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Strong rigidity and periodic minimal surfaces

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A classical question about submanifolds is to decide if given an isometric immersion $\psi : M \longrightarrow N$, this is, up to ambient isometries, the unique way of immerse isometrically M into N. Similarly, when we consider a minimal immersion $\psi : M \longrightarrow \mathbf{R}^3$, we can ask if this is the unique isometric minimal immersion of M into $\mathbf{R}^3$, up to rigid motions. If this is the case, M is said to be (minimally) *rigid*.

Once we have the definition of what is a rigid minimal surface, let us analyze this notion from the point of view of the local or global Geometry: First of all, we should recall that every (conformal) minimal immersion in $\mathbf{R}^3$ can be expressed in terms of complex data, by means of the classical Weierstrass-Enneper representation

$$\begin{aligned} \psi(z) &= \text{Real} \int^z \left(\frac{1}{2}(1-g^2)\omega, \frac{i}{2}(1+g^2)\omega, g\omega \right) \\ &= \text{Real} \int^z \Phi, \qquad z \in M, \end{aligned} \tag{0.1}$$

where g denotes the meromorphic function obtained by stereographic projection of the Gauss map of ψ and ω is a holomorphic one-form on M. Conversely, every choice of a pair (g, ω) on a Riemann surface M yields a conformal minimal immersion $\psi : M \longrightarrow \mathbf{R}^3$ by the formula (0.1), provided that the following compatibility conditions hold:

- Poles(g) =Zeros$(\omega)^2$, (nondegeneracy of the induced metric)
- Real $\displaystyle\int_\Gamma \Phi = 0$, $\forall \Gamma$ closed curve in M. (period problem)

Moreover, the induced metric becomes

$$ds^2 = \frac{1}{4}\left(1 + |g|^2\right)^2 |\omega|^2.$$

in terms of the Weierstrass data. Inmediately we check that every minimal surface in $\mathbf{R}^3$ with Weierstrass data (g, ω) can be locally and isometrically deformed by considering the new pair $(g, e^{i\theta}\omega)$, $\theta \in \mathbf{R}$ on any simply connected neighborhood of a given point of M (minimal surfaces defined by these procedure are called the *associate surfaces* of ψ). Hence the problem of deciding if a given minimal surface is rigid has a global nature, where the associate surfaces are not generically well-defined because of the period problem. Of course, if the whole surface M is simply connected, associate surfaces are globally well-defined and the surface is nonrigid: this is the situation in the Helicoid, given by the Weierstrass data $(g(z) = e^z, \omega = i\, dz)$ on the complex plane C.

Now if one looks for global results about rigidity of minimal surfaces, some kind of idea of abundance of rigid minimal surfaces in $\mathbf{R}^3$ appears: inside the large family of properly embedded minimal surfaces, it is usual to consider two large subfamilies that have been deeply studied, and that we will analyze from the point of view of rigidity:

Minimal surfaces of finite total curvature in $\mathbf{R}^3$.

These surfaces are defined by the condition $\left|\int_M K\, dA\right| < \infty$, where K denotes the Gauss curvature of the surface. A famous theorem of Huber and Osserman insures that these surfaces are conformally parametrized by meromorphic Weierstrass data on finitely punctured compact Riemann surfaces $M = \overline{M} - \{p_1, \ldots, p_r\}$, where each p_i describes the behaviour of the surface at infinity (*ends* of the surface). In particular, properly embedded minimal surfaces of finite total curvature in $\mathbf{R}^3$ have also finite topology, although the Helicoid shows that the converse fails: the Helicoid in $\mathbf{R}^3$ is topologically a sphere minus one point, but its periodicity forces the total curvature to be infinite. A recent work of Collin [3] insures that for properly embedded minimal surfaces in $\mathbf{R}^3$ with more than one end, finite total curvature is equivalent to finite topology. In this setting, Choi, Meeks and White [2] give the following important result about rigidity:

Theorem 1 *If $M \subset \mathbf{R}^3$ is a properly embedded minimal surface with more than one end, then M is rigid. In particular, rigidity holds for any properly embedded minimal surface in $\mathbf{R}^3$ with finite total curvature.*

Again the Helicoid shows that this result cannot be extended to properly embedded minimal surfaces with only one end.

Periodic minimal surfaces.

These are properly embedded minimal surfaces in $\mathbf{R}^3$ invariant by a discrete group G of isometries of the ambient space, so they induce properly embedded minimal surfaces in the complete flat three-manifold $\mathbf{R}^3/G$. By the classification of such three-manifolds, the problem reduces to study the following three cases:

(i) Surfaces which are invariant by a screw motion obtained by composition of a rotation around the vertical x_3-axis with a translation along the same axis (*singly periodic minimal surfaces*).

(ii) Surfaces invariant by two independent translations (*doubly periodic minimal surfaces*).

(iii) Surfaces invariant by three independent translations (*triply periodic minimal surfaces*).

A natural condition that allows us to study these families of surfaces more easily consists of imposing to the induced surface in $\mathbf{R}^3/G$ to have finite topology, which thanks to the works of Meeks and Rosenberg [5, 7] is equivalent to assume finite total curvature in the quotient. These authors also studied the asymptotic geometry of periodic minimal surfaces with quotients of finite topology, proving that *all the ends of a singly periodic minimal surface with finite topology in the quotient are simultaneously asymptotic to parallel planes, flat vertical annuli or ends of Helicoids. If the surface is doubly periodic and has finite topology in the quotient, then all its ends are asymptotic to flat annuli.*

Concerning the problem of rigidity on periodic minimal surfaces with quotients of finite topology, Meeks and Rosenberg showed in a series of works [4, 5] that

> *If $\widetilde{M} \subset \mathbf{R}^3$ is a doubly or triply periodic minimal surface, then $\widetilde{M}$ is rigid.*

They also extended this property to the case of a singly periodic minimal surface, provided that its ends are planar or Scherk type. But rigidity cannot be expected for singly periodic minimal surfaces with helicoidal type ends, as the Helicoid demonstrates. Our main theorem [8] shows that this is the only possible counterexample in the whole family:

Theorem 2 *Let $\widetilde{M} \subset \mathbf{R}^3$ be a properly embedded minimal surface invariant by a discrete group G of isometries of $\mathbf{R}^3$, and such that the quotient surface $M = \widetilde{M}/G \subset \mathbf{R}^3/G$ has finite topology. Then, $\widetilde{M}$ is rigid or it is the Helicoid.*

We will finish by giving a brief idea of the proof of the statement above. Calabi [1] proved that the only possibly well-defined isometric minimal immersions different to a given one are, up to rigid motions, the associate surfaces of this last one. This means that if a properly embedded minimal surface in $\mathbf{R}^3$ is nonrigid, then in a first step at least one of its associate surfaces is well-defined, and now it is easy to see that this holds for any associate surface. A simple analysis of the period problem on the associate surfaces suffices to conclude that not only the real part, but also the imaginary part of the integral along any closed curve of the Weierstrass form of our minimal surface is zero. This is equivalent to imposing that the integral of the unit conormal field along any closed curve (i.e. the *flux* of the surface along such closed curve) vanishes.

On the other hand, for a wider class of minimal surfaces in $\mathbf{R}^3$ than those with zero flux we dispose of a 1-parameter deformation that preserves interesting geometric properties (see details in [9]): suppose that $\psi : M \longrightarrow \mathbf{R}^3$ is a minimal immersion such that the flux along any closed curve in M is a vertical vector and denote by (g, ω) its Weierstrass representation. The verticality of the fluxes is equivalent to imposing that the one-forms ω, $g^2\omega$ are exact. Hence, the pair $(g_\lambda = \lambda g, \omega_\lambda = \frac{1}{\lambda}\omega)$ gives a well-defined minimal immersion $\psi_\lambda : M \longrightarrow \mathbf{R}^3$ with the same third coordinate function as ψ. Moreover, completeness and total curvature are preserved throughout the deformation, but the most useful properties of this deformation deal with the problem of embeddedness. Here we will emphasize the following one:

Lemma 3 *Let $\psi : M \longrightarrow \mathbf{R}^3$ be an immersed minimal surface with vertical flux. If the normal vector of ψ at a point $p \in M$ is vertical, then for all neighborhood D of p there exists a real number $\lambda > 0$ such that $\psi_\lambda|_D$ is not embedded.*

We are now in good conditions for proving our rigidity result: By the former discussion, we can assume that $\widetilde{M} \subset \mathbf{R}^3$ is a properly embedded minimal surface invariant by a screw motion S_θ given by composition of a rotation of angle $\theta \in [0, 2\pi[$ with a nontrivial vertical translation. Suppose that $\widetilde{M}$ is nonrigid and that the quotient surface $M \subset \mathbf{R}^3/S_\theta$ has finite topology and helicoid type ends. Nonrigidity implies that we dispose of a smooth curve $\psi_\lambda : \widetilde{M} \longrightarrow \mathbf{R}^3$ of immersions given by the deformation above. Using the maximum principle at infinity [6] and the fact that the original surface ψ is embedded, one shows that all the ψ_λ are also embeddings, hence ψ has no points with vertical normal vector. In terms of the quotient surface $M \subset \mathbf{R}^3/S_\theta$, this means that the third component of the Weierstrass form has no zeroes on the compactification $\overline{M}$ of M. But this one-form has simple poles exactly at the helicoidal ends of M, so $\overline{M}$ must be a sphere and M has exactly two ends. Now a well-known theorem of Toubiana, Meeks and Rosenberg [7, 10] gives that our surface is a Helicoid.

Remark 4 The argument above uses the vertical flux condition, not the zero flux condition which is equivalent to rigidity. This allows us to state another characterization of the Helicoid:

Theorem 5 *Let $\psi : \widetilde{M} \longrightarrow \mathbf{R}^3$ be a properly embedded minimal surface invariant by a screw motion S_θ and with vertical flux. If the quotient surface $\widetilde{M}/S_\theta$ has finite topology and helicoid type ends, then it is the Helicoid.*

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