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A new variant of the Schwarz–Pick–Ahlfors Lemma

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Abstract:
We prove a "general shrinking lemma" that resembles the Schwarz–Pick–Ahlfors Lemma and its generalizations, but differs in applying to maps of a finite disk into a disk, rather than requiring the domain of the map to be complete. The conclusion is that distances to the origin are all shrunk, and by a limiting procedure we can recover the original Ahlfors Lemma, that all distances are shrunk. The method of the proof is also different in that relates the shrinking of the Schwarz–Pick–Ahlfors–type lemmas to the comparison theorems of Riemannian geometry.

We start by reviewing the history of Schwarz–type lemmas, with remarks about the effects - some beneficial and some not - of successive generalizations.

There are minor variations in the way the Schwarz lemma is usually stated. Here is one of the standard formulations.

Lemma 1 (The Schwarz Lemma). Let $f(z)$ be analytic on a disk $\mid z \mid< R_1$ and suppose that $\mid f(z) \mid< R_2$ and $f(0) = 0$. Then

$$\mid f(z) \mid\leq \frac{R_1}{R_2} \mid z \mid \quad for \quad \mid z \mid< R_1. \tag{0.1}$$

It is also generally noted that strict inequality holds for every $z \neq 0$ unless f is of the special form

$$f(z) = \frac{R_1}{R_2}\, e^{i\alpha} z \tag{0.2}$$

[1]The methods and results of this paper derive from a paper of Antonio Ros[R], and in particular, from Lemma 6 of that paper. A slightly expanded version will appear in the Notices of the AMS. Research at MSRI is supported in part by NSF grant DMS–9701755

As immediate corollaries, one has:

Corollary 1 (Liouville's Theorem). A bounded analytic function in the entire plane is constant.

Proof: R_2 is fixed and R_1 may be chosen arbitrarily large.

Corollary 2 If $R_1 = R_2$, then

$$| f'(0) | \leq 1. \tag{0.3}$$

A slightly less obvious but still elementary corollary is

Corollary 3 If $R_1 = R_2$ and if f maps the boundary to the boundary, then at any point b with $| b |= R_1$ where $f'(b)$ exists, one has

$$| f'(b) | \geq 1. \tag{0.4}$$

The proof follows immediately from the fact that distances to the origin are shrunk under f, and therefore distances from the boundary are stretched. More precisely, for t real, $0 < t < 1$, we have $| f(tb) | \leq t | b |$, so that

$$| f(tb) - f(b) | \geq R_1 - tR_1 =| tb - b |$$

from which (4) follows.

We will return later to the possible significance of this elementary observation. Although we will not make use of it here, we note that a refinement of the above argument gives a stronger and sharp boundary equality; namely, with $R_1 = R_2 = 1$, if a single boundary point b maps to the boundary and if $f'(b)$ exists, then

$$| f'(b) | \geq 1 + \frac{1- | f'(0) |}{1+ | f'(0) |}.$$

(See Osserman[O].)

In 1916, Pick[P] gave a new slant to Schwarz' Lemma that was to have an enormous impact on future developments:

Lemma 2 (Schwarz–Pick Lemma). Let $f(z)$ be a holomorphic map of the unit disk D into the unit disk. Then

$$\hat{\rho}(f(z_1), f(z_2)) \leq \hat{\rho}(z_1, z_2) \ for \ all \ z_1, z_2 \in D \tag{0.5}$$

where $\hat{\rho}$ refers to distances measured in the hyperbolic metric in D.

What Pick observed was that we can compose f with linear fractional transformations that are isometries of the hyperbolic plane, taking z_1 to 0 and $f(z_1)$ to 0. Then (5) reduces to

$$\hat{\rho}(0, f(z_2)) \leq \hat{\rho}(0, z_2). \tag{0.6}$$

But hyperbolic distance to the origin is a monotonic function of Euclidean distance, so that (6) is equivalent to (1) (with $R_1 = R_2 = 1$).

Corollary. If $\| \ \|_H$ denotes norm in the hyperbolic metric, then

$$\| df_z \|_H \leq 1 \ for \ all \ z \in D \tag{0.7}$$

and if γ is any curve in D, then the length of the image of γ under f is less than or equal to the length of γ, both measured in the hyperbolic metric.

For future reference let us note the explicit form of these quantities. The hyperbolic metric is given by

$$d\hat{s}^2 = \left(\frac{2}{1- \mid z \mid^2}\right)^2 \mid dz \mid^2 \tag{0.8}$$

and its Gauss curvature $\hat{K}$ satisfies

$$\hat{K} \equiv -1. \tag{0.9}$$

Integrating (8) yields

$$\hat{\rho}(0, z) = log \ \frac{1+ \mid z \mid}{1- \mid z \mid} = 2 \ tanh^{-1} \mid z \mid . \tag{0.10}$$

Ahlfors' great insight [A] was that the same conclusions would hold far more generally:

Lemma 3 (Schwarz–Pick–Ahlfors Lemma). Let f be a holomorphic map of the unit disk D into a Riemannian surface S endowed with a Riemannian metric ds^2 with Gauss curvature $K \leq -1$. Then the hyperbolic length of any curve in D is at least equal to the length of its image. Equivalently,

$$\rho(f(z_1), f(z_2)) \leq \hat{\rho}(z_1, z_2) \ for \ all z_1, z_2 \in D \tag{0.11}$$

or

$$\| df_z \|_H \leq 1 \ everywhere, \tag{0.12}$$

where the norm is taken with respect to the hyperbolic metric on D and the given metric on the image.

With these results as background, let us give examples of the kind of finite versions one can prove. We recall that a *geodesic disk of radius* R on a surface is the diffeomorphic image of an Euclidean disk of radius R under the exponential map. Equivalently, one has geodesic polar coordinates:

$$ds^2 = d\rho^2 + G(\rho,\theta)^2 d\theta^2, \tag{0.13}$$

where ρ represents distance to the center of the disk, and

$$G(0,\theta) = 0,\ \frac{\partial G}{\partial \rho}(0,\theta) = 1,\ G(\rho,\theta) > 0,\ 0 < \rho < R. \tag{0.14}$$

We shall use the following notation throughout this paper:

Notation: Let f map the disk $\mid z \mid< R$ into a geodesic disk centered at $f(0)$ on a surface S with metric ds^2.Then

$$\rho(p) = distance\ on\ S\ from\ f(0)\ to\ p \tag{0.15}$$

$$\hat{\rho}(p) = distance\ from\ 0\ to\ z\ with\ respect\ to\ a\ metric\ d\hat{s}^2\ on\ \mid z \mid< R. \tag{0.16}$$

Example 1. Let f be a holomorphic map of $\mid z \mid< R_1$ into a geodesic disk of radius R_2 centered at $f(0)$ on a surface S with Gauss curvature $K \leq 0$. Then

$$\rho(f(z)) \leq \frac{R_2}{R_1} \mid z \mid,\ \mid z \mid< R_1. \tag{0.17}$$

Note that this is a direct extension of the original Schwarz Lemma, and it has exactly the same consequences:

Corollary 1. Any holomorphic map of the entire plane into a geodesic disk on a surface with $K \leq 0$ must be constant.

Corollary 2. If $R_2 \leq R_1$, then $\| df_0 \|\leq 1$.

Corollary 3. If $R_1 = R_2$ and if at some point z with $\mid z \mid= R_1$, $\rho(f(z)) = R_1$ and df_z exists, then

$$\| df_z \|\geq 1. \tag{0.18}$$

Remarks. 1. This example is a sligltly more genereal form of the first part of Lemma 6 of Ros[R]; his proof goes through without change.

2. Cor. 1 is false for $K > 0$; stereographic projection is a non-constant conformal map of the entire plane onto a geodesic disk consisting of the sphere minus a point.

Example 2. Let f map $\mid z \mid< r < 1$ into a geodesic disk of radius ρ_2 centered at $f(0)$ on a surface S whose Gauss curvature satisfies $K \leq -1$. Let ρ_1 be the hyperbolic radius of $\mid z \mid= r$; i. e.

$$\rho_1 = log\, \frac{1+r}{1-r}$$

by (10). Then if $\rho_2 \leq \rho_1$ and $d\hat{s}$ is the hyperbolic metric on $\mid z \mid< 1$,

$$\rho(f(z)) \leq \hat{\rho}(z) \; for \;\; \mid z \mid< r. \tag{0.19}$$

Corollary 1. Under the same hypotheses,

$$\parallel df_0 \parallel \leq 1. \tag{0.20}$$

Corollary 2. If furthermore, $\rho_2 = \rho_1$ and f maps the boundary into the boundary, then at any point z on $\mid z \mid= r$ where df_z exists,

$$\parallel df_z \parallel \geq 1. \tag{0.21}$$

Note that in both these examples we can only assert distance shrinking from the center, unlike Schwarz–Pick and its descendants. In fact, as (18) and (21) indicate, the reverse is likely to be true near the boundary. However, the original Ahlfors version of Schwarz–Pick turns out to be a consequence.

Corollary 3. (Schwarz–Pick –Ahlfors). (Lemma 3 above)

Proof: We show that (12) holds if f maps the full disk $\mid z \mid< 1$ into S with $K \leq -1$. If S is not simply-connected we may lift the map f to a map $\tilde{f}$ into the universal covering surface $\tilde{S}$ of S, in which case (12) is equivalent to $\parallel df_z \parallel \leq 1$ everywhere. So we may as well assume that S is simply-connected.

Let z_1, z_2 be any two points in the unit disk. By composing f with an isometry of the hyperbolic plane taking 0 to z_1, we can assume that $z_1 = 0$, and (11) becomes

$$\rho(f(0), f(z)) \leq \hat{\rho}(0, z_2). \tag{0.22}$$

Choose any r_0 such that

$$\mid z_2 \mid< r_0 < 1.$$

Let

$$\rho_0 = max_{|z| \leq r_0}\; \rho(f(z)).$$

Since S is simply connected and $K < 0$, there exist global geodesic coordinates on the disk

$$D_{\rho_0} : \rho(p) < \rho_0.$$

We let $\tilde{f}(\varsigma) = f(r_0\varsigma) : \{\mid \varsigma \mid < 1\} \longrightarrow D_{\rho_0}$. Let $d\tilde{s}$ be the hyperbolic metric in $\mid \varsigma \mid < 1$ and choose r_1, $\mid z_2 \mid < r_1 < r_0$ so that

$$\rho_1 = \tilde{\rho}\left(\frac{r_1}{r_0}\right) \geq \rho_0.$$

(This is always possible, since $\tilde{\rho}(r_1/r_0) \longrightarrow \infty$ as $r_1 \longrightarrow r_0$.) We may now apply our lemma to

$$\tilde{f} : \{\mid \varsigma \mid < \frac{r_1}{r_0}\} \longrightarrow D_{\rho_0}$$

to conclude $\rho(\tilde{f}(\varsigma)) \leq \tilde{\rho}(\varsigma)$ for $\mid \varsigma \mid < r_1/r_0$, and in particular for $\varsigma_2 = z_2/r_0$. Then

$$\rho(f(z_2)) = \rho(\tilde{f}(\varsigma_2)) \leq \tilde{\rho}(\varsigma_2).$$

But r_0 can be chosen arbitrarily close to 1, and

$$r_0 \longrightarrow 1 \Longrightarrow \varsigma_2 \longrightarrow z_2,\ d\tilde{s} \longrightarrow d\hat{s},\ \textit{and}\ \tilde{\rho}(\varsigma_2) \longrightarrow \hat{\rho}(\varsigma_2).$$

This proves (22) and therefore (11), from which (12) follows.

The proof of Example 2 is an obvious analog of the proof of Lemma 2.1 of [OR]. But Example 2 is also a special case of Theorem 1 below: the "general finite shrinking lemma".

Before stating the general shrinking lemma, let us note some of the generalizations of the Ahlfors Lemma that were made subsequently.

Theorem. Yau ([Y], 1973). Let $\hat{S}$ be complete, with $\hat{K} \geq -1$, and let f be a holomorphic map of $\hat{S}$ into S with $K \leq -1$. Then $\parallel df_p \parallel \leq 1$ for all p in $\hat{S}$; i. e., the length of every curve in $\hat{S}$ is greater than or equal to the length of its image.

Theorem. Troyanov ([T], 1991), Ratto, Rigoli and Véron ([RRV], 1994). Let $\hat{S}$ be complete and let f map $\hat{S}$ homorphically into S. Suppose that

$$K(f(p)) \leq \hat{K}(p), \tag{0.23}$$

$$K(f(p)) \leq 0, \tag{0.24}$$

and that certain further restrictions hold on K, $\hat{K}$, weaker than in Yau's theorem. Then $\parallel df_p \parallel \leq 1$ for all p in $\hat{S}$.

We refer to the original papers for the exact hypotheses in each case. What is of interest here is condition (23) which represents the natural culmination of the line of investigation initiated by Ahlfors. The underlying philosophy is that the more negative the curvature, the more a holomorphic map will shrink distances and curve lengths. Note that we are really comparing two metrics on the same domain: the original metric $d\hat{s}^2$ and the pullback of the metric ds^2 under f. In fact all of the Ahlfors-type lemmas may be stated as comparison theorems between two conformally related metrics, and again, the philosophy is that the more negative the curvature, the shorter the curve lengths in the metric.

This type of result seems oddly reminiscent, but in apparent reverse, of the standardcomparison theorems from Riemannian geometry, which say roughly that the more negative the curvature the more certain curves are stretched. Specially, one has:

Lemma 4 (Comparison Lemma). Let ds^2 and $d\hat{s}^2$ be metrics given in geodesic polar coordinates by

$$ds^2 = d\rho^2 + G(\rho,\theta)^2 d\theta$$
$$d\hat{s}^2 = d\rho^2 + \hat{G}(\rho,\theta)^2 d\theta.$$

If

$$K(\rho,\theta) \le \hat{K}(\rho,\theta),\ 0 < \rho < \rho_0, \tag{0.25}$$

then

$$\frac{1}{G}\,\frac{\partial G}{\partial \rho} \ge \frac{1}{\hat{G}}\,\frac{\partial \hat{G}}{\partial \rho} \tag{0.26}$$

and

$$G(\rho,\theta) \ge \hat{G}(\rho,\theta),\ 0 < \rho < \rho_0. \tag{0.27}$$

Note that

$$G(\rho_1,\theta) = \frac{ds}{d\theta}\ \textit{along the geodesic circle}\ \rho = \rho_1, \tag{0.28}$$

so that (27) and (28) imply that

$$L(\rho_1) \ge \hat{L}(\rho_1),\ 0 < \rho_1 < \rho_0 \tag{0.29}$$

where $L(\rho)$, $\hat{L}(\rho)$ refer to the length in their respective metrics of geodesic circles of radius ρ.

An obvious question is what relation, if any, exists between the Ahlfors–type lemmas and the Riemannian comparison theorem. The answer is two-fold; first, there is a heuristic argument, based on (18) and (21), which provides a link between the two, and second, we can use the Riemannian comparison lemma to prove a general finite shrinking lemma which contains our Example 2 above as a special case, and therefore provides a new route to proving the original Ahlfors Lemma.

Let us start with a brief look at the heuristic argument relating the two forms of the comparison. We have a geodesic disk $\hat{D}$ of radius ρ_1 on a surface with

Riemannian metric

$$d\hat{s}^2 = d\rho^2 + \hat{G}(\rho,\theta)^2 d\theta,$$

where for any point P in $\hat{D}$, $\hat{\rho}(P) =$ distance between P and the center O of the disk. We map $\hat{D}$ conformally by f into a surface S with metric ds^2, and assume that the image lies in a geodesic disk D of the same radius centered at the point $f(0)$. Under suitable curvature restrictions we wish to show that

$$\rho(f(P)) \leq \hat{\rho}(P), \ for\ all\ P\ in\ D \tag{0.30}$$

where $\rho(Q) =$ distance on S from $f(0)$ to Q. We introduce geodesic polar coordinates

$$ds^2 = d\rho^2 + G(\rho,\theta)^2 d\theta, \ 0 \leq \rho < \rho_1, \ 0 \leq \theta < 2\,\pi$$

on the image, and the curvature relation we assume is that

$$K(\rho,\theta) \leq \hat{K}(\rho,\theta) \ when\ \rho = \hat{\rho}; \tag{0.31}$$

That is, for each fixed θ, the curvature of the image geodesic disk is at most equal to the curvature of the original at the same distance from the center. Then what we want to show is, inequality (30), is that each geodesic disk $\hat{\rho} < c$, for $c < \rho_1$, maps into the geodesic disk $\rho < c$ in the image. Heuristically, the image disk is likely to be largest when f maps $\hat{D}$ *onto* the full disk D. So let us assume that f is such a map, and f takes the boundary, $\hat{\rho} = \rho_1$, to the boundary $\rho = \rho_1$. Let us further assume that f is defined and conformal in a slightly larger disk $\hat{\rho} < \rho_0$. Then the Riemannian comparison lemma applies, and we have inequality (29), which tell us that *globally*, the map f takes the geodesic circle $\hat{\rho} = \rho_1$ of length $\hat{L}(\rho_1)$ onto a geodesic circle of greater or equal length $L(\rho_1)$; *locally*, by virtue of (28), the inequality (27) tells us that under the map of $\hat{\rho} = \rho_1$ to $\rho = \rho_1$ which relates points with the same angular coordinate θ, we have

$$\frac{ds}{d\hat{s}} \geq 1 \tag{0.32}$$

However, f will not in general preserve θ, so that the inequality (29) tells us only that (32) holds *on average* where s and $\hat{s}$ represent arclength along $\rho = \rho_1$ and $\hat{\rho} = \rho_1$ *under the map* f. The final heuristic assumption is that (32) holds along the whole curve $\rho = \rho_1$, under the map f. Then conformality of f implies that the same inequality also holds in the radial direction, so that along each "radius": $\theta = \theta_0$ of $\hat{D}$, we have

$$\frac{ds}{d\hat{\rho}}\Big|_{\hat{\rho}=\rho_1} \geq 1, \tag{0.33}$$

where $\rho(\hat{\rho})$ is the function whose value is $\rho(f(P))$ at the point P in $\hat{D}$ with coordinates $(\hat{\rho},\theta_0)$. Finally, we may, by a standard type of argument dating at least to Ahlfors' original paper, assume that we have *strict* inequality in (27) and therefore in (32) and (33), and then get weak inequality by going to the limit. Then what (33) tells us is that points in $\hat{D}$ near the boundary $\hat{\rho} = \rho_1$ move *further* from the boundary $\rho = \rho_1$ of D, so that they move *closer* to the center of D; in other words, (30) holds, in fact with strict inequality, for points P in some annular region near the boundary of $\hat{D}$. We are then back to our original situation on a disk of smaller radius in $\hat{D}$, and we may expect the same kind of contraction (30) to extend.

In brief, then, the heuristic connection is that an equality like (31) on Gauss curvature implies an *expansion* of the boundary $\hat{\rho} = \rho_1$ to $\rho = \rho_1$, which by conformality of f implies an expansion in the radial direction from the boundary, or a movement of points *toward* the center, and therefore a *contraction* in the sense of (30).

We have not been able to turn this heuristic argument into a complete proof under the full geenrality of (31), but we can do so for a very broad class of metrics, including those of Examples 1 and 2; namely the case when $d\hat{s}^2$ has circular symmetry.

Theorem 1 (General Finite Shrinking Lemma). Let $\hat{D}$ be a geodesic disk of radius ρ_1 with respect to a metric $d\hat{s}^2$. Assume that $d\hat{s}^2$ is circularly symmetric, so that

(0.34) $$d\hat{s}^2 = d\hat{\rho}^2 + \hat{G}(\hat{\rho}^2)d\theta^2,\ 0 \leq \hat{\rho} < \rho_1,$$

where $\hat{G}$ depends on $\hat{\rho}$ only, and not θ.

Let f be a holomorphic map of $\hat{D}$ into a geodesic disk D of radius ρ_2 on a surface S, with center at the image under f of the center of $\hat{D}$. If $\rho_2 \leq \rho_1$, and if

(0.35) $$K(\rho, \theta) \leq \hat{K}(\hat{\rho})\ for\ \rho = \hat{\rho},$$

then

(0.36) $$\rho(f(P)) \leq \hat{\rho}(P)\ for\ all\ P\ in\ \hat{D}.$$

Proof. First, let us assume, as we may, that the metric (34) is represented as a conformal metric

(0.37) $$d\hat{s}^2 = \hat{\lambda}(r)^2 \mid dz \mid^2,\ \mid z \mid < R \leq \infty,$$

where

(0.38) $$\mid dz \mid^2 = dr^2 + r^2 d\theta^2,\ 0 \leq r < R$$

is the Euclidean metric on the disk. Comparing (34) with (37), (38), we find

(0.39) $$d\hat{\rho} = \hat{\lambda}(r)dr,$$

and

(0.40) $$\hat{G}(\hat{\rho}) = r\hat{\lambda}(r).$$

Thus

$$\hat{\rho} = h(r) =: \int_0^r \hat{\lambda}(t)\ dt,\ 0 \leq r < R, \tag{0.41}$$

where $h(r)$ is a monotone stricly increasing function, with

$$h(R) = \rho_1. \tag{0.42}$$

Hence h has inverse

$$r = H(\hat{\rho}),\ 0 \leq \hat{\rho} < \rho_1, \tag{0.43}$$

also monotone stricly increasing, with

$$H(\rho_1) = R.$$

We next recall that in geodesic polar coordinates:

$$ds^2 = d\rho^2 + G(\rho, \theta)^2 d\theta^2,$$

the Gauss curvature K is given by the formula

$$K(\rho, \theta) = -\frac{1}{G(\rho, \theta)} \frac{\partial^2 G}{\partial \rho^2}(\rho, \theta), \tag{0.44}$$

while the Laplacian Δ with respect to the metric ds^2 of any function $\varphi(\rho)$ is given by

$$\Delta\varphi = \frac{1}{G} \left[\frac{\partial}{\partial\rho}(G\varphi'(\rho))\right] = \varphi''(\rho) + \left(\frac{\partial}{\partial\rho}\log G\right)\varphi'(\rho), \tag{0.45}$$

and in particular, when $\varphi(\rho) = \rho$,

$$\Delta\rho = \frac{\partial}{\partial\rho}\ log\ G = \frac{1}{G}\ \frac{\partial G}{\partial\rho}; \tag{0.46}$$

therefore, (45) can be written

$$\Delta\varphi = \varphi''(\rho) + (\Delta\rho)\varphi'(\rho). \tag{0.47}$$

Comparing (25) and (26) with (35) and (46), we see that our hypotheses imply

$$\Delta\rho|_{\rho=c} \geq = \hat{\Delta}\hat{\rho}|_{\hat{\rho}=c}\ \ for\ 0 < \rho < \rho_2. \tag{0.48}$$

Since the function H in (43) satisfies

$$H'(\hat{\rho}) = \frac{dr}{d\hat{\rho}} = 1/\frac{d\hat{\rho}}{dr} > 0,$$

it follows from (47) and (48) and the definition of H that

$$\Delta\ logH(\rho)|_{\rho=c} \geq \hat{\Delta}\ log\ H(\hat{\rho})|_{\hat{\rho}=c} = \hat{\Delta}\ log\ \ |\ z\ |= 0,$$

since $log \mid z \mid$ is harmonic in the Euclidean metric and therefore in any conformal metric on D. Since f is holomorphic, it follows that the pullback of $logH(\rho)$ to D also satisfies

$$\Delta_z \ logH(\rho(f(z))) \geq 0$$

at all points where $\rho(f(z)) \neq 0$, while the function

$$log \ H(\rho(f(z))) \longrightarrow -\infty$$

as $\rho(f(z)) \longrightarrow 0$. Let

$$u(z) = log \ \frac{H(\rho(f(z)))}{\mid z \mid} \quad for \ 0 <\mid z \mid< R.$$

Then $\Delta_z u \geq 0$ at all points where $0 <\mid z \mid< R$, $\rho(f(z)) \neq 0$. Also, $u \longrightarrow -\infty$ where $\rho(f(z)) \longrightarrow 0$. Hence u is subharmonic for $0 <\mid z \mid< R$. Furthermore near $z = 0$ if we represent the map f by $w = F(z)$ in terms of a local isothermal parameter w near $f(0)$ with $w = 0$ at $f(0)$, then

$$H(\rho(f(z))) \sim \frac{\lambda(0)}{\hat{\lambda}(0)} \mid F'(0) \mid\mid z \mid$$

where $ds^2 = \lambda^2(w) \mid dw \mid^2$. Thus $u(z)$ is bounded at $z = 0$ if $F'(0) \neq 0$, and $u(z) \longrightarrow -\infty$ as $z \longrightarrow 0$ if $F'(0) = 0$. In either case, $u(z)$ is subharmonic in the full disk $\mid z \mid< R$, and by the maximum principle,

$$log \ \frac{H(\rho(f(z)))}{\mid z \mid} = u(z) \leq \lim_{\mid z \mid \longrightarrow R} \ u(z) \leq log \frac{H(\rho_2}{R},$$

or

$$H(\rho(f(z))) \leq \frac{H(\rho_2)}{H(\rho_1)} \ \mid z \mid \leq \mid z \mid \tag{0.49}$$

since $H(\rho_1) = R$, and $\rho_2 \leq \rho_1 \implies H(\rho_2) \leq H(\rho_1)$.

Thus

$$\rho(f(z)) \leq h(\mid z \mid) = \hat{\rho}(z)$$

which is (36), and proves the Theorem.

□

Remarks. We have said that this theorem includes Examples 1 and 2 as special cases, but although that is true of Example 2, the theorem initially implies only the case of Example 1 where $R_2 \leq R_1$. However, it is easy to see, but somewhat awkward to state, what happens when $\rho_2 > \rho_1$. We have

Theorem 2 (More General Finite Shrinking Lemma). Under the same hypotheses and with the same notation as in the GFSL, except that $\rho_2 \neq \rho_1$, we have the inequality

$$\rho(f(z)) \leq h\left(\frac{H(\rho_2)}{H(\rho_1)} \mid z \mid\right) \quad for\ 0 \leq \mid z \mid < R, \tag{0.50}$$

provided, in the case $\rho_2 > \rho_1$, that we make the additional assumption that the metric $d\hat{s}^2$ extends as a circularly symmetric metric to a larger disk $\mid z \mid < R_2$, with $\hat{\rho}(R_2) = \rho_2$, and that the inequality (35) holds whenever $\rho = \hat{\rho} < \rho_2$.

The proof given for Theorem 1 goes through unchanged till the first inequality in (49) which is equivalent to (50).

In the case of Example 1, where $d\hat{s}^2$ is the Euclidean metric, $h(r) = r$, $H(r) = r$, and (50) reduces to (17).

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