$, although they generate an irreducible representation of $SL(2,R)$ with Bargmann index k [11]. The operators $\hat{z}, \hat{z}^\dagger$ seem to have dynamical content as if they were canonically-conjugate (basic) operators. In fact, their (quantum) commutator is no longer the homomorphic image of the corresponding Poisson bracket (see (17)) but

$$
[\hat{z}, \hat{z}^\dagger] = 2(\hat{h} + k\hat{1}) \quad , \tag{21}
$$

where an extra central term comes out.

However, only for a particular value (the quantum value of the anomaly) $k = \frac{1}{4}$, the representation of the Schrödinger group *becomes reducible*, although non-completely reducible and, *on the invariant subspace*, the operators $\hat{z}, \hat{z}^\dagger$ do really express as $\frac{1}{2}\hat{C}^2, \frac{1}{2}\hat{C}^{\dagger^2}$. The invariant subspace is constituted by the solutions to a second-order polarization which exists only for $k = \frac{1}{4}$ [9]:

$$
\mathcal{P}^{HO} = < \tilde{X}^L_C, \tilde{X}^L_h, \tilde{X}^L_z, \tilde{X}^L_{z^*} - \alpha(\tilde{X}^L_{C^*})^2 > \quad , \tag{22}
$$

where the constant α is forced to acquire the value $\alpha = \sqrt{2}k$.

4. The simplest, yet relevant example of topological anomaly: the free particle on the circle

The configuration space (x, t) is the direct product $S^1 \times R$, the fundamental group of which is obviously $\pi^1 = Z$. The relevant symmetry to be used as quantization group is the extended 1+1-Galilei group fibered by

$$T = U(1) \times \{e_k,\ k \in Z\} \equiv U(1) \times \check{T} \ , \tag{23}$$

$\check{T}$ being the subgroup of finite translations on x by an amount of kL, for some spatial period L. As mentioned above, the inequivalent quantizations when a generalized "phase" invariance is involved, are characterized by the irreducible representations of the structure group T:

$$\begin{aligned} \mathcal{D}^{(\epsilon)}(\zeta, e_k) &= \zeta D^{(\epsilon)}(e_k) \ , \ \ \zeta \in U(1) \\ D^{(\epsilon)}(e_k) &= e^{\frac{i}{\hbar}\epsilon k L} \quad \epsilon \in [0, \frac{2\pi\hbar}{L}) \ . \end{aligned} \tag{24}$$

Since the non-triviality of the topology comes from the spatial variable only, let us forget about time and consider the Heisenberg-Weyl group parameterized by (x, p, ζ) with a specially suitable co-cycle [4]:

$$\begin{aligned} x'' &= x' + x \\ p'' &= p' + p \\ \zeta'' &= \zeta'\zeta e^{-\frac{i}{\hbar}xp'} \end{aligned} \tag{25}$$

from which we derive the left- and right-invariant vector fields:

$$\begin{aligned} \tilde{X}^L_x &= \tfrac{\partial}{\partial x} - \tfrac{1}{\hbar} p\, \Xi & \qquad \tilde{X}^R_x &= \tfrac{\partial}{\partial x} \\ \tilde{X}^L_p &= \tfrac{\partial}{\partial p} & \qquad \tilde{X}^R_p &= \tfrac{\partial}{\partial p} - \tfrac{1}{\hbar} x\, \Xi \\ \tilde{X}^L_\zeta &= i\zeta \tfrac{\partial}{\partial \zeta} \equiv \Xi & \qquad \tilde{X}^R_\zeta &= i\zeta \tfrac{\partial}{\partial \zeta} \equiv \Xi \end{aligned} \tag{26}$$

and a polarization subalgebra generated by $\tilde{X}^L_p$ can be chosen.

Starting from a complex function $\Psi(x, p, \zeta)$ on the group, the $U(1)$-function condition just says that $\Psi(x, p, \zeta) = \zeta\psi(x, p)$, whereas the $\check{T}$-function condition establishes the quasi-periodicity condition:

$$\psi^\epsilon(x + kL, p) = e^{\frac{i}{\hbar}\epsilon k L}\psi^\epsilon(x, p) \ . \tag{27}$$

Applying the polarization condition we get:

$$\begin{aligned} \psi^\epsilon(x, p) &= \Phi^\epsilon(x) \\ \Phi^\epsilon(x + L) &= e^{\frac{i}{\hbar}\epsilon L}\Phi^\epsilon(x) \ , \end{aligned} \tag{28}$$

so that, the action of the *quantum operators* is

$$\begin{aligned} \hat{p}\Psi^{\epsilon} &= \zeta\nabla\Phi^{\epsilon} \\ \hat{x}\Psi^{\epsilon} &= \zeta x\Psi^{\epsilon} \ . \end{aligned} \tag{29}$$

We can see that only $\hat{p}$ and *finite* boosts by an amount of $p_k \equiv \frac{2\pi\hbar}{L}k$ are *good* operators. Boosts in general *change the quantization index* ϵ. Then, the question arises of up to whether or not a good position-like operator can be found. To see this question, we realize that the classical function $\eta \equiv e^{i\frac{2\pi}{L}x}$ is periodic and, therefore, $\hat{\eta}^k \equiv (e^{i\frac{2\pi}{L}\hat{x}})^k$ is a *good* position operator for $k \in Z$, although $\hat{\eta}$ is not Hermitian in general (it is rather unitary). However, the operator $\hat{\eta}$ can be decomposed in two Hermitian pieces (the sum of one Hermitian and other anti-Hermitian more precisely):

$$\hat{\eta} = \cos(\frac{2\pi}{L}\hat{x}) + i\sin(\frac{2\pi}{L}\hat{x}) \ . \tag{30}$$

Finally, it is worth mentioning that $< \hat{p}, \hat{\eta}, \hat{\eta}^{\dagger} >$ is a set of good operators which closes, under ordinary commutation, a non-extended oscillator algebra. The operators $\hat{\eta}$ and $\hat{\eta}^{\dagger}$ act as ladder operators on the eigenfunctions of $\hat{p}$, a fact which has been used in [12] to study Quantum Mechanics on the circumference.

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