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The holomorphic fiber bundle structure of some compact complex nilmanifolds

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Abstract:
Strongly non-nilpotent complex structures on compact complex nilmanifolds $\Gamma\backslash G$ are defined and characterized. As an application, a compact complex nilmanifold $\Gamma\backslash G$ with a non-nilpotent complex structure which is not strongly non-nilpotent is described as the total space at the top in a tower of holomorphic principal torus bundles. This description extends the one given in [CFGU1] for the case of $\Gamma\backslash G$ possessing a nilpotent complex structure. The problem of computing the Dolbeault cohomology of $\Gamma\backslash G$ is also discussed.

1. Introduction

A *compact nilmanifold* is a homogeneous space $\Gamma\backslash G$, where G is a simply–connected connected nilpotent Lie group and Γ a lattice of G of maximal rank. Nomizu [N] has proved that the qth de Rham cohomology group $H^q(\Gamma\backslash G)$ of $\Gamma\backslash G$ is canonically isomorphic to the Chevalley–Eilenberg cohomology group $H^q(g)$, g denoting the Lie algebra of G. A key ingredient in Nomizu's proof is the fact that $\Gamma\backslash G$ can be seen as the total space of a principal torus bundle over another compact nilmanifold.

In [CFG] the ascending central series $\{g_l\}_{l\geq 0}$ of g has been used to describe $\Gamma\backslash G$ as the total space at the top in a tower of principal torus bundles. Let us recall that each term in $\{g_l\}_{l\geq 0}$ is an ideal of g defined by

$$(1) \qquad g_0 = \{0\}, \qquad g_l = \{X \in g \mid [X, g] \subseteq g_{l-1}\}, \quad l \geq 1.$$

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Thus g_1 is the center of g, and if g is s-step nilpotent then $g_0 \neq g_1 \neq \cdots \neq g_{s-1} \neq g_s = g_l = g$, for all $l \geq s$. The tower of principal fiberings obtained in [CFG] has $s-1$ steps.

Since $\Gamma\backslash G$ is a parallelizable manifold, it is possible to define geometric structures on it in a natural way. For example, if the real dimension of G is even an almost complex structure J on $\Gamma\backslash G$ can be defined by passing to the quotient a left invariant almost complex structure on G. We say that $\Gamma\backslash G$ is a *compact complex nilmanifold* if it possesses an *integrable* almost complex structure J defined in this way. (It is worthy to remark here that there are even dimensional nilpotent Lie groups with no left invariant complex structures [FGG], and hence the compact manifolds $\Gamma\backslash G$ constructed from them are never compact complex nilmanifolds.)

Such a complex structure J on $\Gamma\backslash G$ can be thought as an endomorphism of g satisfying $J^2 = -I$ and the integrability condition given by the vanishing of its Nijenhuis tensor. But if g is not a *complex* Lie algebra then J may not leave invariant some term neither of the descending central series nor of the ascending central series of g. Therefore, if G is not a *complex* Lie group these central series cannot be used in general to obtain a description of $\Gamma\backslash G$ as a total space of a *holomorphic* principal torus bundle.

Such a fiber structure of $\Gamma\backslash G$ has been obtained by Sakane for the case of G being a complex Lie group [S]. (In this case $\Gamma\backslash G$ is *complex parallelizable* in the sense of Wang [W].) Actually, Sakane has exploited this fact to prove a result similar to Nomizu's theorem for the Dolbeault cohomology $H_{\bar\partial}^{*,*}(\Gamma\backslash G)$ of $\Gamma\backslash G$. Recently, a variation of (1) adapted to the complex structure J of G has been introduced (see [CFGU1]) in order to get a description of a larger class of compact complex nilmanifolds $\Gamma\backslash G$, namely the class of compact complex nilmanifolds whose complex structure is nilpotent (see Definition 1), as the total space at the top in a tower of holomorphic principal torus bundles. As an application, this description of $\Gamma\backslash G$ is used in [CFGU1] to extend Sakane's result for $H_{\bar\partial}^{*,*}(\Gamma\backslash G)$ to any nilpotent complex structure J on a compact nilmanifold $\Gamma\backslash G$.

In the present paper we discuss the situation not considered in [CFGU1], that is the case of J being non-nilpotent. The concept of *strongly non-nilpotent* complex structure on $\Gamma\backslash G$ is introduced (see Definition 2), and the following result is proved (Theorem 9): if $\Gamma\backslash G$ is a compact complex nilmanifold whose complex structure J stems from a non-nilpotent left invariant complex structure which is not strongly non-nilpotent, then it can be realized as the total space at the top in a tower of holomorphic principal bundles whose structure groups are all complex tori; moreover, each base space in this tower is a compact nilmanifold endowed with a non-nilpotent complex structure, and the complex structure on the base space at the bottom of the tower is always strongly non-nilpotent.

The paper is structured as follows. In Section 2 we show some properties of left invariant complex structures J on nilpotent Lie groups and, in Theorem 4, a result of [CFGU2] concerning with a necessary and sufficient condition in dimension 6 for J to be nilpotent is extended to any dimension $2n$. Section 3 is devoted to the construction of examples of compact complex nilmanifolds with non-nilpotent complex structure; moreover, these examples show that the hypothesis in Theorem 4 cannot be weakened. In Section 4, following the steps of [CFGU1], we obtain the desired fiber structure for any compact complex nilmanifold whose complex structure is not strongly non-nilpotent. As an application, the general problem of computing a model for the Dolbeault complex of any compact complex nilmanifold is reduced to solve the problem for the particular class of strongly non-nilpotent complex structures.

2. Nilpotent complex structures

In this section we study necessary and sufficient conditions for a left invariant complex structure on a s-step nilpotent Lie group G to be nilpotent. In particular, we extend to dimension $2n$ a result given in [CFGU2] for dimension 6.

Let g be the Lie algebra of G, and J a left invariant complex structure on G. In [CFGU1] it is introduced the *ascending series* $\{a_l(J)\}_{l\geq 0}$ *of* g *compatible with* J, whose terms $a_l(J)$ are defined by:

$$\begin{cases} a_0(J) = \{0\} \ , \\ a_l(J) = \{X \in g \mid [X,g] \subseteq a_{l-1}(J) \ \text{ and } \ [JX,g] \subseteq a_{l-1}(J)\} \ , \quad l \geq 1. \end{cases}$$

Notice that each term $a_l(J)$ in this series is a J-invariant ideal of g. In particular, $a_1(J)$ is the largest subspace of the center g_1 of g which is invariant under J.

Definition 1. The complex structure J is called *nilpotent* if $a_t(J) = g$ for some integer $t > 0$.

Suppose that J is nilpotent, and let t denote the first positive integer for which $a_t(J) = g$. Then it is easy to see that $\{a_l(J)\}_{l\geq 0}$ satisfies

$$0 \neq a_1(J) \neq \cdots \neq a_{t-1}(J) \neq a_t(J) = a_l(J) = g,$$

for all $l \geq t$. A characterization of a nilpotent complex structure in terms of the structure equations of the Lie group is given in the following

Theorem 1. [CFGU1] *Let G be a nilpotent Lie group of real dimension $2n$. A left invariant complex structure J on G is nilpotent if and only if there exists an ordered basis $\{\omega_i, \overline{\omega}_i; 1 \leq i \leq n\}$ of left invariant complex 1-forms on G satisfying*

$$d\omega_i = \sum_{j<k<i} A_{ijk}\, \omega_j \wedge \omega_k + \sum_{j,k<i} B_{ijk}\, \omega_j \wedge \overline{\omega}_k \ ,$$

for $1 \leq i \leq n$, where each ω_i is of type $(1,0)$ with respect to J.

The only 2-dimensional nilpotent Lie group is, up to isomorphism, the Abelian Lie group $I\!R^2$, and then any left invariant complex structure on it is nilpotent. If a s-step nilpotent Lie group G of real dimension 4 has a left invariant complex structure then s must be ≤ 2 [FGG]. Moreover, there are two nilpotent Lie groups of dimension 4 admitting left invariant complex structures: $I\!R^4$ and $H \times I\!R$, where H is the Heisenberg group.

Theorem 2. [CFGU2] *Let G be a simply–connected Lie group, of real dimension 4, nilpotent in step $s \leq 2$. Then, any left invariant complex structure on G is nilpotent.*

As the examples in Section 3 show, Theorem 2 does not hold in dimension $2n > 4$.

Let us now suppose that $a_q(J) \neq g$ for some q. Since $a_q(J)$ is an ideal of g, then $\widetilde{g}_q = g/a_q(J)$ is a nilpotent Lie algebra. Let $\widetilde{G}_q$ be the simply–connected connected nilpotent Lie group whose Lie algebra is $\widetilde{g}_q$. Since $a_q(J)$ is invariant under J, then there is an induced left invariant almost complex structure $\widetilde{J}_q$ on $\widetilde{G}_q$ defined by:

$$\widetilde{J}_q(\widetilde{X}) = \widetilde{JX}, \tag{1}$$

for $\widetilde{X} \in \widetilde{g}_q$, where $\widetilde{X}$ and $\widetilde{JX}$ denote the class of X and JX, respectively, in the quotient $\widetilde{g}_q$. Notice that $\widetilde{J}_q$ is integrable, and hence it defines a complex structure on $\widetilde{G}_q$, because $N_{\widetilde{J}_q} = \widetilde{N}_J$ and J is integrable, $N_{\widetilde{J}_q}$ and N_J denoting the Nijenhuis tensors of $\widetilde{J}_q$ and J, respectively.

Lemma 3. *With the notation given above, let $\{a_l(\widetilde{J}_q)\}_{l\geq 0}$ be the ascending series of $\widetilde{g}_q$ compatible with $\widetilde{J}_q$. Then*

$$a_l(\widetilde{J}_q) \cong \frac{a_{q+l}(J)}{a_q(J)}\ , \qquad \textit{for all } l \geq 0. \tag{2}$$

Proof.

Since (2) is obvious if $a_q(J) = 0$, assume that $0 \neq a_q(J) \neq g$. Then, (2) being trivial for $l = 0$, we proceed by induction on l. Thus, from $a_{l-1}(\widetilde{J}_q) \cong a_{q+l-1}(J)/a_q(J)$, (1) and the definition of $a_l(\widetilde{J}_q)$, we get

$$\begin{aligned} a_l(\widetilde{J}_q) &= \{\widetilde{X} \in \widetilde{g}_q \mid [\widetilde{X}, \widetilde{g}_q] \subseteq a_{l-1}(\widetilde{J}_q) \text{ and } [\widetilde{J}_q(\widetilde{X}), \widetilde{g}_q] \subseteq a_{l-1}(\widetilde{J}_q)\} \\ &= \{\widetilde{X} \in \widetilde{g}_q \mid [X, g] \subseteq a_{q+l-1}(J) \text{ and } [JX, g] \subseteq a_{q+l-1}(J)\} \\ &\cong a_{q+l}(J)/a_q(J), \end{aligned}$$

and, therefore, (2) holds for all $l \geq 0$. $\square$

Theorem 4. *Let $\Gamma\backslash G$ be a compact complex nilmanifold, of real dimension $2n > 4$, with complex structure J. If $\mathfrak{a}_{n-3}(J) \neq \mathfrak{a}_{n-2}(J)$ then J is nilpotent. Moreover, the converse holds if G is s-step nilpotent with $s > n-3$.*

Proof.

First, we note that $\mathfrak{a}_t(J) = \mathfrak{a}_{t+1}(J)$ for some $t \geq 0$ implies that $\mathfrak{a}_l(J) = \mathfrak{a}_t(J)$ for all $l \geq t$. Therefore, since $\mathfrak{a}_{n-3}(J) \neq \mathfrak{a}_{n-2}(J)$ then

$$(3) \qquad 0 \neq \mathfrak{a}_1(J) \neq \cdots \neq \mathfrak{a}_{n-3}(J) \neq \mathfrak{a}_{n-2}(J).$$

Next, since $\mathfrak{a}_l(J)$ is invariant under J then the real dimension of $\mathfrak{a}_l(J)$ is even. Hence, from (3) it follows that $\dim \mathfrak{a}_l(J) \geq 2l$, for $l = 0, \ldots, n-2$. In particular, $\dim \mathfrak{a}_{n-2}(J) \geq 2n-4$, the quotient Lie algebra $\widetilde{\mathfrak{g}}_{n-2} = \mathfrak{g}/\mathfrak{a}_{n-2}(J)$ has real dimension $\leq 2n - (2n-4) = 4$, and the induced almost complex structure $\widetilde{J}_{n-2}$ given by (1) is integrable on its corresponding Lie group $\widetilde{G}_{n-2}$. Thus, there are two possibilities:

$$\dim \widetilde{\mathfrak{g}}_{n-2} = 4 \quad or \quad \dim \widetilde{\mathfrak{g}}_{n-2} = 2.$$

If $\dim \widetilde{\mathfrak{g}}_{n-2} = 4$, there are again two possibilities:

$$\widetilde{G}_{n-2} \cong I\!R^4 \quad or \quad \widetilde{G}_{n-2} \cong H \times I\!R,$$

where H is the Heisenberg group. Therefore, from Theorem 2 it follows that any left invariant complex structure on $\widetilde{G}_{n-2}$ is nilpotent. In particular, using Lemma 3 we get $\mathfrak{a}_{n-1}(J)/\mathfrak{a}_{n-2}(J) \cong \mathfrak{a}_1(\widetilde{J}_{n-2}) \neq 0$, that is, $\mathfrak{a}_{n-2}(J) \neq \mathfrak{a}_{n-1}(J)$. If $\mathfrak{a}_{n-1}(J) = \mathfrak{g}$ then J is nilpotent on G. Otherwise, $\widetilde{\mathfrak{g}}_{n-1}$ is a Lie algebra of real dimension 2, and the induced almost complex structure $\widetilde{J}_{n-1}$ is integrable on the Lie group $\widetilde{G}_{n-1} \cong I\!R^2$. Since $\widetilde{J}_{n-1}$ is nilpotent we have from Lemma 3 that $\mathfrak{a}_n(J)/\mathfrak{a}_{n-1}(J) \cong \mathfrak{a}_1(\widetilde{J}_{n-1}) \neq 0$, that is, $\mathfrak{a}_{n-1}(J) \neq \mathfrak{a}_n(J) = \mathfrak{g}$ because $\dim \mathfrak{a}_n(J) \geq 2n = \dim \mathfrak{g}$. Therefore, the complex structure J is nilpotent on G.

If $\dim \widetilde{\mathfrak{g}}_{n-2} = 2$, then $\widetilde{G}_{n-2} \cong I\!R^2$. Since $\widetilde{J}_{n-2}$ is nilpotent, taking into account once more Lemma 3, it follows that $\mathfrak{a}_{n-1}(J)/\mathfrak{a}_{n-2}(J) \cong \mathfrak{a}_1(\widetilde{J}_{n-2}) \neq 0$. Thus, by obvious dimensional reasons, $\mathfrak{a}_{n-2}(J) \neq \mathfrak{a}_{n-1}(J) = \mathfrak{g}$ and, therefore, the complex structure J is still nilpotent on G.

Let us suppose now that J is a nilpotent complex structure on a s-step nilpotent Lie group G, with $s > n-3$. Since $\mathfrak{g}$ is not nilpotent in step $\leq n-3$ then $\mathfrak{g} \neq \mathfrak{g}_{n-3}$, where $\mathfrak{g}_{n-3}$ is the $(n-3)$-th term in the ascending central series $\{\mathfrak{g}_l\}$ of $\mathfrak{g}$. Moreover, $\mathfrak{a}_{n-3}(J) \subseteq \mathfrak{g}_{n-3}$ and therefore $\mathfrak{a}_{n-3}(J) \neq \mathfrak{g}$. The nilpotency of J insures the existence of an integer $t \geq n-2$ such that $\mathfrak{a}_t(J) = \mathfrak{g}$, i.e. $\mathfrak{a}_{n-3}(J) \neq \mathfrak{a}_t(J) = \mathfrak{g}$. In particular, $\mathfrak{a}_{n-3}(J) \neq \mathfrak{a}_{n-2}(J)$. $\square$

As a direct consequence of this theorem we get a simple necessary and sufficient condition for J to be nilpotent in (real) dimension 6 .

Corollary 5. [CFGU2] *Let $M^6 = \Gamma\backslash G$ be a compact complex nilmanifold, of (real) dimension 6, with complex structure J. Then, J is nilpotent if and only if $a_1(J) \neq 0$.*

3. Non-nilpotent complex structures

In this section we construct, for each $n \geq 3$, a nilpotent Lie group K^{2n} endowed with a non-nilpotent (left invariant) complex structure J satisfying

$$0 \neq a_1(J) \neq a_2(J) \neq \cdots \neq a_{n-4}(J) \neq a_{n-3}(J) = a_l(J) \neq g, \tag{1}$$

for all $l \geq n-3$. In particular, this family of examples shows that Theorem 4 cannot be extended to step $n-4$.

Example for $n = 3$ [AGS]:

Let K^6 be the simply–connected connected nilpotent Lie group of real dimension 6 determined by the nilpotent Lie algebra K generated by $X_1, Y_1, X_2, Y_2, X_3, Y_3$, for which the nonzero defining bracket relations are the following:

$$[X_1, X_2] = -2X_3, \quad [Y_1, X_2] = -2Y_3, \quad [X_1, X_3] = [Y_1, Y_3] = -2Y_2.$$

The ascending central series $\{K_l\}$ of K is then

$$K_1 = \{Y_2\},\ K_2 = \{Y_2, X_3, Y_3\},\ K_3 = K,$$

that is, K is 3-step nilpotent. Let J be the left invariant almost complex structure on K^6 defined by

$$JX_1 = Y_1, \quad JX_2 = Y_2, \quad JX_3 = Y_3. \tag{2}$$

It is easy to check that the Nijenhuis tensor of J vanishes, and therefore (2) defines a left invariant complex structure on K^6. Since $a_1(J) = 0$, from Corollary 5 it follows that J is non-nilpotent.

The Lie algebra K can be also described by using the exterior differential d on the dual space K^*. Since $d\alpha(X,Y) = -\alpha([X,Y])$, for $\alpha \in K^*$ and $X, Y \in K$, in terms of the basis $\{\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3\}$ of K^* dual to the basis $\{X_1, Y_1, X_2, Y_2, X_3, Y_3\}$ of K the structure equations of K^6 are:

$$\begin{cases} d\alpha_1 = d\beta_1 = d\alpha_2 = 0, \\ d\beta_2 = 2\alpha_1 \wedge \alpha_3 + 2\beta_1 \wedge \beta_3, \\ d\alpha_3 = 2\alpha_1 \wedge \alpha_2, \\ d\beta_3 = 2\beta_1 \wedge \alpha_2. \end{cases} \tag{3}$$

It is worthy to remark that, since the center K_1 of K is 1-dimensional and the term $a_1(J)$ being always the largest J-invariant subspace of K_1, we have $a_1(J) = 0$ for any left invariant complex structure J on K^6 and, therefore, any such J is non-nilpotent.

Example for $n = 4$:

Let K^8 be the simply–connected nilpotent Lie group of real dimension 8 determined by the structure equations (3) together with the equations

$$\left\{ \begin{array}{l} d\alpha_4 = \alpha_1 \wedge \alpha_3 - \beta_1 \wedge \beta_3, \\ d\beta_4 = \beta_1 \wedge \alpha_3 + \alpha_1 \wedge \beta_3. \end{array} \right. \tag{4}$$

If X_4 and Y_4 denote the dual elements of α_4 and β_4, respectively, then the Lie algebra of K^8 is generated by $X_1, Y_1, X_2, Y_2, X_3, Y_3, X_4, Y_4$, and the structure equations of the Lie group in terms of this basis are:

$$[X_1, X_2] = -2X_3,\ [Y_1, X_2] = -2Y_3,\ [X_1, Y_3] = [Y_1, X_3] = -Y_4,$$
$$[X_1, X_3] = -X_4 - 2Y_2,\ [Y_1, Y_3] = X_4 - 2Y_2.$$

Now it is easy to check that the left invariant almost complex structure J defined by

$$JX_1 = Y_1, \quad JX_2 = Y_2, \quad JX_3 = Y_3, \quad JX_4 = Y_4$$

is integrable on K^8. Moreover, $0 \neq a_1(J) = \{X_4, Y_4\} = a_l(J) \neq K$ for all $l \geq 2$. Therefore J is a non-nilpotent complex structure on K^8 satisfying (1) for $n = 4$.

Example for $n \geq 5$:

The previous example for $n = 4$ can be generalized to any $n \geq 5$ in the following way: the Lie group K^8 has been obtained by adding the equations (4) to the equations (3), that is, to the structure equations of K^6 we have added the equations

$$\left\{ \begin{array}{l} d\alpha_n = \alpha_1 \wedge \alpha_{n-1} - \beta_1 \wedge \beta_{n-1}, \\ d\beta_n = \beta_1 \wedge \alpha_{n-1} + \alpha_1 \wedge \beta_{n-1}, \end{array} \right. \tag{5}$$

with $n = 4$. In general, for an arbitrary $n \geq 5$ we construct the nilpotent Lie group K^{2n} of real dimension $2n$ by adding the equations (5) to the structure equations of the nilpotent Lie group $K^{2(n-1)}$.

Then, if $\{X_1, Y_1, \ldots, X_n, Y_n\}$ is the basis for the Lie algebra of K^{2n} dual to the basis $\{\alpha_1, \beta_1, \ldots, \alpha_n, \beta_n\}$, the almost complex structure J defined by $JX_i = Y_i$, for $i = 1, \ldots, n$, is integrable on K^{2n}. Moreover,

$$0 \neq a_1(J) = \{X_n, Y_n\} \neq a_2(J) = \{X_{n-1}, Y_{n-1}, X_n, Y_n\} \neq \cdots$$
$$\cdots \neq a_r(J) = \{X_{n-r+1}, Y_{n-r+1}, \ldots, X_n, Y_n\} \neq \cdots$$
$$\cdots \neq a_{n-3}(J) = \{X_4, Y_4, \ldots, X_n, Y_n\} = a_l(J) \neq K,$$

for all $l \geq n-2$. Therefore J is a non-nilpotent complex structure satisfying (1).

The complex structure J on the Lie group K^{2n} can be easily described in terms of the complex structure equations of K^{2n} as follows. Denote by ω_i the left invariant complex 1-form on K^{2n} of type $(1,0)$ with respect to J defined by

$$\omega_i = \alpha_i + \sqrt{-1}\,\beta_i, \quad 1 \le i \le n.$$

Then, in terms of the complex 1-forms $\omega_1, \omega_2, \omega_3$ and their conjugates, the structure equations of K^6 become

$$\left\{\begin{array}{l} d\omega_1 = 0, \\ d\omega_2 = \sqrt{-1}\,(\omega_1 \wedge \overline{\omega}_3 - \omega_3 \wedge \overline{\omega}_1), \\ d\omega_3 = \omega_1 \wedge \omega_2 + \omega_1 \wedge \overline{\omega}_2. \end{array}\right. \tag{6}$$

For the Lie group K^8 the complex structure equations are then given by (6) together with the equation $d\omega_4 = \omega_1 \wedge \omega_3$. In general, the complex structure equations of K^{2n} will be obtained by adding the equation

$$d\omega_n = \omega_1 \wedge \omega_{n-1} \tag{7}$$

to the complex structure equations of $K^{2(n-1)}$, for $n \ge 4$.

Since the complex structure on K^{2n} is non-nilpotent we know, from Theorem 1, that there does not exist any ordered basis $\{\omega_1, \ldots, \omega_n, \overline{\omega}_1, \ldots, \overline{\omega}_n\}$ of left invariant complex 1-forms on K^{2n} satisfying that $d\omega_i \in \Lambda^{*,*}(\omega_1, \ldots, \omega_{i-1}, \overline{\omega}_1, \ldots, \overline{\omega}_{i-1})$, for $1 \le i \le n$. However, (7) shows that there is "some order" in the structure equations of our examples for $n \ge 4$. This is directly related to the fact that the non-nilpotent complex structure J on K^{2n} satisfies (1) for any $n \ge 4$. In fact, proceeding in an analogous way as in the proof of Theorem 1 (see [CFGU1] for details) it can be proved the following result.

Theorem 6. *Let G be a simply-connected connected nilpotent Lie group of (real) dimension $2n$ with a left invariant complex structure J, and denote by g the Lie algebra of G. Then, $a_q(J) \ne a_{q+1}(J)$ for some $q \ge 0$ if and only if there exists an ordered basis $\{\omega_1, \ldots, \omega_n, \overline{\omega}_1, \ldots, \overline{\omega}_n\}$ of left invariant complex 1-forms on G satisfying*

$$d\omega_i = \sum_{j<k\le n-n_{l+1}} A_{ijk}\,\omega_j \wedge \omega_k + \sum_{j,k\le n-n_{l+1}} B_{ijk}\,\omega_j \wedge \overline{\omega}_k,$$

for $n - n_{l+1} + 1 \le i \le n - n_l$, and for all $0 \le l \le q$, where $2n_l = \dim a_l(J)$.

Notice that the only left invariant complex structures J which are not covered by Theorem 6 are those satisfying $a_l(J) = 0$, for all $l \ge 0$. This fact motivates the following

Definition 2. A left invariant complex structure J on a nilpotent Lie group G is called *strongly non-nilpotent* if $a_1(J) = 0$.

Equivalently, J is strongly non-nilpotent if and only if $\{0\}$ is the only J-invariant subspace of g which is contained in its center. Moreover, from Corollary 5 we have:

Corollary 7. *Any non-nilpotent complex structure on a compact nilmanifold $\Gamma\backslash G$ of (real) dimension 6 is strongly non-nilpotent.*

Remark 8. We refer the reader to [CFGU2] for the construction of a nilpotent Lie group of real dimension 10 with a left invariant complex structure J which is strongly non-nilpotent.

Finally, we must note that since the coefficients in the structure equations (3)–(5) are all rational numbers, a theorem of Mal'cev [M] guarantees the existence of a lattice Γ^{2n} of K^{2n} of maximal rank, for each $n \geq 3$. Therefore, for each $n \geq 3$, $\Gamma^{2n}\backslash K^{2n}$ is a compact complex nilmanifold whose complex structure is non-nilpotent, that is, it comes from a non-nilpotent left invariant complex structure on the Lie group K^{2n} by passing to the quotient.

4. On the holomorphic fiber bundle structure of $\Gamma\backslash G$

A compact complex nilmanifold $\Gamma\backslash G$ with nilpotent complex structure J has been described in [CFGU1] as the total space at the top in a tower of holomorphic principal torus bundles with a complex torus as the base space at the bottom of the tower. In this section we extend this study to the case of J being a non-nilpotent complex structure on $\Gamma\backslash G$, but restricting our attention to those J's which are not strongly non-nilpotent.

Let $\Gamma\backslash G$ be a compact complex nilmanifold with non-nilpotent complex structure J satisfying $a_1(J) \neq 0$, and let g be the Lie algebra of G. Since J is not strongly non-nilpotent, there is an integer $t > 0$ such that

$$(1) \qquad 0 \neq a_1(J) \neq a_2(J) \neq \cdots \neq a_{t-1}(J) \neq a_t(J) = a_l(J) \neq g,$$

for all $l \geq t$. Therefore, the series $\{a_l(J)\}$ induces the sequence of quotient Lie algebras

$$(2) \quad g \longrightarrow g/a_1(J) \longrightarrow \cdots \longrightarrow g/a_q(J) \xrightarrow{\pi_{q+1}} g/a_{q+1}(J) \longrightarrow \cdots \longrightarrow g/a_t(J),$$

where π_{q+1} is the natural surjective homomorphism, and $\ker \pi_{q+1} = a_{q+1}(J)/a_q(J)$. In fact $\pi_{q+1}\colon \widetilde{g}_q \longrightarrow \widetilde{g}_{q+1}$ and $\ker \pi_{q+1} = a_1(\widetilde{J}_q)$, according to the notation and results in Section 2. Therefore, (2) gives the following tower of quotient Lie algebras

$$
\begin{array}{rcl}
a_1(J) = a_1(\widetilde{J}_0) & \hookrightarrow & \widetilde{g}_0 = g \\
& & \downarrow \\
a_1(\widetilde{J}_1) & \hookrightarrow & \widetilde{g}_1 \\
& & \downarrow \\
& & \vdots \\
& & \downarrow \\
a_1(\widetilde{J}_{t-1}) & \hookrightarrow & \widetilde{g}_{t-1} \\
& & \downarrow \\
& & \widetilde{g}_t
\end{array} \tag{3}
$$

where $\widetilde{J}_0 = J$. Since $a_1(\widetilde{J}_q)$ is contained in the center of $\widetilde{g}_q$, it is an Abelian Lie algebra for all $0 \leq q \leq t-1$.

Let $\widetilde{G}_q$ denote the simply–connected connected nilpotent Lie group defined by $\widetilde{g}_q$. From Lemma 3 and (1) it follows that

$$
a_{t-q-1}(\widetilde{J}_q) \neq a_{t-q}(\widetilde{J}_q) = a_{t-q+l}(\widetilde{J}_q) \neq \widetilde{g}_q, \tag{4}
$$

for all $l \geq 0$ and $0 \leq q \leq t-1$. Therefore, the complex structure $\widetilde{J}_q$ on $\widetilde{G}_q$ is non-nilpotent. Moreover, from (4) we have in particular that $a_1(\widetilde{J}_q) \neq 0$, that is, $\widetilde{J}_q$ is not strongly non-nilpotent for $0 \leq q \leq t-1$. However, using again Lemma 3 and (1), we get $a_1(\widetilde{J}_t) \cong a_{t+1}(J)/a_t(J) = 0$, and therefore $\widetilde{J}_t$ is a strongly non-nilpotent complex structure on $\widetilde{G}_t$.

The Abelian Lie algebra $a_1(\widetilde{J}_q)$ with the restriction of the complex structure $\widetilde{J}_q$ becomes a complex Abelian Lie algebra for all $0 \leq q \leq t-1$. Therefore, it defines a simply–connected connected Abelian complex Lie group $A_q \cong \mathbf{C}^{n_{q+1}-n_q}$, where $2n_l = \dim a_l(J)$. Notice that $a_1(\widetilde{J}_q)$ is an ideal in $\widetilde{g}_q$, hence A_q is a closed normal Lie subgroup of $\widetilde{G}_q$ and $\widetilde{G}_{q+1} \cong \widetilde{G}_q/A_q$, for $0 \leq q \leq t-1$.

Therefore, the tower (3) induces by integration a new tower at the level of Lie groups:

$$
\begin{array}{rcl}
\mathbf{C}^{n_1} & \hookrightarrow & \widetilde{G}_0 = G \\
& & \downarrow \\
\mathbf{C}^{n_2-n_1} & \hookrightarrow & \widetilde{G}_1 \\
& & \downarrow \\
& & \vdots \\
& & \downarrow \\
\mathbf{C}^{n_t-n_{t-1}} & \hookrightarrow & \widetilde{G}_{t-1} \\
& & \downarrow \\
& & \widetilde{G}_t
\end{array} \tag{5}
$$

each $\pi_{q+1}: \widetilde{G}_q \longrightarrow \widetilde{G}_{q+1}$ in the tower being a holomorphic map; furthermore, since $[\mathfrak{a}_1(\widetilde{J}_q), \widetilde{\mathfrak{g}}_q] = 0$ there is an induced holomorphic free action of $\mathbf{C}^{n_{q+1}-n_q}$ on $\widetilde{G}_q$ for $0 \leq q \leq t-1$. Thus, the Lie group G appears as the total space at the top in this tower of holomorphic principal bundles, whose structure groups are all Abelian, where the complex structure $\widetilde{J}_q$ on $\widetilde{G}_q$, although non-nilpotent, is not strongly non-nilpotent for $0 \leq q \leq t-1$, and the complex structure $\widetilde{J}_t$ on the base space $\widetilde{G}_t$ at the bottom of (5) being always strongly non-nilpotent.

Moreover, if the real dimension of G is $2n$, then Theorem 4 implies that the number of steps in this tower is always $\leq n-3$.

Theorem 9. *Let $\Gamma\backslash G$ be a compact complex nilmanifold whose complex structure stems from a non-nilpotent (left invariant) complex structure on the Lie group G. If J is not strongly non-nilpotent then $\Gamma\backslash G$ can be described as the total space at the top in a tower of holomorphic principal bundles*

$$
\begin{array}{ccl}
\mathbb{T}^{n_1} & \hookrightarrow & \widetilde{\Gamma}_0\backslash\widetilde{G}_0 = \Gamma\backslash G \\
 & & \downarrow \\
\mathbb{T}^{n_2-n_1} & \hookrightarrow & \widetilde{\Gamma}_1\backslash\widetilde{G}_1 \\
 & & \downarrow \\
 & & \vdots \\
 & & \downarrow \\
\mathbb{T}^{n_{t-1}-n_{t-2}} & \hookrightarrow & \widetilde{\Gamma}_{t-2}\backslash\widetilde{G}_{t-2} \\
 & & \downarrow \\
\mathbb{T}^{n_t-n_{t-1}} & \hookrightarrow & \widetilde{\Gamma}_{t-1}\backslash\widetilde{G}_{t-1} \\
 & & \downarrow \\
 & & \widetilde{\Gamma}_t\backslash\widetilde{G}_t
\end{array}
\tag{6}
$$

whose structure groups are all complex tori $\mathbb{T}^{n_{l+1}-n_l}$ $(0 \leq l \leq t-1)$, where $2n_l = \dim \mathfrak{a}_l(J)$. The complex structure on each compact complex nilmanifold $\widetilde{\Gamma}_l\backslash\widetilde{G}_l$ is that obtained by passing to the quotient the non-nilpotent complex structure $\widetilde{J}_l$ on $\widetilde{G}_l$ given by (1), for $0 \leq l \leq t-1$. Moreover, the complex structure on the base space $\widetilde{\Gamma}_t\backslash\widetilde{G}_t$ at the bottom of the tower is strongly non-nilpotent. Finally, if the (real) dimension of $\Gamma\backslash G$ is $2n$ then the number of steps in the tower is $\leq n-3$.

Proof.

Since $\mathfrak{a}_1(\widetilde{J}_q)$ is contained in the center of $\widetilde{\mathfrak{g}}_q$ then the corresponding Lie subgroup A_q of $\widetilde{G}_q$ is also contained in the center of $\widetilde{G}_q$, for $0 \leq q \leq t-1$. Therefore, if for some $0 \leq q \leq t-1$ we have a lattice $\widetilde{\Gamma}_q$ in $\widetilde{G}_q$ of maximal rank then $A_q \cap \widetilde{\Gamma}_q$ is a lattice in A_q, and thus the compact nilmanifold $(A_q \cap \widetilde{\Gamma}_q)\backslash A_q \cong \mathbb{T}^{n_{q+1}-n_q}$; moreover, $A_q\widetilde{\Gamma}_q$ is closed in $\widetilde{G}_q$ and $\widetilde{\Gamma}_{q+1} = \pi_{q+1}(\widetilde{\Gamma}_q)$ is a lattice in $\widetilde{G}_{q+1}$ (see, for example, [R]). Since $(A_q \cap \widetilde{\Gamma}_q)\backslash A_q \cong \widetilde{\Gamma}_q\backslash A_q\widetilde{\Gamma}_q$, we obtain the holomorphic principal bundle $\mathbb{T}^{n_{q+1}-n_q} \hookrightarrow \widetilde{\Gamma}_q\backslash\widetilde{G}_q \xrightarrow{\pi_{q+1}} \widetilde{\Gamma}_{q+1}\backslash\widetilde{G}_{q+1}$.

Taking into account this argumentation, if Γ is a lattice of G of maximal rank and we denote $\widetilde{\Gamma}_0 \equiv \Gamma$, then $\widetilde{\Gamma}_{q+1} = \pi_{q+1}(\widetilde{\Gamma}_q)$ is a lattice in $\widetilde{G}_{q+1}$ for all $0 \leq q \leq t-1$. Therefore, replacing $\widetilde{G}_q$ by $\widetilde{\Gamma}_q\backslash\widetilde{G}_q$ $(0 \leq q \leq t)$ in the tower (5), the desired tower of holomorphic principal bundles (6) arises. □

Remark 10. Note that the results about the fiber structure of compact complex nilmanifolds in Section 2 of [CFG] must be corrected accordingly. That is, they apply only to special nilpotent complex structures J, namely those J's for which $a_l(J) = g_l$ for all $l \geq 0$.

It is well known that on an arbitrary complex manifold M there exists the so called *Dolbeault complex* of M,

$$\cdots \longrightarrow \Lambda^{p,q-1}(M) \xrightarrow{\bar{\partial}} \Lambda^{p,q}(M) \xrightarrow{\bar{\partial}} \Lambda^{p,q+1}(M) \longrightarrow \cdots, \tag{7}$$

where $\Lambda^{r,s}(M)$ denotes the space of forms of type (r,s) on M, and $\bar{\partial}$ is the operator of type $(0,1)$ in the usual decomposition $d = \partial + \bar{\partial}$ of the exterior differential d. The cohomology groups of (7) are the *Dolbeault cohomology groups* of M.

In the particular case of M being a compact complex nilmanifold $\Gamma\backslash G$, in addition to (7) there is also the complex

$$\cdots \longrightarrow \Lambda^{p,q-1}(g^{\mathbb{C}})^* \xrightarrow{\bar{\partial}} \Lambda^{p,q}(g^{\mathbb{C}})^* \xrightarrow{\bar{\partial}} \Lambda^{p,q+1}(g^{\mathbb{C}})^* \longrightarrow \cdots, \tag{8}$$

where $g^{\mathbb{C}}$ denotes the complexification of the Lie algebra g of G, and $\bar{\partial}$ is the corresponding differential of type $(0,1)$ in the canonical decomposition $d = \partial + \bar{\partial}$ of the Chevalley–Eilenberg differential d extended to $\Lambda^*(g^{\mathbb{C}})^*$. It is natural to ask if there is some relationship between the cohomology groups of (8) and the Dolbeault cohomology groups of $M = \Gamma\backslash G$.

Although the relationship between (7) and (8) has not been completely investigated, as we have mentioned at the begining a result similar to Nomizu's theorem has been obtained by Sakane [S] when the Lie group G is a *complex* Lie group. In [CFGU1] Sakane's result has been extended to the class of compact complex nilmanifolds $\Gamma\backslash G$ whose complex structure J is *nilpotent*, by proving that there is a canonical isomorphism

$$H^{p,q}_{\bar{\partial}}(\Gamma\backslash G) \cong H^{p,q}_{\bar{\partial}}(g^{\mathbb{C}}),$$

between the cohomology groups $H^{p,q}_{\bar\partial}(g^{\mathbb{C}})$ of (8) and those of (7) for $M=\Gamma\backslash G$. In fact, in [CFGU1] it is proved that the natural inclusion of differential bigraded algebras

$$i\colon (\Lambda^{*,*}(g^{\mathbb{C}})^*,\bar\partial)\longrightarrow(\Lambda^{*,*}(\Gamma\backslash G),\bar\partial)$$

induces an isomorphism at the cohomology level or, in other words, $(\Lambda^{*,*}(g^{\mathbb{C}})^*,\bar\partial)$ is a *minimal model* (in the sense of [NT]) for the Dolbeault complex of $\Gamma\backslash G$. This gives a partial answer to the following natural question:

Question:
Is $(\Lambda^{*,*}(g^{\mathbb{C}})^*,\bar\partial)$ a model for the Dolbeault complex $(\Lambda^{*,*}(\Gamma\backslash G),\bar\partial)$ of any compact complex nilmanifold $\Gamma\backslash G$?

As far as we know, there is no complete answer to this question. The result in [CFGU1] gives a positive answer only for the class of compact complex nilmanifolds $\Gamma\backslash G$ with nilpotent complex structure J, but the question remains still unsolved for the class of non-nilpotent J's. Nevertheless, the techniques developed in [CFGU1] together with the fiber structure obtained in Theorem 9 allow us to reduce the question to give an affirmative answer in the case of $\Gamma\backslash G$ being a compact complex nilmanifold with a strongly non-nilpotent complex structure.

Actually, let us suppose that $\Gamma\backslash G$ is a compact complex nilmanifold with non-nilpotent J. From Theorem 9 we have a description of $\Gamma\backslash G$ as the total space of the tower (6), where the complex structure on the space $\widetilde\Gamma_t\backslash\widetilde G_t$ at the bottom is strongly non-nilpotent. If $(\Lambda^{*,*}(\widetilde g_t^{\mathbb{C}})^*,\bar\partial)$ were a model (not necessarily minimal) for the Dolbeault complex of $\widetilde\Gamma_t\backslash\widetilde G_t$, then the iterative procedure from the bottom to the top of the tower used in Section 4.3 of [CFGU1] in the case of a nilpotent J can be easily adapted to the tower (6) in order to get that $(\Lambda^{*,*}(g^{\mathbb{C}})^*,\bar\partial)$ is a model for the Dolbeault complex of $\Gamma\backslash G$.

The main tools to be used in such iterative procedure are a Hirsch-type Lemma and a suitable principal connection on each bundle

$$\mathbb{T}^{n_{q+1}-n_q}\hookrightarrow\widetilde\Gamma_q\backslash\widetilde G_q\xrightarrow{\pi_{q+1}}\widetilde\Gamma_{q+1}\backslash\widetilde G_{q+1},\qquad for\ 0\le q\le t-1,$$

which allows to apply the mentioned Hirsch Lemma. Such suitable connection also exists in this case and is defined as follows. Since $a_q(J)\neq a_{q+1}(J)$, Theorem 6 implies in particular the existence of a basis $\{\omega_1,\ldots,\omega_{n-n_q}\}$ of left invariant complex 1-forms on $\widetilde G_q$ of type (1,0) such that

$$d\omega_i=\sum_{j<k\le n-n_{q+1}}A_{ijk}\,\omega_j\wedge\omega_k+\sum_{j,k\le n-n_{q+1}}B_{ijk}\,\omega_j\wedge\overline\omega_k,\tag{9}$$

for $n-n_{q+1}+1\le i\le n-n_q$. (For the sake of simplicity we denote with the same symbols the 1-forms $\omega_1,\ldots,\omega_{n-n_{q+1}}$ in $\widetilde g_q$ and those induced in $\widetilde g_{q+1}$; moreover, $\omega_1,\ldots,\omega_{n-n_l}$ descend to $\widetilde\Gamma_l\backslash\widetilde G_l$, $l=q$ or $q+1$, and we also denote with the same symbols the forms on the quotient.) If $\{Z_1,\ldots,Z_{n_{q+1}-n_q}\}$ is a basis of left invariant holomorphic vector fields of type (1,0) on $\mathbb{T}^{n_{q+1}-n_q}$, then

$$\omega = \sum_{i=1}^{n_{q+1}-n_q} \omega_{n-n_{q+1}+i} \otimes Z_i$$

defines a principal connection on the bundle. Notice that ω has type (1,0) and is compatible with the holomorphic structure in the bundle. Moreover, from (9) it follows that the curvature from Ω of ω on the base $\widetilde{\Gamma}_{q+1}\backslash\widetilde{G}_{q+1}$ is given by

$$\Omega = \sum_{i=1}^{n_{q+1}-n_q} \left(\sum_{j<k\leq n-n_{q+1}} A_{n-n_{q+1}+i\,j\,k}\, \omega_j \wedge \omega_k + \sum_{j,k\leq n-n_{q+1}} B_{n-n_{q+1}+i\,j\,k}\, \omega_j \wedge \overline{\omega}_k \right) \otimes Z_i.$$

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