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ON LORENTZIAN PARA-SASAKIAN MANIFOLDS

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Abstract. The present paper deals with Lorentzian para-Sasakian (briefly LP-Sasakian) manifolds with conformally flat and quasi conformally flat curvature tensor. It is shown that in both cases, the manifold is locally isometric with a unit sphere $S^n(1)$. Further it is shown that an LP-Sasakian manifold with $R(X,Y).C = 0$ is locally isometric with a unit sphere $S^n(1)$.

Introduction

In 1989, K. Matsumoto [2] introduced the notion of Lorentzian para Sasakian manifold. I. Mihai and R. Rosca [3] defined the same notion independently and thereafter many authors [4], [5] studied LP-Sasakian manifolds. In this paper, we investigate LP-Sasakian manifolds in which

$$C = 0 \tag{1}$$

where C is the Weyl conformal curvature tensor. Then we study LP-Sasakian manifolds in which

$$\widetilde{C} = 0 \tag{2}$$

where $\widetilde{C}$ is the quasi conformal curvature tensor. In both the cases, it is shown that an LP-Sasakian manifold is isometric with a unit sphere $S^n(1)$. Finally, an LP-Sasakian manifold with

$$R(X,Y).C = 0 \tag{3}$$

has been considered, where $R(X,Y)$ is considered as a derivation of the tensor algebra at each point of the manifold of tangent vectors, X, Y. It is easy to see that $R(X,Y).R = 0$ implies $R(X,Y).C = 0$. So it is meaningful to undertake the study of manifolds satisfying the condition (3). In this paper it is proved that if in a Lorentzian para-Sasakian manifold (M^n, g) $(n > 3)$ the relation (3) holds, then it is locally isometric with a unit sphere $S^n(1)$. (n has been taken > 3 because it is known that $C = 0$ when $n = 3$).

1. Preliminaries

A differentiable manifold of dimension n is called Lorentzian para-Sasakian [2], [3] if it admits a $(1,1)$-tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and a Lorentzian metric g which satisfy

$$\eta(\xi) = -1 \tag{4}$$

$$\phi^2 = I + \eta(X)\xi \tag{5}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y) \tag{6}$$

$$g(X,\xi) = \eta(X), \qquad \nabla_X \xi = \phi X \tag{7}$$

$$(\nabla_X \phi)Y = [g(X,Y) + \eta(X)\eta(Y)]\xi + [X + \eta(X)\xi]\eta(Y) \tag{8}$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g.

It can easily be seen that in an LP-Sasakian manifold the following relations hold:

$$\phi\xi = 0 \qquad \eta(\phi X) = 0 \tag{9}$$

$$\operatorname{rank}\phi = n-1. \tag{10}$$

Also, an LP-Sasakian manifold M is said to be η-Einstein if its Ricci tensor S is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) \tag{11}$$

for any vector fields X, Y where a, b are functions on M.

Further, on such an LP-Sasakian manifold with (ϕ,η,ξ,g) structure, the following relations hold [4], [5]:

$$g(R(X,Y)Z,\xi) = \eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Z) \tag{12}$$

$$R(\xi,X)Y = g(X,Y)\xi - \eta(Y)X \tag{13}$$

$$R(\xi,X)\xi = X + \eta(X)\xi \tag{14}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y \tag{15}$$

$$S(X,\xi) = (n-1)\eta(X) \tag{16}$$

$$S(\phi X,\phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y) \tag{17}$$

for any vector fields X, YZ where $R(X,Y)Z$ is the Riemannian curvature tensor.

The above results will be used in the next sections.

2. LP-Sasakian manifolds with $C = 0$

The conformal curvature tensor C is defined as

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z - \frac{1}{n-2}\{g(Y,Z)QX - g(X,Z)QY \\ &\quad + S(Y,Z)X - S(X,Z)Y\} + \frac{r}{(n-1)(n-2)}\{g(Y,Z)X - g(X,Z)Y\}, \end{aligned} \tag{18}$$

where

$$S(X,Y) = g(QX,Y).$$

Using (1) we get from (18)

$$R(X,Y)Z = \frac{1}{n-2}\{g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y\} - \frac{r}{(n-1)(n-2)}\{g(Y,Z)X - g(X,Z)Y\}. \tag{19}$$

Taking $Z = \xi$ in (18) and using (7), (15) and (16), we find

$$\eta(Y)X - \eta(X)Y = \frac{1}{n-2}\{\eta(Y)QX - \eta(X)QY\} + \frac{n-1}{n-2}\{\eta(Y)X - \eta(X)Y\} - \frac{r}{(n-1)(n-2)}\{\eta(Y)X - \eta(X)Y\}.$$

Taking $Y = \xi$ and using (4) we get

$$QX = \left(\frac{1}{n-1} - 1\right)X + \left(\frac{r}{n-1} - 1\right)\eta(X)\xi. \tag{20}$$

Thus the manifold is η-Einstein.

Contracting (20) we get

$$r = n(n-1). \tag{21}$$

Using (21) in (20) we find

$$QX = (n-1)X. \tag{22}$$

Putting (22) in (19) we get after a few steps

$$R(X,Y)Z = g(Y,Z)X - g(X,Y)Y. \tag{23}$$

Thus a conformally flat LP-Sasakian manifold is of constant curvature. The value of this constant is $+1$. Hence we can state

Theorem 1. *A conformally flat LP-Sasakian manifold is locally isometric to a unit sphere $S^n(1)$.*

3. LP-Sasakian manifolds with $\widetilde{C} = 0$

The quasi conformal curvature tensor $\widetilde{C}$ is defined as

$$C(X,Y)Z = aR(X,Y)Z + b\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\} - \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)\{g(Y,Z)X - g(X,Z)Y\} \tag{24}$$

where a, b are constants such that $ab \neq 0$ and

$$S(Y,Z) = g(QY,Z).$$

Using (2), we find from (24)

$$R(X,Y)Z = -\frac{b}{a}\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\} + \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)\{g(Y,Z)X - g(X,Z)Y\}. \tag{25}$$

Taking $Z=\xi$ in (18) and using (7), (15) and (16), we get

$$\eta(Y)X-\eta(X)Y = -\frac{b}{a}\{\eta(Y)QX-\eta(X)QY\} \left\{\frac{r}{an}\left(\frac{a}{n-1}+2b\right)-\frac{b}{a}(n-1)\right\}\{\eta(Y)X-\eta(X)Y\}. \tag{26}$$

Taking $Y=\xi$ and applying (4) we have

$$QX = \left\{\frac{r}{bn}\left(\frac{a}{n-1}+2b\right)-(n-1)-\frac{a}{b}\right\}X + \left\{\frac{r}{bn}\left(\frac{a}{n-1}+2b\right)-\frac{a}{b}-2(n-1)\right\}\eta(X)\xi. \tag{27}$$

Contracting (27), we get after a few steps

$$r=n(n-1). \tag{28}$$

Using (28) in (27), we get

$$QX=(n-1)X. \tag{29}$$

Finally, using (29), we find from (25)

$$R(X,Y)Z=g(Y,Z)X-g(X,Z)Y.$$

Thus we can state

Theorem 2. *A quasi conformally flat LP-Sasakian manifold is locally isometric with a unit sphere $S^n(1)$.*

4. LP-Sasakian manifolds satisfying $R(X,Y).C=0$

Using (7), (13) and (16) we find from (18)

$$\eta(C(X,Y)Z)=\frac{1}{n-2}\left[\left(\frac{r}{n-1}-1\right)\{g(Y,Z)\eta(X) -g(X,Z)\eta(Y)\}-\{S(Y,Z)\eta(X)-S(X,Z)\eta(Y)\}\right]. \tag{30}$$

Putting $Z=\xi$ in (30) and using (7), (16) we get

$$\eta(C(X,Y)\xi)=0. \tag{31}$$

Again, taking $X=\xi$ in (30), we get

$$\eta(C(\xi,Y)Z) = \frac{1}{n-2}\left[\{S(Y,Z)+(n-1)\eta(Y)\eta(Z)\} -\left(\frac{r}{n-1}-1\right)\{g(Y,Z)+\eta(Y)\eta(Z)\}\right]. \tag{32}$$

Now

$$(33)\quad (R(X,Y)C)(U,V)W = R(X,Y)C(U,V)W - C(R(X,Y)U,V)W - C(U,R)(X,Y)V)W - C(U,V)R(X,Y)W.$$

Using (3), we find from above

$$g[R(\xi,Y)C(U,V)W,\xi] - g[C(R(\xi,Y)U,V)W,\xi] - g[C(U,R(\xi,Y)V)W,\xi] - g[C(U,V)R(\xi,Y)W,\xi] = 0.$$

Using (7) and (13) we get

$$(34)\quad -`C(U,V,W,Y) - \eta(Y)\eta(C(U,V)W) - G(Y,U)\eta(C(\xi,V)W) + \eta(U)\eta(C(Y,V)W) - g(Y,V)\eta(C(U,\xi)W) + \eta(V)\eta(C(U,Y)W) - g(Y,W)\eta(C(U,V)\xi) + \eta(W)\eta(C(U,V)Y) = 0,$$

where

$$`C(U,V,W,Y) = g(C(U,V)W,Y).$$

Putting $U = Y$ in (34) we find

$$(35)\quad -`C(U,V,W,U) - \eta(U)\eta(C(U,V)W) + \eta(U)\eta(C(U,V)W) + \eta(V)\eta(C(U,U)W) + \eta(W)\eta(C(U,V)U) - g(U,U)\eta(C(\xi,V)W) - g(U,V)\eta(C(U,\xi)W) - g(U,W)\eta(C(U,V)\xi) = 0.$$

Let $\{e_i : i = 1,\dots,n\}$ be an orthonormal basis of the tangent space at any point, then the sum for $1 \le i \le n$ of the relations (35) for $U = e_i$ gives

$$(1-n)\eta(C(\xi,V)W) = 0$$

$$(36)\quad \eta(C(\xi,V)W) = 0 \qquad \text{as } n > 3.$$

Using (31) and (36), (34) takes the form

$$(37)\quad -`C(U,V,W,Y) - \eta(Y)\eta(C(U,V)W) + \eta(U)\eta(C(Y,V)W) + \eta(V)\eta(C(U,Y)W) + \eta(W)\eta(C(U,V)Y) = 0.$$

Using (30) in (37) we get

$$(38)\quad -`C(U,V,W,Y) + \eta(W)\frac{1}{n-2}\left[\left(\frac{r}{n-1} - 1\right)\{\eta(U)g(V,Y) - \eta(V)g(U,Y)\} - \{\eta(U)S(V,Y) - \eta(V)S(U,Y)\}\right] = 0.$$

In virtue of (36), (32) reduces to

$$(39)\quad S(Y,Z) = \left(\frac{r}{n-1}1\right)g(Y,Z) + \left(\frac{r}{n-1} - n\right)\eta(Y)\eta(Z).$$

Using (39), (37) reduces to

$$(40)\quad -`C(U,V,W,Y) = 0,$$

i.e.

$$C(U,V)W = 0. \tag{41}$$

Hence the manifold is conformally flat. Using Theorem 1, we state

Theorem 3. *If in an LP-Sasakian manifold M^n $(n > 3)$ the relation $R(X,Y).C = 0$ holds, then it is locally isometric with a unit sphere $S^n(1)$.*

For a conformally symmetric Riemannian manifold [1], we have $\nabla C = 0$. Hence for such a manifold $R(X,Y).C = 0$ holds. Thus we have the following corollary of the above theorem:

Corollary 1. *A conformally symmetric LP-Sasakian manifold M^n $(n > 3)$ is locally isometric with a unit sphere $S^n(1)$.*

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