Proceedings Ist International Meeting
on Geometry and Topology
Braga (Portugal)

Public. Centro de Matemática
da Universidade do Minho
p. 63–73, 1998

An Euclidean Characterization of the Regions of an Arrangement of Pseudospheres

Ilda P. F. da Silva

Consider a finite set, $\mathcal{H} = \{H_e\}_{e\in E}$, of hyperplanes in $I\!R^d$ and let $\Delta(\mathcal{H})$ denote the cell decomposition of $I\!R^d$ into polyhedral cones determined by $\mathcal{H}$. The maximal cells of $\Delta(\mathcal{H})$ correspond to the connected components or *regions* of $\mathcal{H}^c = I\!R^d \setminus \cup_{e\in E} H_e$ and can be identified with sign vectors in the following way:

For every $e \in E$ distinguish the two (open) halfspaces defined by H_e denoting one by H_e^+ and the other by H_e^-. Each region R of $\mathcal{H}^c$ is then an intersection of the form $R = \bigcap_{e\in E^+} S_e^+ \cap \bigcap_{e\in E^-} S_e^-$ for some partition $E^+ \uplus E^- = E$ and can be identified with the sign vector $X_R \in \{+,-,0\}^E$ defined by $X_R(e) = +\ if R \subseteq H_e^+$, $X_R(e) = -\ if R \subseteq H_e^-$.

The set $\mathcal{R}(\mathcal{H}) := \{X_R \in \{+,-,0\}^E : R\ is\ a\ region\ of\ \mathcal{H}^c\}$ encodes all the information about the cell decomposition $\Delta(\mathcal{H})$.

It is a major problem in combinatorics to characterize those families $\mathcal{R} \subseteq \{+,-,0\}^E$ for which there exists a family $\mathcal{H}$ of hyperplanes such that $\mathcal{R} = \mathcal{R}(\mathcal{H})$. By taking the intersection of every hyperplane of $\mathcal{H}$ with the sphere S^{d-1} of $I\!R^d$, this problem is clearly equivalent to the problem of characterizing combinatorically the regions of the S^{d-1} determined by a set of equators.

This point of view leads to the following more general (and easier) problem : *characterize those families $\mathcal{T} \subseteq \{+,-,0\}^E$ for which there exists an arrangement $\mathcal{A}$ of pseudospheres (pseudoequators) of S^d such that $\mathcal{T}$ represents the regions of S^d determined by $\mathcal{A}$.*

In this communication we present the results obtained in [dS] which provide combinatorial characterizations for the set $\mathcal{T}$ of sign vectors defining the regions of an arrangement of pseudospheres.

These characterizations are stated in part II, Theorem 1. They improve all the previous characterizations by W. Bienia and R. Cordovil [B-C], K. Handa [H] and I. da Silva [dS]. At the same time they generalize to arrangements of pseudospheres (or oriented matroids) in general the characterizations given by J. Lawrence [L] for the particular case of arrangements of pseudospheres in general position (or uniform oriented matroids).

Our main concern will be to present in a selfcontained way the ideas behind Theorem 1 since detailed proofs can be found in [dS]. In section I we recall, in a convenient way, some standard facts about oriented matroids and in section II we give the results and point out some further consequences. One of the consequences is Theorem 1', an Euclidean Representation Theorem for Oriented Matroids, which characterizes oriented matroids as subsets of vertices of a cube in $I\!R^E$ with certain symmetry properties.

Notation: A *sign vector* on E is an element of

$$\{+,-,0\}^E := \{X : E \to \{+,-,0\}\}.$$

For every $X \in \{+,-,0\}^E$ and $\epsilon \in \{+,-,0\}$, $X^\epsilon = \{e \in E : X(e) = \epsilon\}$. The *support of* X is $spt(X) = X^+ \uplus X^-$ and the *zero-set of* X is X^0.
Given $X \in \{+,-,0\}^E$, for every subset A of E:
$X \setminus A$ denotes the *restriction of* X *to* $E \setminus A$ and ${}_{-A}X$ denotes the *sign vector obtained from* X *reversing signs on* A i.e. ${}_{-A}X(e) = -X(e)$ *if* $e \in A$ and ${}_{-A}X(e) = X(e)$ *otherwise*. The opposite $-X$ of X is $-X ={}_{-E} X$.
The *composition*, $X \circ Y$, of two sign vectors $X, Y \in \{+,-,0\}^E$ is the sign vector defined by $X \circ Y(e) := X(e)$, *if* $X(e) \neq 0$ and $X \circ Y(e) = Y(e)$ *otherwise*. Note that this operation is not commutative. In fact $X \circ Y = Y \circ X$ iff $X(e) = Y(e) \quad \forall e \in sp(X) \cap sp(Y)$. More generally given $X \in \{+,-,0\}^A$, $Y \in \{+,-,0\}^B$ we consider $X \circ Y \in \{+,-,0\}^{A \cup B}$ defined as before, by taking X, Y as sign vectors on $A \cup B$ with $X(e) = 0$, $\forall e \in B \setminus A$ and $Y(e) = 0$, $\forall e \in A \setminus B$.
When we refer to set $\mathcal{T} \subseteq \{+,-,0\}^E$ as a *poset* we are considering $\mathcal{T} := (\mathcal{T}, \preceq)$ where the *partial order* $\preceq$ is defined by: $X \preceq Y$ iff $X^+ \subseteq Y^+$ and $X^+ \subseteq Y^+$.

I. Arrangements of pseudospheres and oriented matroids

Let S^d denote the standard d-sphere of $I\!R^{d+1}$: $S^d = \{x \in I\!R^{d+1} : ||x|| = 1\}$ and identify the standard S^{d-1} sphere with the equator $S^d_0 = \{x \in S^d : x_{d+1} = 0\}$ of S^d. A $d-sphere$ is any topological space homeomorph to S^d.

A *pseudosphere* of S^d is the image $h(S^d_0)$ of S^d_0 by a homeomorphism $h : S^d \to S^d$. Pseudospheres are very well behaved $(d-1)-subspheres$ of S^d: the complement $S^d \setminus S$ of a pseudosphere S has two connected components, the *(open) sides* or *(open) pseudohemispheres* of S that we denote by S^+ and S^- which are homeomorphic to $(d-1)$-balls.

An *arrangement of pseudospheres* in S^d is a finite multiset $\mathcal{A} = \{S_e\}_{e\in E} \cup \{S^d\}_{l\in L}$ of $|E|$- pseudospheres and $|L|$-copies of S^d satisfying the following two conditions:
(A1) For all $A \subseteq E$, $S_A = \cap_{e\in A} S_e$ is a sphere.
(A2) For all $A \subseteq E$, $e \in E$ such that $S_A \not\subseteq S_e$ if S_e^+ and S_e^- denote the two sides of S_e then $S_A \cap S_e$ is a pseudosphere of S_A with sides $S_A \cap S_e^+$ and $S_A \cap S_e^-$.

An arrangement of pseudospheres is *simple* when it is a finite *set* $\mathcal{A} = \{S_e\}_{e\in E}$ of pseudospheres : different elements of E index different pseudospheres and it contains no copies of S^d. The definition we gave is the convinient one to deal with general families of sign vectors. To simplify we will say let $\mathcal{A} = \{S_e\}_{e\in E}$ be an arrangement of pseudopheres, knowing that some of the S_e may be the whole sphere S^d.

An arrangement of pseudospheres is *essential* when $\cap_{e\in E} S_e = \emptyset$.

A *signed arrangement of pseudospheres* is an arrangement $\mathcal{A} = \{S_e\}_{e\in E}$ of pseudospheres together with a fixed choice of the positive side S_e^+ and negative side S_e^- for every pseudosphere.

Any arrangement $\mathcal{A} = \{S_e\}_{e\in E}$ of pseudospheres of S^d determines a cell decomposition $\Delta(\mathcal{A})$ of S^d. Each cell of $\Delta(\mathcal{A})$ is a nonempty intersection of the form:

$$\sigma = \bigcap_{e\in X^+} S_e^+ \cap \bigcap_{e\in X^-} S_e^- \cap \bigcap_{e\in X^0} S_e, \ for\ some\ 3-partition\ X^+ \uplus X^- \uplus X^0 = E$$

and can be identified with the sign vector $X_\sigma \in \{+,-,0\}^e$ defined by:

$$X_\sigma(e) = \begin{cases} + & if & \sigma \subseteq S_e^+ \\ - & if & \sigma \subseteq S_e^- \\ 0 & if & \sigma \subseteq S_e \end{cases}$$

Note that for every $e \in E$ such that $S_e = S^d$, $X_\sigma(e) = 0$. The cell decomposition $\Delta(\mathcal{A})$ of S^d is then described by the family $\mathcal{L}(\mathcal{A}) := \{X_\sigma \in \{+,-,0\}^e : \sigma\ is\ a\ cell\ of\ \Delta(\mathcal{A})\}$ and the *poset* $\mathcal{F}(\Delta(\mathcal{A}))$ *of closed cells of* $\Delta(\mathcal{A})$ *ordered by containment, is isomorphic to the poset* $\mathcal{L}(\mathcal{A}) = (\mathcal{L}(\mathcal{A}), \preceq)$.

The *Topological Representation Theorem for Oriented Matroids* characterizes combinatorically in terms of $\mathcal{L}(\mathcal{A})$ cell decompositions of spheres by arrangements of pseudospheres.

An *Oriented Matroid* over a (finite) set E is a pair $\mathcal{M} = (E, \hat{\mathcal{L}})$ where $\hat{\mathcal{L}} \subseteq \{+,-,0\}^E$ satisfies the following four axioms:
(O1) $0 \in \hat{\mathcal{L}}$
(02) $X \in \hat{\mathcal{L}}$ implies $-X \in \hat{\mathcal{L}}$
(O3) If $X, Y \in \hat{\mathcal{L}}$ then $X \circ Y \in \hat{\mathcal{L}}$.

(O4) For all $X, Y \in \hat{\mathcal{L}}$, $e \in E$ such that $X(e) = -Y(e)$ there is $Z \in \hat{\mathcal{L}}$ such that $Z(e) = 0$ and $Z(f) = X(f) = Y(f)$ for every $f \in E$ such that $X(f) = Y(f)$.

The poset $(\hat{\mathcal{L}}, \preceq)$ is called the *big face lattice* of the oriented matroid $\mathcal{M}$ and the set $\mathcal{L} = \hat{\mathcal{L}} \setminus \{0\}$ is the *poset of covectors* of $\mathcal{M}$.

An *oriented matroid is simple* if it contains no *loops*, elements $e \in E$ such that $X(e) = 0, \forall X \in \mathcal{L}$ and no *parallel elements* pair of distinct elements $e, f \in E$ such that $X(e) = X(f), \forall X \in \mathcal{L}$ or $X(e) = -X(-f), \forall X \in \mathcal{L}$.

Topological Representation Theorem for Oriented Matroids (Theorem 5.2.1 [OM])

$\mathcal{L} \subseteq \{+,-,0\}^E$ *is the set of covectors of an oriented matroid of rank* r *over* E *if and only if* $\mathcal{L} = \mathcal{L}(\mathcal{A})$ *for some (essential) arrangement of pseudospheres* $\mathcal{A}$ *of the sphere* S^{r-1}*. Moreover, for any such arrangement* $\mathcal{A}$*,* $\mathcal{F}(\Delta(\mathcal{A})) \simeq (\mathcal{L}(\mathcal{A}), \preceq)$.

Note that in the correspondence between $\mathcal{F}(\Delta(\mathcal{A}))$ and $(\mathcal{L}(\mathcal{A}), \preceq)$, the minimal covectors - the *cocircuits* of $\mathcal{M}$ - correspond to the *pairs of opposite points* (0-dimensional spheres) of $\mathcal{F}(\Delta(\mathcal{A}))$ and the maximal covectors - the *topes* of $\mathcal{M}$ - to the connected componentes of $S^d \setminus \cup_{e \in E'} S_e$ (E' the subset of E that indexes pseudopheres).

2. Lattice of flats, closure operator and rank function of an oriented matroid

Let $\mathcal{M} = (E, \hat{\mathcal{L}})$ be an oriented matroid over a set E, where $\hat{\mathcal{L}} \subseteq \{+,-,0\}^E$ is the big face lattice of $\mathcal{M}$.

Consider the family $\mathcal{F}$ of zero-sets of the sign vectors of $\mathcal{L}$:

$$\mathcal{F} = \{F \subseteq E : F = X^0, \ for\ some\ X \in \hat{\mathcal{L}}\}$$

The poset $\mathcal{F} = (\mathcal{F}, \subseteq)$ is a *geometric lattice* - the lattice of *flats* of $\mathcal{M}$ or, to be more precisely the lattice of flats of the (unoriented) matroid that underlies $\mathcal{M}$. Note that the maximum element of $\mathcal{F}$ is allways E (because $0 \in \hat{\mathcal{L}}$) but the minimum element $\hat{0}$ of $\mathcal{F}$ may not be the emptyset, it contains all the loops of oriented matroid.

From standard matroid theory we recall that *the rank - r(F) -of a flat* $F \in \mathcal{F}$ is the length of any maximal chain of $\mathcal{F}$ between $\hat{0}$ and F. The *rank of the matroid* $\mathcal{M}$, $r(\mathcal{M}) := r(E)$.

The *closure operator of* $\mathcal{M}$ is the operator $cl_{\mathcal{M}} : \mathcal{P}(E) \to \mathcal{P}(E)$ that associates to every $A \subseteq E$ the smallest flat $cl_{\mathcal{M}}(A) \subseteq E$ that contains A, that is to say, $cl_{\mathcal{M}}(A) := \cap_{F \in \mathcal{F}, A \subseteq F} F$. For every $A \subseteq E$ the rank of A is defined as $r(A) := r(cl_M(A))$.

For an arrangement $\mathcal{A} = \{S_e\}_{e \in E}$ of pseudospheres of S^d the notions of flat, rank and closure have the following interpretation:

- $F \subseteq E$ is *a flat of* $\mathcal{A}$ iff $\forall e \in E \setminus F, \ S_F = \cap_{f \in F} S_f \not\subseteq S_e$.
- *the rank of a flat* F is related with the dimension of the sphere S_F in the following way: $r(F) = d - dim S_F$.

- *the closure* $cl_M(A)$ *of a subset* A *is defined as* $cl_{\mathcal{M}}(A) = A \cup \{e \in E \setminus A : S_A = \cap_{a \in A} S_a \subseteq S_e\}$.

3. Minors of an Oriented Matroid

Let $\mathcal{M} = (E, \hat{\mathcal{L}})$ be an oriented matroid over a set E.

For every $A \subseteq E$ the following two families of sign vectors satisfy the axioms (O1) to (O4) of the definition of oriented matroid:

$$\mathcal{L} \setminus A := \{X \setminus A \in \{+, -, 0\}^E : X \in \mathcal{L}\}$$

$$\mathcal{L}/A := \{X \setminus A \in \{+, -, 0\}^E : X \in \mathcal{L} \text{ and } A \subseteq X^0\}$$

The oriented matroid $\mathcal{M} \setminus A = (E \setminus A, \mathcal{L} \setminus A)$ is the *restriction to* $E \setminus A$ *of the oriented matroid* $\mathcal{M}$. The oriented matroid $\mathcal{M}/A = (E \setminus A, \mathcal{L}/A)$ is the *contraction by* A *of the oriented matroid* $\mathcal{M}$.

If $\{S_e\}_{e \in E}$ is an arrangement of pseudospheres that represents $\mathcal{M}$ then $\{S_e\}_{e \in E \setminus A}$ is the arrangement of pseudopheres that represents $\mathcal{M} \setminus A$ and $\{S_A \cap S_e\}_{e \in E \setminus A}$ is the arrangement of pseudospheres of the sphere $S_A = \cap_{e \in A} S_e$ that represents $\mathcal{M}/A$. When we contract by A, each element $e \in cl_{\mathcal{M}}(A) \setminus A$ is a loop of $\mathcal{M}/A$ and is represented by a copy of S_A.

The following is a standard result in oriented matroids:

Theorem A. (Prop. 3.3.2, 3.3.3 of [OM])
Let $\mathcal{M} = (E, \mathcal{L})$ *be an oriented matroid of rank* r *over* E. *Then,*
i) For every $A \subseteq E$, *and any total order* $(a_1, \ldots, a_k)$ *of* A:

$$\mathcal{M}/A = (((\mathcal{M})/a_1)/a_2)/\ldots)/a_k$$

Moreover, the set $\mathcal{T}/A$ *of topes of* $\mathcal{M}/A$ *is given by.*

$$\mathcal{T}/A = Max\mathcal{L}/A = \{X \in \mathcal{L}/A : X^0 = cl_{\mathcal{M}}(A)\}$$

ii) $\forall A, B \subseteq E$ *such that* $A \cap B = \emptyset$

$$(\mathcal{M} \setminus A)/B = (\mathcal{M}/B) \setminus A$$

4. Tope graph and supertopes of an oriented matroid

Let $\mathcal{M} = (E, \mathcal{L})$ be an oriented matroid and denote by $\mathcal{T} \subseteq \{+, -, 0\}^E$ its set of *topes*: $\mathcal{T} := Max\mathcal{L}$. Note that all the topes have the same support. If $\mathcal{M}$ has no *loops* then the support of every tope is E and we have $\mathcal{T} \subseteq \{+, -\}^E$.

A *supertope of* $\mathcal{M}$ is a subset of topes of the form

$$\mathcal{T}_X := \{T \in \mathcal{T} : X \preceq T\} \text{ for some restriction } X \text{ of a tope of } \mathcal{T}$$

Note that in terms of arrangements of pseudospheres a supertope is the set of topes contained in an intersection of pseudohemispheres.

The *Tope Graph* $G_{\mathcal{M}}$ of the oriented matroid $\mathcal{M}$ is the graph with vertex set $\mathcal{T}$, two topes $T, T' \in \mathcal{T}$ define an edge of $G_{\mathcal{M}}$ if and only if $T' =_{-cl_{\mathcal{M}}(e)} T$.

In terms of arrangements of pseudospheres the tope graph of a (simple) arragement $\{S_e\}_{e\in E}$ of pseudospheres in S^d is the graph whose vertices correspond to the connected components of $S^d \setminus \cup_{e\in E} S_e$ two components T, T' define an edge iff they are separated by a unique pseudosphere.

As Theorem A, the next theorem expresses properties of the tope set $\mathcal{T}$:

Theorem B (Prop. 4.4.8, 4.3.6 [OM])
Let $\mathcal{M} = (E, \mathcal{L})$ be an oriented matroid of rank r over E. Then,
i) the tope graph $G_{\mathcal{M}}$ of $\mathcal{M}$ is r-connected.
ii) the subgraph, $G_{\mathcal{M}}(\mathcal{T}_X)$, of $G_{\mathcal{M}}$ generated by the vertices of a supertope is connected.

5. Uniform Oriented Matroids - Lopsidness and Topes

A *uniform oriented matroid of rank r over E* is an oriented matroid whose lattice of flats has the special form: $\mathcal{F} = \{F \subseteq E : |F| < r\} \cup \{E\}$.

In terms of arrangements of pseudospheres, uniform oriented matroids correspond to *arrangements of pseudospheres in general position* in S^{r-1} i.e. those *simple* arrangements $\mathcal{A}$ of pseudospheres for which every k- dimensional subpseudosphere $S_A = \cap_{e\in A} S_e$ is the intersection of *exactly* $r-k-1$ pseudospheres.

J. Lawrence [L] obtained several characterizations for the set of topes of a uniform oriented matroid based on the notion of lopsidness.

A subset $\mathcal{K} \subseteq \{+,-\}^E$ is *lopsided* if for every partition $A \uplus B = E$ one (and only one) of the following conditions is satisfied:

either there is $X \in \{+,-\}^A$ such that $X \circ \{+,-\}^B \subseteq \mathcal{K}$

or there is $Y \in \{+,-\}^B$ such that $X \circ \{+,-\}^A \subseteq \mathcal{K}^c$, where $\mathcal{K}^c = \{+,-\} \setminus \mathcal{K}$.

Theorem C (J. Lawrence [L])
For a set $\mathcal{T} \subseteq \{+,-\}^E$ satisfying $\mathcal{T} = -\mathcal{T}$, the following two conditions are equivalent:
L1) $\mathcal{T}$ is the set of topes of a uniform oriented matroid.
L2) $\mathcal{T}$ is lopsided.

II. Axioms for Topes of an Oriented Matroid

The results obtained in [dS] depend on the following proposition which says how to recover $\mathcal{L}$ directly from $\mathcal{T}$ (without making use of orthogonality of oriented matroids).

Proposition 1. ([dSO],[dS])
Suppose that $\mathcal{M} = (E, \mathcal{L})$ is an oriented matroid whose set of topes is $\mathcal{T} = Max\mathcal{L}$. Then, the set $\mathcal{L}$ of all covectors of $\mathcal{M}$ is given by:

$$\mathcal{L} = \{X \in \{+, -, 0\}^E : X \circ \mathcal{T} \subseteq \mathcal{T}\}$$

There are two obvious conditions that any collection $\mathcal{T} \subseteq \{+, -, 0\}^E$ of sign vectors must satisfy in order to be the collection of topes of an oriented matroid:
(T1) $\forall T, T' \in \mathcal{T}$, $spt(T) = spt(T')$
(T2) $T \in \mathcal{T}$ then $-T \in \mathcal{T}$

The first condition expresses the fact that degenerate pseudospheres (copies of the sphere) of an arrangement of pseudospheres play no significant role in the definition of $\Delta(\mathcal{A})$. The second condition that every region of $\Delta(\mathcal{A})$ has an apposite region.

The characterizations obtained by Bienia and Cordovil [B-C] and by Handa [H] are essentially converses of, respectively, Theorems A and B of the last section.

Note that both theorems use explicitely the notion of contraction of an oriented matroid and associated with it the notion of closure operator which first of all needs to be generalized for a family of sign subsets which only satisfies (T1) and (T2).

This generalization can be made essentially in two different ways:
- define contraction by a single element set and then, by induction, contraction by *any ordered set* as in [B-C] and [H].
- combine Proposition 1 with the definition of contraction of an oriented matroid to obtain a definition of contraction by *any unordered set.* This is the definition we adopted. It is more difficult to handle but easier to test.

Definition 1. *Given $\mathcal{T} \subseteq \{+, -, 0\}^E$ satisfying conditions (T1) and (T2), for every subset $A \subseteq E$ the contraction of $\mathcal{T}$ by A is the family $\mathcal{T}/A$ of sign vectors on $E \setminus A$ defined by:*

$$\mathcal{T}/A = Max\{V \in \{+, -, 0\}^{E \setminus A} : V \circ \mathcal{T} \subseteq \mathcal{T}\}$$

Note that all the elements $V \in \mathcal{T}/A$ have the same support $B \subseteq E \setminus A$. As we know from Theorem A, when $\mathcal{T}$ is the set of topes of an oriented matroid $\mathcal{M}$ the following equality holds $B = cl_{\mathcal{M}}(A)$. Hence we can generalize the notion of closure operator in the following way:

Definition 2. *The $\mathcal{T}$-operator of a family $\mathcal{T} \subseteq \{+, -, 0\}^E$ satisfying conditions (T1) and (T2) is the operator $[\ ]_{\mathcal{T}} : \mathcal{P}(E) \to \mathcal{P}(E)$ that associates to every $A \subseteq E$ the subset $[A]_{\mathcal{T}} \subseteq E$ defined by $[A]_{\mathcal{T}} :=$ complement in E of the support of any vector $V \in \mathcal{T}/A$.*

For a general $\mathcal{T}$ satisfying (T1) and (T2), the $\mathcal{T}-$ *operator is not in general a closure operator.* This is related closely with the fact that: for a subset A with $|A| > 1$ we may have $\mathcal{T}/A \neq (((\mathcal{T}/a_1)/a_2)\ldots)/a_k$ for some total order $(a_1, \ldots, a_n)$ of A.

Note that for any element $a \in A$, $[a]_\mathcal{T} = \{b \in E : T(b) = T(a), \forall T \in \mathcal{T}\} \cup \{b \in E : T(b) = -T(a), \forall T \in \mathcal{T}\}$.

The notion of $\mathcal{T}-$*operator*, in particular the $\mathcal{T}-$"closure" of a single element set leads to the next generalization of tope graph:

Definition 3. *The $\mathcal{T}$ − graph, $G_\mathcal{T}$, of a family $\mathcal{T} \subseteq \{+,-,0\}^E$ satisfying conditions (T1) and (T2), is the graph whose vertices are the elements of $\mathcal{T}$, a pair $\{T, T'\}$ of vertices is an edge of the graph iff $T' =_{-[e]_\mathcal{T}} T$ for some $e \in E$.*

We are now able to state the main results:

Theorem 1. (Theorems 3.2, 3.19, 4.1 of [dS])
Suppose that $\mathcal{T} \subseteq \{+,-,0\}^E$ is a family of sign vectors satisfying
(T1) $\mathcal{T} = -\mathcal{T}$ and (T2) $T \in \mathcal{T}$ then $-T \in \mathcal{T}$
Then the following four conditions are equivalent:
(1) *$\mathcal{T}$ is the set of topes of an oriented matroid over E.*
(2) *$\forall A, B \subseteq E$ such that $A \cap B = \emptyset$*

$$\mathcal{T} \setminus A/B = \mathcal{T}/B \setminus A$$

(3) *$\forall A \subseteq E$ and $X \in \{+,-,0\}^{E\setminus A}$*

$$(\mathcal{T}/A)_X \text{ is connected}$$

(4) *For every $X \in \{+,-,0\}^A$, restriction of a vector of $\mathcal{T}$*

$$\mathcal{T}_X \setminus A = -\mathcal{T}_X \setminus A \;\; \text{implies} \;\; \mathcal{T}_X \setminus A = \mathcal{T} \setminus A$$

Example: Consider

$$\mathcal{T} = \{1^+2^+3^+4^+, 1^+2^+3^-4^-, 1^+2^-3^+4^- \text{ and their opposites}\} \subseteq \{+,-\}^{[4]}.$$

$\mathcal{T}$ is not the set of topes of an oriented matroid because:
(2) $(\mathcal{T} \setminus 4)/3 = \{1^+2^+, 1^-2^-\} \neq (\mathcal{T}/3) \setminus 4 = \emptyset$ or, because:
(3) $\mathcal{T}_{1+2+} = \{1^+2^+3^+4^+, 1^+2^+3^-4^-\}$ is not connected or yet, because:
(4) $\mathcal{T}_{1+2+} \setminus \{1,2\} = \{3^+4^+, 3^-4^-\} \neq \mathcal{T} \setminus A = \{3^+4^+, 3^-4^-, 3^+4^-, 3^-4^+\}$.

Axioms (2) and (3) may be considered as converses of Theorems A and B, respectively. Axiom (4) is a generalization of Theorem C to all oriented matroids.

The "disagreable" length of the proofs given in [dS] compared with Lawrence's proofs [L] in the uniform case can be explained by the fact that while it is easy to deduce from the definition of lopsidness that: a subset $\mathcal{K} \subseteq \{+,-\}^E$ *is lopsided* $\Longleftrightarrow$ $\mathcal{K}^c = \{+,-\}^E \setminus \mathcal{K}$ *is lopsided* and then use (sign vector) orthogonality arguments, it is not true that if a family $\mathcal{T}$ satisfies axiom (4) then its complement $\mathcal{T}^c$ also satisfies axiom (4). The definition of the orthogonal family $\mathcal{T}^*$ is in fact part of the proof of the implications (1)$\Longrightarrow$(2) and (3)$\Longrightarrow$(1) in [dS] (it explicitly done in [dSO]).

The characterization given by Axiom (4) can be restated as an *Euclidean Representation Theorem for oriented matroids* in the following way:

Denote by $C(E)$ the hypercube $C(E) := [-1,+1]^E \subset I\!R^E$ and identify $\{+,-\}^E$ with the set $\{-1,+1\}^E$, the set of vertices of the cube $C(E)$.

For every $\mathcal{T} \subseteq \{-1,1\}^E$ and $X \in \{-1,1\}^A$ with $A \subseteq E$,

$$\mathcal{T}_X := \{T \in \mathcal{T} : X \preceq T\}$$

corresponds to the intersection, $\mathcal{T} \cap F$, of $\mathcal{T}$ that with the face $F_X = X \circ C(E \setminus A)$ of $C(E)$ and $\mathcal{T} \setminus A$ to the orthogonal projection of $\mathcal{T}$ into the face F_X.

Then,

Theorem 1.' (Euclidean representation Theorem for Oriented Matroids)

A subset $\mathcal{T} \subset \{-1,1\}^E$ of vertices of the the cube $C(E)$ is the set of topes of an oriented matroid (without loops) if and only if $\mathcal{T}$ satisfyes the following two properties:
(T'1) *$\mathcal{T}$ is symmetrical with respect to the origin of $I\!R^E$.*
(T'2) *Whenever the intersection, $\mathcal{T} \cap F$, of $\mathcal{T}$ with a face F of the cube $C(E)$ is symmetrical with respect to the center of the face F, then $\mathcal{T} \cap F$ coincides with the orthogonal projection of $\mathcal{T}$ into F.*

For uniform oriented matroids we get the following interpretation of lopsidness:

Corollary 1.' (Uniform oriented matroids)

A subset $\mathcal{T} \subset \{-1,1\}^E$ of vertices of the the cube $C(E)$ is the set of topes of a uniform oriented matroid if and only if $\mathcal{T}$ satisfyes the following two properties:
(T'1) *$\mathcal{T}$ is symmetrical with respect to the origin of $I\!R^E$.*
(T'2) *Whenever the intersection, $\mathcal{T} \cap F$, of $\mathcal{T}$ with a face F of the cube $C(E)$ is symmetrical with respect to the center of the face F then $\mathcal{T} \cap F$ is the set of all vertices of F.*

Such as oriented matroids are combinatorial abstractions of arrangements of linear hyperplanes in $I\!R^d$, affine oriented matroids are combinatorial abstractions of arrangements of affine hyperplanes in $I\!R^d$. Theorem 1 together with the following Proposition characterizes affine oriented matroids in terms of topes:

Proposition 2. (Affine Oriented Matroids)

For a subset $\mathcal{W} \subseteq \{+,-,0\}^E$ the following two conditions are equivalent:
(1) *$\mathcal{W}$ is the set of topes of an affine oriented matroid.*
(2) *$\mathcal{T} := g^+ \circ \mathcal{W} \cup g^- \circ \mathcal{W} \subseteq \{+,-,0\}^{E \cup g}$ is the set of topes of an oriented matroid over $E \cup g$.*

Connected with this result is the following natural generalization of one the problems proposed by J. Lawrence at the end of his paper [L]:

Problem: *Suppose that $\mathcal{W} \subseteq \{+,-\}^E$ satisfies property (T4) of Theorem 1. Can we conclude that $\mathcal{W}$ is a supertope of some oriented matroid on a set E' such that $E \subseteq E'$?*

Finally we would like to mention that Theorem 1' characterizing oriented matroids in terms of subsets of vertices of a cube satisfying certain symmetry properties can be regarded as a *Zonotopal Representation Theorem for Oriented Matroids*. For that recall that zonotopes are those convex polytopes whose faces are centrally symmetric and also that the face lattice of a zonotopes is the dual of the lattice of closed faces of a decomposition of $I\!R^d$ by an arrangement of hypeperlanes. This point of view is further developped in [dS-M].

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Ilda P. F. da Silva
CMAF
Universidade de Lisboa
Av. Prof. Gama Pinto 2
1699 Lisboa Codex
Portugal
Email: isilva@alf1.cii.fc.ul.pt