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Geodesic Spheres, Tubes and Flow Space Forms

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1 Introduction

Real, complex, quaternionic and Sasakian space forms are the simplest examples of Riemannian manifolds in the framework of Riemannian, complex, quaternionic and contact geometry. Their geometric properties have been studied extensively and a lot of characterizations are well-known. At many occasions they are considered as model spaces for comparison purposes.

Furthermore, Sasakian manifolds are endowed with a unit Killing vector field. (See [B] for more details.) We have studied the geometry of Riemannian manifolds equipped with such a vector field, generalizing in this way many aspects of Sasakian geometry to what we called *flow geometry*. In particular, we have treated the notion of a flow space form, or more specifically, that of a normal flow space form, in the framework of flow geometry. In particular, we have studied the curvature, the classification and some of the properties of these spaces in [5], where one can see that although there are many similarities with Sasakian geometry, there are also some remarkable differences. For example, there exist normal flow space forms which do not have an analogue in Sasakian geometry.

As is well-known, there is a strong relationship between the properties of the curvature of the Riemannian manifold and the properties of the extrinsic and intrinsic geometry of geodesic spheres and tubes. This has been illustrated extensively in [2], [8], [13]. In this paper, we give a short survey about the main results of our research. In particular, we use extrinsic geometric properties of geodesic spheres and tubes about geodesics to characterize normal flow space forms. This has been treated in full detail in [6], [7]. Our method is based on the explicit determination of the Jacobi vector fields on normal flow space forms.

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2 Normal flow space forms

Let (M, g) be an n-dimensional, smooth, connected Riemannian manifold with $n \geq 2$. Let ∇ denote the Levi Civita connection of (M, g) and R the corresponding Riemannian curvature tensor given by $R_{UV} = \nabla_{[U,V]} - [\nabla_U, \nabla_V]$ for $U, V \in \mathfrak{X}(M)$, the Lie algebra of smooth vector fields on M.

Further, let $\mathfrak{F}_\xi$ denote an *isometric flow* [11] on (M, g) generated by a *unit* Killing vector field ξ. In this case, the flow lines of $\mathfrak{F}_\xi$ are geodesics. A vector which is orthogonal to ξ is called a *transversal* or *horizontal* vector and a geodesic which is orthogonal to ξ is called a transversal or horizontal geodesic.

$\mathfrak{F}_\xi$ determines locally a Riemannian submersion. So, one can use the O'Neill's integrability tensors. Specifically, we put $HU = -\nabla_U \xi$ and $h(U, V) = g(HU, V)$ for all $U, V \in \mathfrak{X}(M)$. Since ξ is a Killing vector field, h is skew-symmetric and moreover, $h = -d\eta$, where η is the dual one-form of ξ with respect to g. Furthermore, the curvature tensor R of (M, g) satisfies

$$R(X, \xi, Y, \xi) = g(HX, HY) = -g(H^2X, Y)$$

for all horizontal vector fields X, Y. Here, we use the notation $R(X, Y, Z, W) = g(R_{XY}Z, W)$. It follows that the ξ-*sectional curvature* $K(X, \xi)$ of the two-plane spanned by X and ξ is non-negative for all horizontal X. Moreover, $K(X, \xi)$ is strictly positive for all X if and only if H is of maximal rank $n - 1$ in which case n is necessarily odd. Then η is a contact form and the flow $\mathfrak{F}_\xi$ is called a *contact flow*.

$\mathfrak{F}_\xi$ is called a *normal flow* [3] if, for all horizontal X, Y, the curvature transformations R_{XY} leave the horizontal subspaces invariant. This is equivalent to the condition $R(X, Y, X, \xi) = 0$. Here, we note that a Sasakian manifold is a Riemannian manifold equipped with a normal flow $\mathfrak{F}_\xi$ such that $K(X, \xi) = 1$ for all horizontal X.

Next, we collect some facts about a remarkable subclass of such spaces which generalizes those of (locally) φ-symmetric spaces from Sasakian geometry [10] and (locally) Hermitian symmetric spaces from complex geometry. Let $m \in (M, g)$ and denote by $\sigma = \sigma_m : [-\delta, \delta] \to M$ a geodesic flow line through $m = \sigma(0)$ where δ is sufficiently small. A local diffeomorphism s_m of M defined in a neighborhood $\mathcal{U}$ of m is said to be a (*local*) *reflection with respect to* σ if for every transversal geodesic $\gamma(s)$, where $\gamma(0)$ lies in the intersection of $\mathcal{U}$ and σ, we have $(s_m \circ \gamma)(s) = \gamma(-s)$ for all s with $\gamma(\pm s) \in \mathcal{U}$, s being the arc length of γ. A Riemannian manifold (M, g) equipped with a flow $\mathfrak{F}_\xi$ such that each local reflection s_m is an isometry, is called a *locally Killing-transversally symmetric space* (briefly, a *locally* KTS-*space*). These spaces may be characterized as follows.

Proposition 1 [3] *$(M, g, \mathfrak{F}_\xi)$ is a locally* KTS*-space if and only if the flow $\mathfrak{F}_\xi$ is normal and moreover,*

$$(\nabla_X R)(X, Y, X, Y) = 0$$

for all horizontal X, Y.

Proposition 2 [3] *Let $\mathfrak{F}_\xi$ be a normal flow on (M, g). Then $(M, g, \mathfrak{F}_\xi)$ is a locally* KTS*-space if and only if each base space $\tilde{\mathcal{U}}$ of a local Riemannian submersion $\pi: \mathcal{U} \to \tilde{\mathcal{U}} = \mathcal{U}/\xi$ is a locally symmetric space.*

So, $(M, g, \mathfrak{F}_\xi)$ is a locally KTS-space if and only if $\mathfrak{F}_\xi$ is a normal flow which is transversally modelled on a locally symmetric space or equivalently, according to the terminology used in [12], $\mathfrak{F}_\xi$ is a normal transversally symmetric foliation. Note that a locally KTS-space is a locally homogeneous space.

For the contact case, we mention the following useful result.

Proposition 3 [4] *Any complete contact locally* KTS*-space $(M, g, \mathfrak{F}_\xi)$ is transversally modelled on a (global) Hermitian symmetric space $\tilde{M}$. Moreover, if $\tilde{M} = \tilde{M}_0 \times \tilde{M}_1 \times \ldots \times \tilde{M}_r$ is its de Rham decomposition, where $\tilde{M}_0 = \mathbf{C}^p$, then there exist $r + p$ real numbers $c_1, \ldots, c_r, \mu_1, \ldots, \mu_p$ such that on each distinguished chart $\mathcal{U} \subset M$, the smooth distributions $\mathcal{H}_i$, $i = 0, 1, \ldots, r$, obtained by the horizontal lifts of the tangent vectors of $\tilde{M}_i$ verify*

(i) *$\mathcal{H}(m) = \mathcal{H}_0(m) \oplus \mathcal{H}_1(m) \oplus \ldots \oplus \mathcal{H}_r(m)$ is an H-invariant orthogonal decomposition of the horizontal subspace $\mathcal{H}(m)$, for each $m \in \mathcal{U}$;*

(ii) *each sectional curvature $K(\mathcal{H}_j, \xi)$, $j = 1, \ldots, r$, is a positive constant equal to c_j^2.*

Finally, we turn to the consideration of the normal flow space forms. A plane section in T_mM, $m \in M$, is called an *H-section* if there exists a horizontal X in T_mM such that $\{X, HX\}$ is a basis of the plane section. The sectional curvature $K(X, HX)$ of an H-section is called the *H-sectional curvature* corresponding to X. In [5], it is proved that the H- and ξ- sectional curvatures on a complete contact locally KTS-space $(M, g, \mathfrak{F}_\xi)$ determine completely its curvature. A Riemannian manifold (M, g) equipped with a contact flow $\mathfrak{F}_\xi$ is said to be a *flow space form* if the H-sectional curvature is pointwise constant, that is, the sectional curvature of the two-plane spanned by X and HX, for horizontal $X \in T_mM$, is independent of X for each point $m \in M$. Moreover, a *normal* flow space form has globally constant H-sectional curvature if and only if it is a locally KTS-space.

Normal flow space forms have been studied in [5] where two cases are considered according to whether the ξ-sectional curvature is constant or not. When the ξ-sectional curvature is constant (case 1), each normal flow space form is obtained by a homothetic change of metric from a Sasakian space form. Hence, we have

Theorem 4 [5] *A Riemannian manifold (M, g) of dimension ≥ 5 which is equipped with a normal flow $\mathfrak{F}_\xi$ such that the ξ-sectional curvature is a non-vanishing constant, is a flow space form if and only if for any horizontal X, $R_{XHX}X$ is proportional to HX.*

In what follows we shall denote such spaces by $M(c^2, k)$ where c^2 denotes the constant ξ-sectional curvature and k the constant H-sectional curvature.

The situation is totally different for the case of normal flow space forms with non-constant ξ-sectional curvature and globally constant H-sectional curvature. This is the case 2. In fact, we have

Theorem 5 [5] *Let $\mathfrak{F}_\xi$ be a normal contact flow on a complete $(2n+1)$-dimensional Riemannian manifold (M, g) with non-constant ξ-sectional curvature. Then the H-sectional curvature is globally constant if and only if $\mathfrak{F}_\xi$ is transversally modelled on $\mathbf{C}P^{n_1}(h_1) \times \mathbf{C}H^{n_2}(h_2)$ where $|h_2| < h_1$, $n_1 + n_2 = n$ and the ξ-sectional curvatures $c_i^2 = K(X_i, \xi)$, $i = 1, 2$, satisfy*

$$c_i^2 = (-1)^{i+1} h_i \frac{h_1 - h_2}{3(h_1 + h_2)}.$$

Further, the H-sectional curvature k is a strictly negative constant satisfying

$$k = 2\frac{h_1 h_2}{h_1 + h_2} = -6\frac{c_1^2 c_2^2}{c_1^2 + c_2^2}$$

and the curvature tensor is given by

$$\begin{aligned}
R_{UV}W = \sum_{i=1}^{2} \Big\{ & \frac{h_i}{4}\Big(g(X_i, Z_i)Y_i - g(Y_i, Z_i)X_i\Big) \\
& +(-1)^i \frac{3(h_1 + h_2)}{4(h_1 - h_2)}\Big(g(HY_i, Z_i)HX_i - g(HX_i, Z_i)HY_i - 2g(HX_i, Y_i)HZ_i\Big) \\
& +(-1)^i \frac{h_i(h_1 - h_2)}{3(h_1 + h_2)}\Big\{\Big(g(Y_i, Z_i)\eta(U) - g(X_i, Z_i)\eta(V)\Big)\xi \\
& \quad +\eta(W)\Big(\eta(V)X_i - \eta(U)Y_i\Big)\Big\}\Big\} \\
& +g(HV, W)HU - g(HU, W)HV - 2g(HU, V)HW
\end{aligned}$$

for vector fields $U = \sum_{i=1}^{2} X_i + \eta(U)\xi, V = \sum_{i=1}^{2} Y_i + \eta(V)\xi, W = \sum_{i=1}^{2} Z_i + \eta(W)\xi$ on M.

Such flow space forms will be denoted by $M(n_1, n_2; h_1, h_2)$, since these parameters determine exactly this normal flow space form. Furthermore, we have completely classified these spaces when the H-sectional curvature is globally constant and the manifold is complete and simply connected. More specifically, we have

Theorem 6 [5] *Let (M, g) be a complete, simply connected $(2n+1)$-dimensional Riemannian manifold equipped with a normal flow such that the H-sectional curvature is globally constant and the ξ-sectional curvature is non-constant. Then (M, g) is the universal covering of the quotient space*

$$\frac{S^{2n_1+1} \times \left(\dfrac{SU(1,n_2)^{\sim}}{SU(n_2)} / \mathbf{Z}[l]\right)}{S^1}$$

where n_1 and n_2 are integers, $n = n_1 + n_2$, and the length of the fibers l is given by

$$l = 8\pi \left(\frac{h_1 - h_2}{3h_1(h_1 + h_2)} \right)^{1/2},$$

where h_1, h_2 are real numbers with $|h_2| < h_1$.

Next, we give a useful characterization of normal flow space forms of this second kind.

Theorem 7 [6] *Let $(M, g, \mathfrak{F}_\xi)$ be a complete contact locally* KTS*-space such that at each point $m \in M$ the self-adjoint operator $R_{\xi\cdot}\xi$ (or equivalently, $-H_m^2$) has two eigenspaces $\mathcal{V}_1(m)$ and $\mathcal{V}_2(m)$ with $\mathcal{H}(m) = \mathcal{V}_1(m) \oplus \mathcal{V}_2(m)$ and where* $\dim \mathcal{V}_i(m) = 2n_i \geq 4$, $i = 1, 2$. *Then $(M, g, \mathfrak{F}_\xi)$ is a flow space form (with non-constant ξ-sectional curvature) if and only if*

$$R_{X_1 H X_1} X_1 + R_{X_2 H X_2} X_2 \ \text{ and } \ R_{X H^\perp X} X$$

are proportional to HX for all $X = X_1 + X_2 \in \mathcal{V}_1(m) \oplus \mathcal{V}_2(m)$ and where $H^\perp X$ denotes a vector orthogonal to HX in the plane spanned by HX_1 and HX_2 when $\|X_1\| \|X_2\| \neq 0$.

3 The shape operator of geodesic spheres

Now, we characterize normal flow space forms by using properties of the shape operator of their small geodesic spheres.

Let m be a point in a $(2n+1)$-dimensional Riemannian manifold (M, g), γ a geodesic such that $\gamma(0) = m$ and $\gamma'(0) = u$ where u is a unit vector. Further, let $\{e_1, \ldots, e_{2n+1}\}$ be an orthonormal basis of $T_m M$ with $u = e_1$ and let

$\{E_1, \ldots, E_{2n+1}\}$ be its parallel translated basis along γ. Next, let X_a, $a = 2, \ldots, 2n+1$, be the $2n$ Jacobi vector fields with initial conditions

$$X_a(0) = 0 \quad , \quad X_a'(0) = e_a \quad , \quad a = 2, \ldots, 2n+1$$

and put

$$X_a = AE_a = \sum_{b=2}^{2n+1} A_a^b E_b.$$

Let $G_m(s)$ denote the geodesic sphere with center $m \in M$ and radius s, contained in a normal neighborhood of m. The shape operator S_m of $G_m(s)$ is given by

$$S_m(p)X = (\nabla_X \gamma')(p)$$

where $p = \gamma(s) = \exp_m(su)$ and $X \in T_pG_m(s)$. Then we have

$$S_m(p) = A'A^{-1}(s) \tag{3.1}$$

and S_m satisfies the Riccati equation

$$S_m' + S_m^2 + R_\gamma = 0. \tag{3.2}$$

See [8], [13] for more details.

Using (3.1) and the expressions of the Jacobi vector fields obtained for normal flow space forms, we get an explicit expression for the shape operator of a small geodesic sphere in these spaces [6]. More precisely, for normal flow space forms of case 1, that is, with constant ξ-sectional curvature c^2, when we take the orthonormal basis $\{e_1, \ldots, e_{2n+1}\}$ with $u = e_1$, $\|Hu\|^{-1}Hu = e_{2n}$, $\xi = e_{2n+1}$, then the shape operator admits the following matrix expression with respect to the parallel translated basis $\{E_1, \ldots, E_{2n+1}\}$:

$$S_m(\exp_m(su)) = \begin{pmatrix} fI_{2n-1} & 0 & 0 \\ 0 & S_{2n2n} & S_{2n2n+1} \\ 0 & S_{2n+12n} & S_{2n+12n+1} \end{pmatrix}$$

where

$$f = \begin{cases} \dfrac{\sqrt{h}}{2} \cot \dfrac{\sqrt{h}}{2} s & , \quad h > 0, \\ \dfrac{1}{s} & , \quad h = 0, \\ \dfrac{\sqrt{-h}}{2} \coth \dfrac{\sqrt{-h}}{2} s & , \quad h < 0 \end{cases}$$

and S_{2n2n}, $S_{2n2n+1} = S_{2n+12n}$, $S_{2n+12n+1}$ are functions depending only on s.

Then, from (3.2) and Theorem 4 we have may deduce

Theorem 8 [6] *A Riemannian manifold* (M, g) *with* $\dim M \geq 5$ *and equipped with a normal flow* $\mathfrak{F}_\xi$ *such that the* ξ*-sectional curvature is a non-vanishing constant, is a flow space form if and only if, for any horizontal* γ, $S_m H\gamma'$ *(or* $S_m\xi$*, respectively) belongs to the plane* $\{\xi, H\gamma'\}$ *for all* $m \in M$ *and all sufficiently small* s.

For the more general situation of locally KTS-spaces, properties of the small geodesic spheres may also be used to decide whether the ξ-sectional curvature is constant or not. In fact, using Jacobi vector fields, we have proved in [6] the following

Theorem 9 *A complete contact locally* KTS*-space* $(M, g, \mathfrak{F}_\xi)$ *has constant* ξ*-sectional curvature if and only if there exists a point* $m \in M$ *such that* $G_m(s)$ *is umbilical at its intersection points with the flow line through* m.

Next, we consider $(M, g, \mathfrak{F}_\xi) = M(n_1, n_2; h_1, h_2)$, that is, a normal flow space form of the second kind. Then each horizontal subspace $\mathcal{H}(m)$ is decomposed into two H-invariant factors $\mathcal{H}_1(m)$ and $\mathcal{H}_2(m)$. Again, taking a suitable orthonormal basis $\{e_1, \ldots, e_{2n+1}\}$ when u belongs to $\mathcal{H}_1$ or to $\mathcal{H}_2$, we obtain the following matrix expressions for the shape operator at $\exp_m(su)$ with respect to the corresponding parallel translated basis along γ [6]:

A. $u \in \mathcal{H}_1$

$$S_m(\exp_m(su)) = \begin{pmatrix} \frac{\sqrt{h_1}}{2}\cot\frac{\sqrt{h_1}}{2}sI_{2n_1-2} & 0 & 0 \\ 0 & \frac{1}{s}I_{2n_2} & 0 \\ 0 & 0 & B \end{pmatrix};$$

B. $u \in \mathcal{H}_2$

$$S_m(\exp_m(su)) = \begin{pmatrix} \frac{1}{s}I_{2n_1} & 0 & 0 \\ 0 & \frac{\sqrt{-h_2}}{2}\coth\frac{\sqrt{-h_2}}{2}sI_{2n_2-2} & 0 \\ 0 & 0 & B \end{pmatrix}$$

where in both cases $B = (B_{ab})$ is a symmetric 2×2 matrix which only depends on s.

Furthermore, when u has non-vanishing projections on $\mathcal{H}_1$ and $\mathcal{H}_2$, we put

$$\begin{cases} u = e_1 \ , \ \ \|Hu\|^{-1}Hu = e_{2n} \ , \ \ \xi = e_{2n+1}, \\ \|u_1\|^{-1}\|u_2\|u_1 - \|u_2\|^{-1}\|u_1\|u_2 = u^\perp = e_{2n-2}, \\ \|Hu\|^{-1}(\|Hu_1\|^{-1}\|Hu_2\|Hu_1 - \|Hu_2\|^{-1}\|Hu_1\|Hu_2) = \|Hu\|^{-1}H^\perp u = e_{2n-1} \end{cases}$$

where $\{e_2, \ldots, e_{2n_1-1}\} \subset \mathcal{H}_1(m)$ and $\{e_{2n_1}, \ldots, e_{2n-3}\} \subset \mathcal{H}_2(m)$. In this case the shape operator at $\exp_m(su)$ admits the following expression:

C. $u = u_1 + u_2 \in \mathcal{H}_1 \oplus \mathcal{H}_2$

$$S_m(\exp_m(su)) = \begin{pmatrix} f_1 I_{2n_1-2} & 0 & 0 & 0 \\ 0 & f_2 I_{2n_2-2} & 0 & 0 \\ 0 & 0 & \frac{1}{s} & 0 \\ 0 & 0 & 0 & C \end{pmatrix}$$

where

$$f_1 = \frac{\|u_1\|\sqrt{h_1}}{2} \cot \frac{\|u_1\|\sqrt{h_1}}{2} s \quad , \quad f_2 = \frac{\|u_2\|\sqrt{-h_2}}{2} \coth \frac{\|u_2\|\sqrt{-h_2}}{2} s$$

and $C = (C_{ab})$ is a symmetric 3×3 matrix with entries only depending on s.

Hence, considering the cases A, B and C and applying (3.2) and Theorem 7, we may prove the following characterization for the case 2.

Theorem 10 [6] *Let $(M, g, \mathfrak{F}_\xi)$ be a complete contact locally* KTS*-space such that the shape opertor of $G_m(s)$ at $p = \exp_m(s\xi)$, for all $m \in M$ and all sufficiently small s, has two eigenspaces $\mathcal{V}_1$ and $\mathcal{V}_2$ with $T_pG_m(s) = \mathcal{V}_1 \oplus \mathcal{V}_2$ and* $\dim \mathcal{V}_i = 2n_i \geq 4$, $i = 1, 2$. *Then $(M, g, \mathfrak{F}_\xi)$ is a flow space form (with non-constant ξ-sectional curvature and globally constant H-sectional curvature) if and only if the following two conditions are satisfied:*

(i) *$S_m(\exp_m(su_i))$ and $S_m(\exp_m(su))$ preserve the vector spaces spanned by $\{\xi, Hu_i\}$ and $\{H^\perp u, Hu, \xi\}$, respectively;*

(ii) *if $M^i = (M^i_{jk})_{1 \leq j,k \leq 2}$ and $N = (N_{lm})_{1 \leq l,m \leq 3}$ denote the corresponding matrices associated to these endomorphisms with respect to the orthonormal bases $\{\|Hu_i\|^{-1}Hu_i, \xi\}$ and $\{\|Hu\|^{-1}H^\perp u, \|Hu\|^{-1}Hu, \xi\}$, respectively, then we have*

$$(M^1_{11})^2 + (M^1_{12})^2 + {M^1_{11}}' - 2\|Hu_1\|M^1_{12} = (M^2_{11})^2 + (M^2_{12})^2 + {M^2_{11}}' - 2\|Hu_2\|M^2_{12},$$

$$\sum_{l=1}^{3} (N_{1l})^2 + N'_{11} = 0$$

for all unit vectors $u_i \in \mathcal{V}_i$ and all unit $u \in \mathcal{V}_1 \oplus \mathcal{V}_2$, $u \notin \mathcal{V}_1$ or $\mathcal{V}_2$.

4 The shape operator of geodesic tubes

Now, we turn the consideration of tubes about geodesics and first recall some basic facts about their treatment. We refer to [8], [9], [13] for more details and further references.

Let $\sigma : [a,b] \to M$ be a smooth embedded curve in (M,g) and denote by $\mathcal{P}_\sigma(s)$ the tubular hypersurface or just the tube of radius s about σ, where s is sufficiently small. In what follows we suppose that σ is a unit speed geodesic. Then $\mathcal{P}_\sigma$ is called a *geodesic tube* about σ.

Fermi coordinate systems are well-adapted to treat the geometry of these tubes. They are defined as follows. Let $\sigma : [a,b] \to M$ be such a geodesic and let $\{e_1 = \sigma'(a), e_2, \ldots, e_n\}$ be an orthonormal basis of $T_{\sigma(a)}M$. Further, let $E_1 = \sigma'$ and let $E_2, \ldots, E_n$ be the normal vector fields along σ which are parallel with respect to the Levi Civita connection ∇ and such that $E_i(a) = e_i$, $i = 2, \ldots, n$. Then the Fermi coordinates $(x_1, \ldots, x_n)$, with respect to $\sigma(a)$ and the frame field $\{E_1, \ldots, E_n\}$, are defined by

$$x_1\Big(\exp_{\sigma(t)} \sum_{j=2}^{n} t_j E_j\Big) = t - a \ , \ x_i\Big(\exp_{\sigma(t)} \sum_{j=2}^{n} t_j E_j\Big) = t_i, \quad i = 2, \ldots, n.$$

Next, let $p = \exp_{\sigma(t)}(sv)$, $v \in \sigma(t)^\perp$, $\|v\| = 1$. Then $p \in \mathcal{P}_\sigma(s)$. Further, let γ be the unit speed geodesic connecting $\sigma(t)$ and p and adapt the frame field $\{E_1, \ldots, E_n\}$ along σ such that $E_2(t) = \gamma'(0)$. Moreover, let $\{F_1, \ldots, F_n\}$ be the frame field along γ obtained by parallel translation of $\{E_1(t), \ldots, E_n(t)\}$. Furthermore, let $X_1, X_3, \ldots, X_n$ be the Jacobi vector fields along γ which satisfy the following set of initial conditions:

$$\begin{cases} X_1(0) = E_1(t) \ , \ X_1'(0) = \Big(\nabla_{\gamma'} \dfrac{\partial}{\partial x_1}\Big)(\sigma(t)) = 0, \\ X_i(0) = 0 \ , \ X_i'(0) = E_i(t) \ , \quad i = 3, \ldots, n. \end{cases}$$

Then we have

$$X_1(s) = \frac{\partial}{\partial x_1}(\gamma(s)) \ , \ X_i(s) = s\frac{\partial}{\partial x_i}(\gamma(s)) \ , \ i = 3, \ldots, n.$$

Now, define the endomorphism-valued function $B : s \mapsto B(s)$ by

$$X_i(s) = BF_i(s) \ , \ i = 1, 3, \ldots, n.$$

Then this function satisfies the Jacobi equation $B'' + R_\gamma \circ B = 0$, where $R_\gamma = R_{\gamma' .}\gamma'$, and the shape operator S_σ of $\mathcal{P}_\sigma(s)$ may be expressed in terms of B by

$$S_\sigma(p) = (B'B^{-1})(s). \tag{4.1}$$

Moreover, by differentiation along γ, we obtain that S_σ satisfies the following Riccati type differential equation:

$$S'_\sigma + S^2_\sigma + R_\gamma = 0. \tag{4.2}$$

4.1 Tubes about flow lines

Let $(M, g, \mathfrak{F}_\xi)$ be a Riemannian manifold equipped with a normal flow $\mathfrak{F}_\xi$ and let σ be a segment of the flow line through $m \in M$. Further, let $p = \exp_m(su)$ where u is a unit horizontal vector and denote by $\kappa_\sigma(p)$ the normal curvature of the geodesic of $\mathcal{P}_\sigma(s)$ tangent to $v = \|Hu\|^{-1}Hu$ at p, that is,

$$\kappa_\sigma(p) = g(S_\sigma(p)v, v).$$

Here, κ_σ is a real-valued function on each $\mathcal{P}_\sigma(s)$.

Theorem 11 [7] *A Riemannian manifold (M, g) with* $\dim M = 2n + 1 \geq 5$ *and equipped with a normal flow $\mathfrak{F}_\xi$ such that the ξ-sectional curvature is a non-vanishing constant, is a flow space form if and only if for any horizontal u, $S_\sigma(p)Hu$ belongs to the plane $\{\xi, Hu\}$ for all m and all small tubes $\mathcal{P}_\sigma(s)$ about the flow line through m. Moreover, $h = -(\kappa^2_\sigma + \kappa'_\sigma)$ is constant on every tube $\mathcal{P}_\sigma(s)$ and $k = h - 3c^2$ is the constant H-sectional curvature.*

Next, let $(M, g, \mathfrak{F}_\xi) = M(n_1, n_2; h_1, h_2)$ be a complete normal flow space form with non-constant ξ-sectional curvature and globally constant H-sectional curvature. Using the solutions of the Jacobi equation on normal flow space forms given in [6], we obtain the shape operator of $\mathcal{P}_\sigma(s)$ by applying (4.1) (see [7]).

A. $u = u_1 \in \mathcal{H}_1$

Let $\{e_1, \ldots, e_{2n+1}\}$ be an orthonormal basis at m such that $e_1 = \xi$, $e_2 = u_1$, $e_3 = \|Hu_1\|^{-1}Hu_1$ and where $\{e_2, \ldots, e_{2n_1+1}\}$ and $\{e_{2n_1+2}, \ldots, e_{2n+1}\}$ are orthonormal bases of $\mathcal{H}_1(m)$ and $\mathcal{H}_2(m)$, respectively. With respect to the parallel basis $\{F_1, F_3, \ldots, F_{2n+1}\}$, $S_\sigma(\exp_m(su_1))$ has the matrix form

$$S_\sigma(\exp_m(su_1)) = \begin{pmatrix} A & 0 & 0 \\ 0 & \frac{\sqrt{h_1}}{2}\cot\frac{\sqrt{h_1}}{2}sI_{2n_1-2} & 0 \\ 0 & 0 & \frac{1}{s}I_{2n_2} \end{pmatrix};$$

B. $u = u_2 \in \mathcal{H}_2$

Taking now $\{e_1, \ldots, e_{2n+1}\}$ such that $e_1 = \xi$, $e_2 = u_2$, $e_3 = \|Hu_2\|^{-1}Hu_2$ and with $\{e_4, \ldots, e_{2n_1+3}\}$, $\{e_2, e_3, e_{2n_1+4}, \ldots, e_{2n+1}\}$ as bases of $\mathcal{H}_1(m)$ and $\mathcal{H}_2(m)$, respectively, we have

$$S_\sigma(\exp_m(su_2)) = \begin{pmatrix} A & 0 & 0 \\ 0 & \frac{1}{s}I_{2n_1} & 0 \\ 0 & 0 & \frac{\sqrt{-h_2}}{2}\coth\frac{\sqrt{-h_2}}{2}sI_{2n_2-2} \end{pmatrix}$$

where in both cases A is a symmetric 2×2 matrix.

C. $u = u_1 + u_2 \in \mathcal{H}_1 \oplus \mathcal{H}_2$

In this case, we put

$$e_1 = \xi \;\;,\;\; e_2 = u \;\;,\;\; e_3 = \|Hu\|^{-1}Hu,$$
$$e_4 = \|Hu\|^{-1}H^\perp u \;\;,\;\; e_5 = \|u_1\|^{-1}\|u_2\|u_1 - \|u_2\|^{-1}\|u_1\|u_2,$$

$\{e_6, \ldots, e_{2n_1+3}\} \subset \mathcal{H}_1(m)$ and $\{e_{2n_1+4}, \ldots, e_{2n+1}\} \subset \mathcal{H}_2(m)$. Then, with respect to the parallel frame field $\{F_1, F_3, \ldots, F_{2n+1}\}$, $S_\sigma(\exp_m(su))$ is of the form

$$S_\sigma(\exp_m(su)) = \begin{pmatrix} B & 0 & 0 & 0 \\ 0 & \frac{1}{s} & 0 & 0 \\ 0 & 0 & f_1 I_{2n_1-2} & 0 \\ 0 & 0 & 0 & f_2 I_{2n_2-2} \end{pmatrix}$$

where $f_1 = \frac{\|u_1\|\sqrt{h_1}}{2}\cot\frac{\|u_1\|\sqrt{h_1}}{2}s$, $f_2 = \frac{\|u_2\|\sqrt{-h_2}}{2}\coth\frac{\|u_2\|\sqrt{-h_2}}{2}s$ and B is a symmetric 3×3 matrix.

Now, we define on each tube $\mathcal{P}_\sigma(s)$ the real-valued function k by

$$k(p) = -\Big(3\|Hu\|^2 + \kappa_\sigma^2 + \kappa'_\sigma\Big)(p) \quad , \quad p = \exp_m(su).$$

Then, using the expressions for the shape operator, (4.2) and Theorem 7, we get

Theorem 12 [7] *Let $(M, g, \mathfrak{F}_\xi)$ be a complete, contact locally* KTS-*space such that at each point m in M the self-adjoint operator $R_{\xi}.\xi$ (or equivalently, $-H_m^2$) has two eigenspaces $\mathcal{V}_1(m)$ and $\mathcal{V}_2(m)$ with $\mathcal{H}(m) = \mathcal{V}_1(m) \oplus \mathcal{V}_2(m)$ and where* $\dim \mathcal{V}_i(m) = 2n_i \geq 4$, $i = 1, 2$. *Then $(M, g, \mathfrak{F}_\xi)$ is a flow space form (with non-constant ξ-sectional curvature and globally constant H-sectional curvature) if and only if, for all unit vectors $u_i \in \mathcal{V}_i$ and $u \in \mathcal{V}_1 \oplus \mathcal{V}_2$, the shape operator S_σ of any small tube $\mathcal{P}_\sigma$ about the flow lines σ satisfies:*

(i) $S_\sigma(\exp_m(su_i))Hu_i \in \{\xi, Hu_i\}$ *and* $S_\sigma(\exp_m(su))$ *preserves the vector space spanned by* $\{\xi, Hu, H^\perp u\}$;

(ii) $k(\exp_m(su_1)) = k(\exp_m(su_2))$ *and* $N'_{33} + N^2_{23} + N^2_{33} = 0$ *where* N_{ij}, $1 \leq i,j \leq 3$, *are the entries of the matrix associated to* $S_\sigma(\exp_m(su))$ *with respect to the orthonormal basis* $\{\xi, \|Hu\|^{-1}Hu, \|Hu\|^{-1}H^\perp u\}$.

4.2 Tubes about horizontal geodesics

Now, we consider small tubes $\mathcal{P}_\sigma(s)$ about a horizontal geodesic σ and put $\sigma' = u$ where u is a unit vector. Next, let γ be the unit speed geodesic of M meeting σ orthogonally at $\gamma(0) = m$ and tangent to the horizontal vector v with $v(m) = \|Hu\|^{-1}Hu(m)$.

First, we recall the following theorem which generalizes the similar one for Sasakian space forms.

Theorem 13 [7] *Let* (M, g) *be a Riemannian manifold with* $\dim M = 2n+1 \geq 5$ *and equipped with a normal flow* $\mathfrak{F}_\xi$ *such that the* ξ*-sectional curvature is a non-vanishing constant. Then, with the conventions made above,* M *is a flow space form if and only if for all horizontal geodesics* σ*, the shape operator* S_σ *along* γ *preserves the plane spanned by* $\{\xi, Hv\}$.

Next, we consider complete normal flow space forms of the second kind. Let σ_i, $i = 1, 2$, or σ be the horizontal geodesics through $m \in M$ tangent to the unit vector $u_i \in \mathcal{H}_i(m)$ or $u = u_1 + u_2 \in \mathcal{H}_1(m) \oplus \mathcal{H}_2(m)$, respectively. Further, let γ_i, $i = 1, 2$, or γ denote the unit speed geodesics of M meeting σ_i or σ orthogonally at $m = \sigma(t)$ and which are tangent to v_i or v, respectively, where $v_i(m) = \|Hu_i\|^{-1}Hu_i(m)$ and

$$v(m) = \Big(\|Hu_1\|^{-1}\cos\theta Hu_1 + \|Hu_2\|^{-1}\sin\theta Hu_2\Big)(m)$$

with $\theta = \arctan(-\|Hu_1\|/\|Hu_2\|)$. Then $H^\perp v$ is proportional to u at m. Using again the explicit formulas given in [6], we obtain the shape operator of the tubes $\mathcal{P}_\sigma(s)$ at the points $\gamma_i(s)$ or $\gamma(s)$. We have

A. $u = u_1 \in \mathcal{H}_1$

Let $\{e_1, \ldots, e_{2n+1}\}$ be an orthonormal basis at m such that $e_1 = u_1$, $e_2 = v_1$, $e_3 = \xi$ and where $\{e_1, e_2, e_4, \ldots, e_{2n_1+1}\}$ and $\{e_{2n_1+2}, \ldots, e_{2n+1}\}$ are bases of $\mathcal{H}_1(m)$ and $\mathcal{H}_2(m)$, respectively. With respect to the corresponding parallel frame field $\{F_1, F_3, \ldots, F_{2n+1}\}$, we have

$$S_\sigma(\exp_m(sv_1)) = \begin{pmatrix} A & 0 & 0 \\ 0 & \frac{\sqrt{h_1}}{2}\cot\frac{\sqrt{h_1}}{2}sI_{2n_1-2} & 0 \\ 0 & 0 & \frac{1}{s}I_{2n_2} \end{pmatrix};$$

B. $u = u_2 \in \mathcal{H}_2$

Here, we take a basis with $e_1 = u_2, e_2 = v_2, e_3 = \xi$ and such that $\{e_4, \ldots, e_{2n_1+3}\}$ and $\{e_1, e_2, e_{2n_1+4}, \ldots, e_{2n+1}\}$ are bases of $\mathcal{H}_1(m)$ and $\mathcal{H}_2(m)$, respectively. Then we have with respect to the corresponding parallel frame field:

$$S_\sigma(\exp_m(sv_2)) = \begin{pmatrix} A & 0 & 0 \\ 0 & \frac{1}{s}I_{2n_1} & 0 \\ 0 & 0 & \frac{\sqrt{-h_2}}{2}\coth\frac{\sqrt{-h_2}}{2}sI_{2n_2-2} \end{pmatrix}$$

where in both cases A is a symmetric 2×2 matrix.

C. $u = u_1 + u_2 \in \mathcal{H}_1 \oplus \mathcal{H}_2$

Let $\{e_1, \ldots, e_{2n+1}\}$ be an orthonormal basis at m such that

$$e_1 = u \ , \ e_2 = v \ , \ e_3 = \|Hv\|^{-1}Hv,$$
$$e_4 = \xi \ , \ e_5 = \|u_1\|^{-1}\|u_2\|u_1 - \|u_2\|^{-1}\|u_1\|u_2$$

and where $\{e_6, \ldots, e_{2n_1+3}\} \subset \mathcal{H}_1(m)$, $\{e_{2n_1+4}, \ldots, e_{2n+1}\} \subset \mathcal{H}_2(m)$. With respect to the corresponding parallel frame field, S_σ has the form

$$S_\sigma(\exp_m(su)) = \begin{pmatrix} B & 0 & 0 & 0 \\ 0 & \frac{1}{s} & 0 & 0 \\ 0 & 0 & f_1I_{2n_1-2} & 0 \\ 0 & 0 & 0 & f_2I_{2n_2-2} \end{pmatrix}$$

where $f_1 = \frac{\|u_1\|\sqrt{h_1}}{2}\cot\frac{\|u_1\|\sqrt{h_1}}{2}s$, $f_2 = \frac{\|u_2\|\sqrt{-h_2}}{2}\coth\frac{\|u_2\|\sqrt{-h_2}}{2}s$ and B is a symmetric 3×3 matrix.

Then, using Theorem 7 and (4.2), we obtain the following

Theorem 14 [7] *Let $(M, g, \mathfrak{F}_\xi)$ be a complete, contact locally* KTS*-space such that at each point $m \in M$ the self-adjoint operator $R_{\xi .}\xi$ (or equivalently, $-H_m^2$) has two eigenspaces $\mathcal{V}_1(m)$ and $\mathcal{V}_2(m)$ with $\mathcal{H}(m) = \mathcal{V}_1(m) \oplus \mathcal{V}_2(m)$ and where* $\dim\mathcal{V}_i(m) = 2n_i \geq 4$, $i = 1, 2$. *Then $(M, g, \mathfrak{F}_\xi)$ is a flow space form (with non-constant ξ-sectional curvature and globally constant H-sectional curvature) if and only if for all unit speed geodesics σ_i, σ such that $\sigma_i'(0) = u_i \in \mathcal{V}_i(m)$ and $\sigma'(0) = u \in \mathcal{V}_1(m) \oplus \mathcal{V}_2(m)$ and with the conventions as above, the following conditions are satisfied by the shape operators S_{σ_i} and S_σ :*

(i) $S_{\sigma_i}(\exp_m(sv_i))$ *and* $S_\sigma(\exp_m(sv))$ *preserve the vector spaces spanned by* $\{\xi, Hv_i\}$ *and* $\{\xi, Hv, H^\perp v\}$, *respectively;*

(ii) *if* $M^i = (M^i_{jk})_{1\le j,k\le 2}$ *and* $N = (N_{lm})_{1\le l,m\le 3}$ *denote the corresponding matrices associated to these endomorphisms with respect to the orthonormal bases* $\{\xi, \|Hv_i\|^{-1}Hv_i\}$ *and* $\{\xi, \|Hv\|^{-1}Hv, \|Hv\|^{-1}H^{\perp}v\}$, *then we have*

$$(M^1_{12})^2+(M^1_{22})^2+{M^1_{22}}'-2\|Hv_1\|M^1_{12} = (M^2_{12})^2+(M^2_{22})^2+{M^2_{22}}'-2\|Hv_2\|M^2_{12},$$

$$\sum_{l=1}^{3}(N_{l3})^2 + N'_{33} = 0.$$

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