Proceedings Ist International Meeting
on Geometry and Topology
Braga (Portugal)

Public. Centro de Matemática
da Universidade do Minho
p. 175–183, 1998

Some Global Properties of Surfaces in 4-Space

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Introducction

We make in this paper a rewiew of global results concerning surfaces immersed in $I\!R^4$, obtained by application of the Poincaré - Hopf index formula to some line fields geometrically defined on them.

We introduce in sections 1 and 2 all the geometrical concepts and basic facts that are needed to understand these results. The main references for these sections are [4] and [6]. Section 3 contains a description of the line fields that will be considered later and their singularities (ref. [2],[3] and [4]). Finally, section 4 exhibits the main results, consisting in several formulae relating the Euler number of a given surface immersed in 4-space and its numbers of different geometrically relevant points. We point out that the complete proofs of Theorems 3 and 4 can be found in [2], whereas Theorems 2 and 5, although easily proven as a consequence of the material contained in [4], are new. We also discuss the relation of Corollary 2 with the Carathéodory's conjecture for surfaces in 3-space (*Any 2-sphere embedded in $I\!R^3$ has at least 2 umbilic points*) and according to this, we push forward an analogous conjecture for 2-spheres embedded in $I\!R^4$.

1. Preliminary geometrical concepts

Let $f : M^2 \to R^4$ be a smooth immersion. The curvature ellipse at a point $f(m)$ of $f(M^2)$ is constructed as follows ([4]):

Consider the unit circle in $T_{f(m)}f(M)$, parametrized by the angle θ in $[0, 2\pi]$. Let γ_θ be the normal section of $f(M)$ in the direction θ (i.e., the curve obtained

[1] Work partially supported by DGICYT grant no. PB93-0707.

[2] Work partially supported by CNPq grant no. 300066/88-0.

by intersecting $f(M)$ with the 3-space obtained as the direct sum of the normal plane, $N_{f(m)}f(M)$, of $f(M)$ at $f(m)$ and the straight line spanned by the tangent direction θ). Let η_θ be the curvature vector of γ_θ at f(m). (It lies in $N_{f(m)}f(M)$).

As θ varies from 0 to 2π, the extremum of η_θ describes an ellipse in $N_{f(m)}f(M)$ which is called **curvature ellipse** of $f(M)$ at $f(m)$.

A point $f(m)$ of $f(M)$ is said to be **hyperbolic**, **parabolic** or **elliptic** according to whether $f(m)$ lies outside, on or inside the curvature ellipse of $f(M)$ at $f(m)$. When the curvature ellipse at $f(m)$ degenerates to a segment, the point $f(m)$ is said to be a **semiumbilical centre** of $f(M)$. An **inflection point** is a parabolic semiumbilical centre (i.e., a point such that the corresponding curvature ellipse is a segment and it lies on the line determined by this segment). Inflection points can be **imaginary**, **real** or **flat** according to they lie off, inside or at one of the end points of the segment.

A direction θ in $T_{f(m)}f(M)$ is called **asymptotic** if η_θ is parallel to $\frac{\partial \eta_\theta}{\partial \theta}$. There are exactly 2, 0 or 1 asymptotic directions respectively at a hyperbolic, elliptic or parabolic point of $f(M)$, the only exceptions being the inflection points, for which all the directions are asymptotic.

A tangent vector θ at $f(m)$, such that the end of the vector η_θ is one of the vertices of the curvature ellise is called **principal direction** at $f(m)$.

The vector joining the point $f(m)$ with the center of the curvature conic of $f(M)$ at $f(m)$ is known as the **mean curvature vector** of $f(M)$ at $f(m)$. A point is said to be **minimal** if the mean curvature vector vanishes at it. A surface is said to be **minimal** if all its points are minimal.

An immediate consequence of the above definitions is:

a) *Minimal surfaces do not have hyperbolic points,*

b) *Inflection points of minimal surfaces are of real type.*

It can also be seen ([4], pag. 28) that for a generic subset of immersions the minimal points are isolated.

2. Geometry and singularities of height functions

Consider the family of height functions on M associated to an immersion f,

$$\begin{array}{rcccl} \lambda(f) & : & M \times S^3 & \longrightarrow & I\!R \\ & & (m,v) & \longmapsto & f_v(m) = f(m).v \end{array}$$

We quote now some results from [6] concerning generic immersions:

1) A point $f(m)$ is elliptic if and only if m is a nondegenerate critical point

for any height function associated to a normal direction to $f(M)$ at $f(m)$.

2) A point $f(m)$ is hyperbolic if and only if m is a degenerate critical point (generically of type A_3 along some curves and A_4 at isolated points of M) of exactly two height functions on M. The corresponding normal vectors b_1 and b_2 are called **binormals** of M at $f(m)$.

3) A point $f(m)$ is a parabolic point if and only if there is a unique height function f_b having a degenerate critical point at m (generically of type A_2 and possibly of type A_3 or D_4^1 at isolated points).

4) A parabolic point $f(m)$ is an inflection point if and only if m is an umbilic singularity (D_4^1) for the height function in the (unique) binormal direction at $f(m)$. Flat inflection points do not occur on generic surfaces.

Remark: A semiumbilical centre is a singularity of type D_k (generically D_4^1 along curves, and D_5 on isolated points) for some distance squared function on M ([8]). So, inflection points can also be seen as a particular case of semiumbilical centres at which the "osculating hypersphere" becomes a "osculating hyperplane".

5) The set Δ of parabolic points of $f(M)$ is made of a union of immersed (with normal crossings at inflection points of real type) closed curves and isolated points corresponding to inflections of imaginary type.

6) The curves of parabolic points separate $f(M)$ into hyperbolic and elliptic regions (M_h and M_e respectively). The imaginary inflection points are isolated points completely surrounded by hyperbolic points.

7) The hyperbolic region of a closed surface is always nonempty.

Consequently, we arrive to the following conclusion:

If a minimal surface in 4-space is generic in the sense of [6], then it has no inflection points.

In this sense, it turns up to be interesting to investigate :

A) Are there any minimal surfaces generic in the sense of [6]?

B) What is the "typical" structure of a minimal (non necessarily generic) surface at an inflection point? (Can these be isolated? Do they form closed curves?)

3. Line fields on M

The asymptotic directions define 2 line fields on the hyperbolic region, M_h, of M that collapse onto a unique one over all the points of the parabolic set Δ. Their integral lines are called **asymptotic lines** of $f(M)$.

If we put $f : (M^2, 0) \to (\mathbb{R}^4, 0)$ locally in the Monge form,

$$f(x,y) = (x, y, f_1(x,y), f_2(x,y)),$$

it can be shown that the differential equation of the asymptotic lines of $f(M)$ has the form:

$$A(x,y)(x)^2 + B(x,y)(x)(y) + C(x,y)(y)^2 = 0,$$

where

$$A(x,y) = f_{1xy}f_{2xx} - f_{1xx}f_{2xy}$$

$$B(x,y) = f_{1yy}f_{2xx} - f_{1xx}f_{2yy}$$

$$C(x,y) = f_{1yy}f_{2xy} - f_{1xy}f_{2yy}$$

For a generically embedded surface, the inflection points of imaginary type are the singular points (Darbouxian umbilics) of the fields of asymptotic directions and they have indexes $\pm 1/2$ ([2]).

The inverse stereographic projection from $I\!R^3$ to S^3 maps curvature lines of surfaces in 3-space into asymptotic lines of their spherical images, considered as surfaces in 4-space. (A proof of this assertion can be found in [7]). From this, and the fact that there are exactly two (mutually orthogonal) principal directions of curvature at each point of a surface in 3-space, we have

Theorem 1

If a generic closed surface in 4-space lies on a 3-sphere, then it has exactly two **orthogonal** *asymptotic directions at each point except at most at a finite number of them (at which all the directions are asymptotic).*

Conjecture: *This is also a sufficient condition for a surface to lie on the 3-sphere.*

There are also some other relevant line fields on any surface M immersed in $I\!R^4$:

A) If the curvature ellipse at a point p of M is not a circle, then the two ends of the major axis determine two well defined (orthogonal) lines in the tangent plane at p (principal directions). And so do the two ends of the minor axis. All these together define 4 line fields over M whose singularities are those points of M at which the curvature ellipse becomes a circle (or a point, considered as a particular case). The integral lines are called **curvature lines** and their singularities **umbilics**.

These fields were defined by Little ([4]), who proved that the index of their critical points is $\pm 1/4$. Their corresponding differential equations and the generic configurations of the curvature lines around the umbilics have been studied by Gutierrez, Guadalupe, Guiñez and Tribuzy ([3]).

B) The line trough the mean curvature vector in the normal plane at a point p meets the curvature ellipse in two points. These determine a pair of orthogonal lines in the tangent plane. So we have again a pair of direction fields, well defined all over M except at the points when either the mean curvature vector vanishes (minimal points) or the ellipse degenerates into a radial segment (inflection points). J. Little ([4], pg. 286) also found that the index of these singular points is $\pm 1/2$.

4. Counting distinguished points on M

Let S be a smooth compact surface (possibly with boundary), and suppose that V is a smooth line field on S with finitely many critical points, all contained in the interior of S, and finitely many inner and outer contact points (i.e., boundary points of S such that the trajectory of V through them is internally or externally tangent to the boundary of S at the considered point).

The sum, $Ind(V)$, of the indices at the critical points of V is given by the formula

$$Ind(V) = \chi(S) + (s - n)/2$$

where s and n denote the numbers of inner and outer contact points respectively, and $c(S)$ represents the Euler number of S. (See [5]).

Observe that if $\partial S = 0$ then $s = n = 0$, and this becomes the classical Poincaré-Hopf formula.

An immediate consequence of this formula is that given any line field on a surface S with non-vanishing Euler number, then there is at least one singularity of the field on S. By applying this principle to the field associated to the mean curvature directions J. Little obtained ([4], Theorem 1.14):

Any compact surface with non-vanishing Euler number must have at least either a minimal or an inflection point.

On the other hand, by using the same argument on the fields of curvature lines, we can conclude that any surface with non-vanishing Euler number must always have umbilic points. Moreover, by taking into account the index of these umbilics, we get:

Theorem 2

If $M^2 \to \mathbb{R}^4$ is a generic immersion of a closed surface M in $\mathbb{R}^4$, then

$4|\chi(M)| \leq \#\{umbilics\}$, *where* $\chi(M)$ *denotes the Euler number of the surface* M.

Corollary 1 (**Eight umbilics theorem**)

Any generic immersion of a 2-sphere in $\mathbb{R}^4$ *has at least 8 umbilics.*

We say that the surface $f(M)$ is **locally convex** if it has a locally support hyperplane at each point. It was also shown in [6] that a generic surface in 4-space is locally convex if and only if all its points, except at most a finite number of them, are hyperbolic. Hence there are exactly 2 asymptotic directions at all their points, except at isolated imaginary inflections which are the critical points of the two (globally defined) fields of asymptotic lines. And as a consequence it follows

Theorem 3 ([2])

If $M^2 \to \mathbb{R}^4$ *is a generic locally convex immersion of a closed surface* M *in* $\mathbb{R}^4$, *then* $2|\chi(M)| \leq \#\{inflection\ points\}$, *where* $\chi(M)$ *denotes the Euler number of the surface* M.

Corollary 2 (**Four inflections theorem**)

Any generic locally convex immersion of a compact surface with non-vanishing Euler number has at least 4 inflection points.

Remark:

A particular case of convex surface in $\mathbb{R}^4$ is given by the images of surfaces in 3-space trough the inverse of the stereographic projection $p : \mathbb{R}^3 \to S^3$. In fact, the singular points (umbilics) of the principal curvature direction fields of any surface M in 3-space are taken by p to the singular points (inflections) of the asymptotic direction fields on $p(M)$ considered as a surface in 4-space. Consequently, the above corollary generalizes the result obtained by Feldman ([1]) on the number of umbilics for generic closed surfaces in 3-space. Taking into account that Feldman's result gives a proof of Carathéodory's conjecture for **generically embedded** 2-spheres in 3-space, we propose the following:

Caratheodory's conjecture in 4-space:

Any locally convex 2-sphere embedded in $\mathbb{R}^4$ *has at least 2 inflection points.*

Again, by considering the stereographic projection it is easy to see that Caratheodory's conjecture in 4-space contains, as a particular case, the one in 3-space.

Suppose now that f is a generic immersion of a non-necessarily convex surface in $\mathbb{R}^4$. Let $f(m)$ be a parabolic point, with binormal and asymptotic directions b and θ respectively. We have then that the function f_b has a degenerate singularity at m that, generically, can be either of fold or cusp type. We can distinguish between these two possibilities according to the relative possition of the direction θ with respect to the curve Δ of parabolic points as follows (see [2]):

i) $f(m)$ is a fold singularity (A_2) of f_b if and only if θ is transversal to the curve Δ of parabolic points.

ii) $f(m)$ is a cusp singularity (A_3) of f_b if and only if θ is tangent to the curve Δ of parabolic points.

The points in the second case will be called **parabolic cusps**. Generically, they are isolated in Δ.

When $f(M)$ is not locally convex, the fields of asymptotic directions are only defined on the hyperbolic, (M_h), and parabolic, (Δ), regions, that is, in the closure, $cl(M_h)$, of the hyperbolic region. In the case that there are not inflection points of real type, this is a smooth surface with boundary, so we can apply the above formula to the fields of asymptotic directions and taking into account that the parabolic cusps are the inner and outer contact points of the asymptotic directions field, we can conclude

Theorem 4 ([2])

If $M^2 \to I\!\!R^4$ is a generic immersion of a (non-necessarily orientable) closed surface M in $I\!\!R^4$ with no inflection points of real type, then

$$2|\chi(cl(M_h))| \leq \#\{\textit{inflection points}\} + \#\{\textit{parabolic cusps}\}.$$

Corollary 3

If $c(cl(M_h)) \neq 0$, then M has either inflection points or parabolic cusps.

Some interesting examples of surfaces in 4-space are obtained by considering S^1-bundles over closed curves in $I\!\!R^4$, the fibres lying in the normal space of the curve at each point. These are tori that may be convex or not, depending on the choice of the circles in the normal bundle of the curve.

A non convex one is provided by the following torus:

$$(x,y) \to (f_1(x,y), f_2(x,y), f_3(x,y), f_4(x,y)),$$

where

$$f_1(x,y) = (1 - 1/10cos(y))cos(x) + 1/10sin(x)sin(y),$$

$$f_2(x,y) = (1 - 1/10cos(y))sin(x) - 1/10cos(x)sin(y),$$

$$f_3(x,y) = (1 - 2/5cos(y))cos(2x) + 4/5sin(2x)sin(y),$$

$$f_4(x,y) = (1 - 2/5cos(y))sin(2x) - 4/5cos(2x)sin(y).$$

This has a parabolic set made of two non homotopically trivial closed curves, and does not present any inflection points, neither parabolic cusps. So, in this case, M_h is homeomorphic to an annulus and $N = 0$.

Finally, by applying the above criteria to the line field induced by the mean curvature directions on a surface M, we can assert

Theorem 5

If $M^2 \to I\!R^4$ is a generic immersion of a closed surface M in $I\!R^4$,

$$2|\chi(cl(M))| \leq \#\{\textit{inflection points}\} + \#\{\textit{minimal points}\}.$$

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