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A note on inversive invariants of curves

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1 Introduction

A. Fialkow [F] constructed, by using the notion of conformal derivative, a system of conformal Frenet equations for curves in n-space. This lead to the definition of a complete set of conformal invariants known as conformal curvatures. In fact, Fialkow obtained a series of equations that allow us to write the conformal curvatures of a given space curve in terms of its metric curvatures and their derivatives, but the exact expressions for these conformal curvatures are not explicitly given in his paper (except for the one that meassures the "deviation" of the curve from a circle) and appear to be difficult to calculate in general. On the other hand, Cairns, Sharpe and Webb [C-S-W] used the properties of the conformal transformations in order to exhibit the algebra of conformal invariants of a curve in 3-space. This is generated by the infinitesimal conformal arc-length (1-form) and the conformal curvature and torsion functions.

In fact, Cairns and Sharpe had previously used in [C-S] the idea that inversive maps preserve circles and the Coxeter distance [C] between them in order to obtain a explicit expression for the conformal arc-lenght of a plane curve, $\alpha : I\!R \to I\!R^2 \cup \{\infty\}$,

$$\nu_\alpha = \sqrt{|\ k'(s)\ |}ds,$$

where $k'(s)$ represents the derivative with respect to euclidean arc-length of the curvature function $k(s)$ of the plane curve.

We use a similar argument in order to get explicit expressions for inversive invariants of curves in n-spaces in terms of their centres and radii of osculating m-spheres.

The underlying philosophy to our calculations is the following: In the same way that the Coxeter distance between couples of circles in the extended plane is preserved by inversive maps [C], so is the distance between couples of (n-1)-spheres in $I\!R^n$, for $n > 2$. Moreover, there are other inversive invariants associated to couples of m-spheres in $I\!R^n$, $1 \leq m \leq n-1$, and applying this over the whole family of osculating m-spheres of a given curve in $I\!R^n$ must provide 1-forms which are inversive invariants of the curves.

2 Calculation of invariants

Let α be a smooth regular curve in $I\!R^{n+1} \cup \{\infty\}$ parametrized by arc-lenghts. We define the **generalized evolute**, $e_\alpha(s)$, of the curve α as the curve of the centers of the osculating (n-1)-spheres of α.

For instance, for $n = 3$ we have

$$e_\alpha(s) = \alpha(s) + \frac{1}{k(s)} N(s) + \frac{-k'(s)}{k(s)^2 t(s)} B(s),$$

where $k(s)$, $t(s)$ are the curvature and torsion functions and $N(s)$, $B(s)$ are the normal and binormal respectively of α at $z(s)$.

Theorem 1[R-S]

Given a curve $\alpha : I\!R \to I\!R^{n+1} \cup \{\infty\}$ parametrized by arc-length, the 1-form:

$$\omega_\alpha = \left| \frac{\|e'_\alpha(s)\| r_{n-1,\alpha}(s)}{(r_{n,\alpha})^2(s)} \right| ds,$$

where $\|e'_\alpha(s)\|$ is the speed of "generalized evolute" of α, $r_{n,\alpha}(s)$ is the radius of the osculating n-sphere of α and $r_{n-1,\alpha}(s)$ is the radius of the osculating (n-1)-sphere of α is an inversive invariant.

Remark 1

The functional coefficient of the above 1-form measures, for all n, how much the curve is far from being locally contained in a n-sphere (i.e. of being flat in the conformal sense), we call it the **conformal torsion** of α. In fact, we have that the curve α is spherical if and only if it vanishes everywhere. On the other hand, we observe that it coincides, for $n = 3$, with the conformal torsion in the sense of Fialkow as displayed by Cairns, Sharpe and Webb in [C-S-W].

Definition

We say that s is a **vertex** of α if $\|e'_\alpha(s)\| = 0$.

Notice that the vertices of α are points at which the osculating hypersphere has a contact of order at least $n + 2$ with the curve and they correspond to singular points ("cusps") of the generalized evolute. In this sense they generalize the idea of vertex of plane curves. Uribe [U] observes that these points are also the extremals of the function $r_{n,\alpha}(s)$. It follows immediately from the above theorem

Corollary

Two inversively equivalent curves in $I\!R^n$ have the same number of vertices.

More remarks Given a curve $\gamma : I\!R \to I\!R^n$, let $\tilde{\gamma} : I\!R \to S^n \to I\!R^{n+1}$, $\tilde{\gamma} = i(\xi(\gamma))$, be the image of γ throught the inverse, ξ , of the stereographic projection from $S^n - \{(0, ..., 0, 1\}$ to $I\!R^n$, considered as a curve in (n+1)-space. It can be seen that ξ "transports" contacts of γ with hyperspheres in $I\!R^n$ into contacts of $\tilde{\gamma}$ with hyperplanes in $I\!R^{n+1}$ (see [R]). As a consequence of this it follows that s_0 is a vertex of γ if and only if s_0 is a zero of the n-th metric curvature function of $\tilde{\gamma}$ (see also [U]). So in view of the above comments, we can write:

s_0 is a zero of the conformal torsion of γ if and only if s_0 is a zero of the metric torsion of $\tilde{\gamma}$.

Now, if instead of working with the family of osculating hyperspheres we consider the family of osculating circles of a curve in $I\!R^n$, similar arguments can be applied in order to get further invariants. In fact, it can be seen that there are 3 independent inversive invariants for couples of circles in $I\!R^n$, $n > 3$ and 2 of them for $n = 3$. (We thank A. Montesinos Amilibia for pointing this out to us and calculating the invariants).

In the case of $n = 3$, given two circles $C_1(c_1, r_1)$ and $C_2(c_2, r_2)$ in $I\!R^3$, where c_i, r_i and N_i are their center, their radii and their normal vector respectively, the invariants are given by

$$K_1 = (\frac{(c_2 - c_1) \cdot N_2}{r_1})^2 + (\frac{(c_1 - c_2) \cdot N_1}{r_2})^2 + (\frac{| c_1 - c_2 |^2 - {r_1}^2 - {r_2}^2}{2r_1 r_2})^2 + (N_1 \cdot N_2)^2,$$

$$\begin{aligned} K_2 = {} & ((N_1 . N_2)^2 + (\frac{(c_2 - c_1) \cdot N_2}{r_1})^2)((\frac{| c_1 - c_2 |^2 - {r_1}^2 - {r_2}^2}{2r_1 r_2})^2 + (\frac{(c_1 - c_2) \cdot N_1}{r_2})^2) \\ & -((\frac{| c_1 - c_2 |^2 - {r_1}^2 - {r_2}^2}{2r_1 r_2})(\frac{(c_2 - c_1) \cdot N_2}{r_1}) - (\frac{(c_1 - c_2) \cdot N_1}{r_2})(N_1 \cdot N_2))^2, \end{aligned}$$

where we have denoted by $A \cdot B$ the inner product of vectors A and B.

From them, we obtain the 1-form

$$\nu_\alpha = \sqrt[4]{k'^2(s) + k^2(s) t^2(s)} ds$$

which happens to be the **infinitesimal conformal arc-length** of the space curve α (see [C-S-W]).

So this seems to generalize the procedure used in [C-S] in order to obtain the infinitesimal conformal arc-lenght of plane curves.

It is interesting to observe that both expressions for the infinitesimal conformal arc-length of curves in $I\!R^2$ and $I\!R^3$ can be written in the following unifying manner

$$\nu_\alpha = \sqrt{\frac{\|c_\alpha{}'(s)\|}{(r_\alpha)^2(s)}}ds,$$

where $\|c'_\alpha(s)\|$ is the speed of the curve described by the centers of the osculating circles and $r_\alpha(s)$ is the osculating radius of α.

Based on this fact we conjecture that the inversive invariant associated to the family of osculating circles of a curve in $I\!R^n$ provides the infinitesimal conformal arc-length of the given curve, and that this can be expressed in general by the above formula.

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