

**SYMMETRIC SELF-ADJUNCTIONS:
A JUSTIFICATION OF BRAUER'S REPRESENTATION OF
BRAUER'S ALGEBRAS ***

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A classic result of representation theory is Brauer's construction of a diagrammatical (geometrical) algebra whose matrix representation is a certain given matrix algebra, which is the commutating algebra of the enveloping algebra of the representation of the orthogonal group. The purpose of this paper is to provide a motivation for this result through the categorical notion of symmetric self-adjunction.

1. Introduction

In [1] Richard Brauer introduced a class of diagrammatical algebras and found a matrix representation for them. These algebras have been in focus again after more than fifty years since Temperley-Lieb algebras, the subalgebras of Brauer's algebras, have started to play an important role in knot theory and low-dimensional topology via the polynomial approach to knot invariants (see [14], [18] and [19]). Brauer's algebras arose as a side product of investigations concerning a representation of the orthogonal group $\mathcal{O}(n)$, and the representation of these algebras was established following Brauer's

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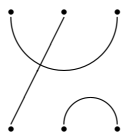
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remark that if one associates (in a natural way) diagrams to matrices from a particular class, then the product of these matrices corresponds to the operation of “composition” of such diagrams. We give more details concerning Brauer’s introduction of these algebras in Section 2. So, for a given matrix algebra, Brauer had constructed a diagrammatical algebra whose matrix representation turned out to be this matrix algebra. We find such a representation insufficiently justified.

On the other hand, there are several results concerning diagrammatical characterization of various kinds of free adjunctions and related notions from category theory (see for example [3], [5], [6], [10] and [17], [11], [13]). Combining the fact that the symmetric self-adjunction freely generated by a singleton set of objects is isomorphic to the category of Brauer’s diagrams and the fact that a symmetric self-adjunction exists in the skeleton of the category of finite dimensional vector spaces over a field \mathcal{F} , one can find a matrix representation of Brauer’s diagrams that coincides with Brauer’s representation. We find this is a natural justification of Brauer’s representation.

2. Brauer’s algebras and their representation

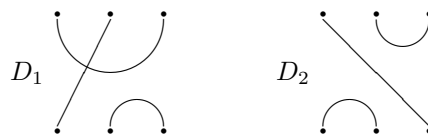
For every $n \in \mathbf{N}^+$, Brauer’s algebra B_n over a field \mathcal{F} of characteristic 0 is a vector space whose basis consists of $(2n - 1)!!$ diagrams, which we call *Brauer’s n -diagrams* or just *n -diagrams*. Every n -diagram consists of n vertices in the *top row* and n vertices in the *bottom row*. Each of these $2n$ vertices is connected by a *thread* with exactly one of the remaining $2n - 1$ vertices. For example,



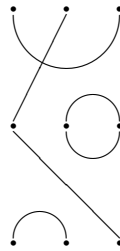
is a 3-diagram.

So, addition and multiplication by scalars is formal in B_n and as a vector space, B_n is isomorphic to $\mathcal{F}^{(2n-1)!!}$. For the structure of algebra in B_n it is sufficient to define multiplication of n -diagrams. (We call this multiplication *composition* and denote it by \circ .) To define the n -diagram $D_2 \circ D_1$ for two n -diagrams D_1 and D_2 , we have to identify the bottom row of D_1 with the top row of D_2 so that the top row of D_1 becomes the top row of $D_2 \circ D_1$ and the bottom row of D_2 becomes the bottom row of $D_2 \circ D_1$.

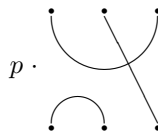
The threads of $D_2 \circ D_1$ are obtained by concatenating the threads of D_1 and D_2 . The number $k \geq 0$ of *circular components* that may occur in this procedure reflects in the scalar p^k which multiplies the resulting n -diagram (p is here a fixed positive integer and the choice to represent a circle in a diagram by multiplying the rest of the diagram by p is forced by the matrix algebra in which Brauer represented B_n). For example, let D_1 and D_2 be the following 3-diagrams:



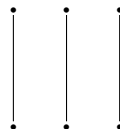
After identification of the bottom row of D_1 with the top row of D_2 we have



and $D_2 \circ D_1$ is the following element of B_3



Let I_n be the n -diagram in which for every $i \in \{1, \dots, n\}$ we have that the i -th vertex from the top row is connected with the i -th vertex from the bottom row. For example I_3 is



It is pretty obvious that I_n is the unit for composition and associativity of composition is straightforward when we rely on such an informal (pictorial)

definition of \circ . For a formal proof of associativity of \circ one may consult [7] and [8]. We explain below how these algebras arose in the work of Brauer.

Let \mathcal{F} be a field of characteristic 0 and let \mathcal{G} be $\mathcal{O}(p)$ (group of orthogonal linear transformations of the p -dimensional vector space \mathcal{F}^p over \mathcal{F}). Every member of \mathcal{G} is given by an orthogonal $p \times p$ matrix G ($G^{-1} = G^T$) with entries from \mathcal{F} .

Brauer was particularly interested in the following representation of \mathcal{G} :

$$M(G) = G^{\otimes n} = \underbrace{G \otimes \dots \otimes G}_{n\text{-fold Kronecker product}} \in \text{End}(\mathcal{F}^{p^n}).$$

(That M is a representation, i.e. that $M(G_1G_2) = M(G_1)M(G_2)$, follows from the functoriality of \otimes .)

Let \mathcal{M} be the group $\{M(G) \mid G \in \mathcal{G}\}$ and let \mathcal{A} be the *enveloping algebra* of \mathcal{M} , i.e.,

$$\mathcal{A} = \{c_1M(G_1) + \dots + c_kM(G_k) \mid k \in \mathbf{N}^+, c_i \in \mathcal{F}, G_i \in \mathcal{G}, 1 \leq i \leq k\}.$$

Brauer's goal was to characterize elements of this algebra. For this purpose he used the *commutating algebra* \mathcal{B} of \mathcal{A} , i.e.,

$$\mathcal{B} = \{B \in \text{End}(\mathcal{F}^{p^n}) \mid (\forall A \in \mathcal{A}) AB = BA\}.$$

The algebra \mathcal{B} is a matrix algebra whose elements are $p^n \times p^n$ matrices of the form

$$[b_{i,j}]_{p^n \times p^n} = \begin{bmatrix} b_{1,1} & b_{1,2} & \dots & b_{1,p^n} \\ b_{2,1} & b_{2,2} & \dots & b_{2,p^n} \\ \dots & \dots & \dots & \dots \\ b_{p^n,1} & b_{p^n,2} & \dots & b_{p^n,p^n} \end{bmatrix}$$

with entries from \mathcal{F} .

We explain now some technical notation that we are going to use below. There are p^n functions from $\{1, \dots, n\}$ to $\{1, \dots, p\}$. Each of these functions can be envisaged as a sequence of length n of elements of $\{1, \dots, p\}$. The set of these sequences may be ordered lexicographically so that $(1, 1, \dots, 1)$ is the first and (p, \dots, p) is the last (p^n -th) in this ordering. We use this ordering to identify the elements of $\{1, \dots, p^n\}$ with the functions from $\{1, \dots, n\}$ to $\{1, \dots, p\}$. So, for $i \in \{1, \dots, p^n\}$ and $k \in \{1, \dots, n\}$, by $i(k) \in \{1, \dots, p\}$ we mean the image of k by the i -th function from $\{1, \dots, n\}$ to $\{1, \dots, p\}$.

Let

$$u^i = \begin{bmatrix} u_1^i \\ \vdots \\ u_p^i \end{bmatrix} \quad \text{and} \quad v^j = \begin{bmatrix} v_1^j \\ \vdots \\ v_p^j \end{bmatrix}$$

for $i, j \in \{1, \dots, n\}$ be $2n$ vectors of \mathcal{F}^p and let \mathcal{G} acts on them, so that for $G \in \mathcal{G}$ we have $(u^i)' = Gu^i$ and $(v^j)' = Gv^j$. Then the following holds.

Theorem 2.1 (Brauer) *The function*

$$J(u^1, \dots, u^n, v^1, \dots, v^n) = \sum_{1 \leq i, j \leq p^n} b_{i,j} u_{i(1)}^1 \cdots u_{i(n)}^n v_{j(1)}^1 \cdots v_{j(n)}^n$$

is an invariant of \mathcal{G} (i.e., $J(u^1, \dots, v^n) = J((u^1)', \dots, (v^n)')$) iff the matrix $[b_{i,j}]_{p^n \times p^n}$ belongs to \mathcal{B} .

From the main theorem of invariant theory concerning the orthogonal group case (see [20], Chapter II, Section A.9) it follows that

$$J(u^1, \dots, u^n, v^1, \dots, v^n) = \sum_{1 \leq i, j \leq p^n} b_{i,j} u_{i(1)}^1 \cdots u_{i(n)}^n v_{j(1)}^1 \cdots v_{j(n)}^n$$

is an invariant of \mathcal{G} iff $J(u^1, \dots, u^n, v^1, \dots, v^n)$ is a linear combination of products of scalar products of the form

$$(w^1 w^2) \cdot (w^3 w^4) \cdots (w^{2n-1} w^{2n})$$

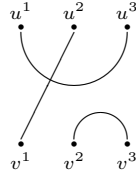
where $w^1 w^2 \cdots w^{2n}$ is a permutation of vectors $u^1 \cdots u^n v^1 \cdots v^n$.

Up to commutativity of multiplication and scalar product there are $(2n-1)!!$ different terms of this form. It is natural to associate with every such term an n -diagram. In this diagram the vertices from the top row represent the vectors u^1, \dots, u^n and vertices from the bottom row represent the vectors v^1, \dots, v^n . Then every thread of the diagram shows which pairs of vectors occur in the scalar products of the term. However, this correspondence is a bijection (up to commutativity of multiplication and scalar product). This means that starting from an arbitrary n -diagram one can find a term of the above form representing an invariant of \mathcal{G} from which a matrix from \mathcal{B} can be extracted. In this way Brauer obtained a function that maps n -diagrams to matrices from \mathcal{B} .

Brauer remarked that the result of composition of two n -diagrams is mapped to the product of matrices corresponding to these diagrams. This

is the core of Brauer’s representation of Brauer’s algebra since, by linearity, the above correspondence between n -diagrams and matrices can be extended to a representation of B_n in a unique way.

We illustrate this representation by an example in which $p = 2$, $n = 3$ and D is the 3-diagram:



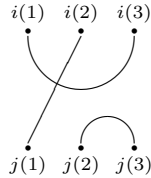
Then the following term corresponds to D

$$(u^1 u^3) \cdot (u^2 v^1) \cdot (v^2 v^3) = (u_1^1 \cdot u_1^3 + u_2^1 \cdot u_2^3) \cdot (u_1^2 \cdot v_1^1 + u_2^2 \cdot v_2^1) \cdot (v_1^2 \cdot v_1^3 + v_2^2 \cdot v_2^3).$$

After distributions at the right-hand side of this equation we obtain a term of the form

$$\sum_{1 \leq i, j \leq 8} b_{i,j} u_{i(1)}^1 u_{i(2)}^2 u_{i(3)}^3 v_{j(1)}^1 v_{j(2)}^2 v_{j(3)}^3$$

where $b_{ij} = 1$ iff $i(1) = i(3)$, $i(2) = j(1)$ and $j(2) = j(3)$, otherwise $b_{ij} = 0$. Roughly speaking, $b_{ij} = 1$ if and only if i as a ternary sequence of elements of $\{1, 2\}$ above j as a ternary sequence of elements of $\{1, 2\}$ as in the picture below



is ready to “accept” D , in the sense that linked elements of $\{1, 2\}$ are equal.

The Temperley-Lieb algebra TL_n is a subalgebra of B_n whose basis consists of non-intersecting n diagrams. The number of such diagrams is $(2n)!/(n!(n + 1)!)$ (the n -th Catalan number). It is proved in [12] (see also [4]) that the restriction to TL_n of Brauer’s representation of B_n is faithful for $p \geq 2$, which means that this representation is an embedding of Temperley-Lieb algebras into $\text{End}(\mathcal{F}^{p^n})$. However, this cannot always be the case for Brauer’s representation of B_n since $(2n - 1)!!$ as the dimension of B_n may exceed p^{2n} as the dimension of $\text{End}(\mathcal{F}^{p^n})$.

We are going now to generalize the notion of Brauer’s n -diagram in the sense that we allow different number of vertices in its top and bottom row. So, let an m - n -diagram be a diagram like Brauer’s n -diagram save that

it has m vertices in the top row and n vertices in the bottom row for m not necessarily equal to n . Then we can take instead of just n -diagrams for a particular $n \in \mathbf{N}^+$, the class of m - n -diagrams for all $m, n \in \mathbf{N}$ and define the composition of an m - n -diagram and an n - q -diagram analogously to what we had for two n -diagrams. So the result of this composition is an m - q -diagram multiplied by a scalar of the form p^k which reflects the number k of circular components that arise after concatenating the threads of these diagrams. In this way we obtain the category Br_p whose objects are natural numbers and whose arrows are m - n -diagrams with coefficients of the form p^k for fixed $p \geq 1$. In [9], Section 2.3, the category Br related to Br_p (case $p = 1$) is defined in a more formal way. One can call this generalization of B_n a *categorification* of multiplicative submonoids of Brauer's algebras generated out of the basis. We shall see in the following section that the category Br_p is strongly connected to the notion of symmetric self-adjunction of [10] in a way that it may be called the *geometry* of symmetric self-adjunctions.

All the above shows that Brauer's representation of Brauer's algebras "works". But one may still ask why does it work? Or, what mathematics underlies this representation? We try to answer these questions in the following sections.

3. Symmetric self-adjunctions

In the hierarchy of categorial notions, one of the topmost positions is reserved for the notion of adjunction. This notion can be defined equationally in the following manner: an *adjunction* is a 6-tuple $\langle \mathcal{A}, \mathcal{B}, F, G, \varphi, \gamma \rangle$ where \mathcal{A} and \mathcal{B} are categories; $F: \mathcal{B} \rightarrow \mathcal{A}$ and $G: \mathcal{A} \rightarrow \mathcal{B}$ are functors; $\varphi: FG \rightarrow 1_{\mathcal{A}}$ and $\gamma: 1_{\mathcal{B}} \rightarrow GF$ are natural transformations such that the following *triangular equations* hold in \mathcal{A} and \mathcal{B} respectively

$$\begin{aligned}\varphi_{FB} \circ F\gamma_B &= 1_{FB}, \\ G\varphi_A \circ \gamma_{GA} &= 1_{GA}.\end{aligned}$$

The definition above is equational in the sense that it is possible to present the notions of category, functor and natural transformation equationally. Such an equational definition guarantees the existence of some free structures on which we rely in this section.

In [3], Section 4.10.1, an m - n -diagram is associated to every canonical arrow f of an adjunction. These diagrams are called the *set of links* of f and are denoted by $\Lambda(f)$. For example,

$$\Lambda(\varphi_A) \text{ is } \begin{array}{c} F G A \\ \frown \\ A \end{array} \quad \text{and} \quad \Lambda(\gamma_B) \text{ is } \begin{array}{c} B \\ \frown \\ G F B \end{array}$$

and $\Lambda(g \circ f) = \Lambda(g) \circ \Lambda(f)$ (where \circ on the right-hand side denotes the composition of m - n -diagrams). We illustrate the soundness of Λ with the first of the triangular equations; this yields the following picture:

$$\begin{array}{ccc} & F B & \\ \Lambda(F\gamma_B) \swarrow & \frown & \Lambda(1_{FB}) \downarrow \\ & F G F B & \\ \Lambda(\varphi_{FB}) \swarrow & \frown & \\ & F B & \end{array}$$

As we have already mentioned, the equational definition of adjunction guarantees the existence of the adjunction freely generated by a pair of sets of objects (discrete categories). Then Λ gives rise to functors from both categories involved in this freely generated adjunction to the category Br_p . It is proved in [3], Proposition in Section 4.10.1, that both of these functors are faithful. However, not all the m - n -diagrams are covered by the arrows of freely generated adjunction. It is easy to see that all the m - n -diagrams corresponding to these arrows are of Temperley-Lieb kind (they are non-intersecting diagrams) and even not all the diagrams of the Temperley-Lieb kind are covered by this correspondence. This was a motivation for a step leading from the notion of adjunction to a more specific notion of self-adjunction (see [5], [6] and references therein, see also [10] for a more gradual introduction of this notion). A *self-adjunction* (also called \mathcal{L} -adjunction in [5]) may be introduced as a quadruple $\langle \mathcal{A}, F, \varphi, \gamma \rangle$ such that $\langle \mathcal{A}, \mathcal{A}, F, F, \varphi, \gamma \rangle$ is an adjunction. (So, $F : \mathcal{A} \rightarrow \mathcal{A}$ is an endofunctor adjoint to itself.)

As in the case of adjunction, one may construct the self-adjunction freely generated by an arbitrary set of objects. Then Λ , as before, gives rise to a functor from the category involved in this freely generated self-adjunction to the category Br_p . This time, the functor is not faithful because a simple counting of circular components that occur in compositions of m - n -diagrams is not sufficient. The faithfulness of this functor requires some adjustments in the category Br_p ; namely, one must take into account not just the number of circular components, but also their positions in the diagram (See [5] for the definition of friezes and \mathcal{L} -equivalence between them). It is shown in [5] that the arrows of freely generated self-adjunction cover

by Λ all the diagrams of the Temperley-Lieb kind.

Since all the intersecting m - n -diagrams are still out of the range of Λ we can make a step forward, to arrive at the notion of symmetric self-adjunction, which is defined as follows. A *symmetric self-adjunction* is a quintuple $\langle \mathcal{A}, F, \varphi, \gamma, \chi \rangle$ for which $\langle \mathcal{A}, F, \varphi, \gamma \rangle$ is a self-adjunction, χ is a natural transformation from $F \circ F$ to $F \circ F$ such that the equations

$$\begin{aligned} \chi_A \circ \chi_A &= 1_{FFA}, & \chi_{FA} \circ F\chi_A \circ \chi_{FA} &= F\chi_A \circ \chi_{FA} \circ F\chi_A, \\ \varphi_A \circ \chi_A &= \varphi_A, & \chi_A \circ \gamma_A &= \gamma_A, \\ \varphi_{FA} \circ F\chi_A &= F\varphi_A \circ \chi_{FA}, & \chi_{FA} \circ F\gamma_A &= F\chi_A \circ \gamma_{FA}. \end{aligned}$$

are satisfied.

If we extend Λ to cover all the canonical arrows of a symmetric self-adjunction by defining $\Lambda(\chi_A)$ to be



then Λ gives rise to a functor from the category involved in the symmetric self-adjunction freely generated by a set of objects, to the category Br_p . It is shown in [10] that this functor is faithful and, moreover, if the set of generating objects is a singleton, then this functor is an isomorphism.

4. Symmetric self-adjunction of the category $\mathbf{Mat}_{\mathcal{F}}$

In this section we discuss an example of symmetric self-adjunction. To find such an example we start with the category $\mathbf{Vect}_{\mathcal{F}}$ of vector spaces over the field \mathcal{F} . For a given vector space $V \in \mathbf{Vect}_{\mathcal{F}}$, the functor $F: \mathbf{Vect}_{\mathcal{F}} \rightarrow \mathbf{Vect}_{\mathcal{F}}$ which acts on objects as $FU = V \otimes U$ has the right adjoint $G: \mathbf{Vect}_{\mathcal{F}} \rightarrow \mathbf{Vect}_{\mathcal{F}}$ which maps a vector space W to the vector space of all linear transformations from V to W .

If we replace $\mathbf{Vect}_{\mathcal{F}}$ by $\mathbf{Vect}_{\mathcal{F}}^{fd}$, the compact closed category of finite dimensional vector spaces over \mathcal{F} (for the notion of compact closed category see [15], [16] or [2]), then the right adjoint of $F = V \otimes _$ becomes $V^* \otimes _$, where V^* is the dual vector space of V . Since V and V^* are isomorphic in $\mathbf{Vect}_{\mathcal{F}}^{fd}$ we have that this isomorphism leads to the isomorphism of functors $V \otimes _$ and $V^* \otimes _$. So, $V \otimes _$ becomes a self-adjoint functor. It is easy to see that the natural transformation χ (symmetry) with all the required equations is present in $\mathbf{Vect}_{\mathcal{F}}^{fd}$ with $F = V \otimes _$ and, hence, we obtain an example of symmetric self-adjunction.

We can simplify the category $\mathbf{Vect}_{\mathcal{F}}^{fd}$ by passing to its skeleton $\mathbf{Mat}_{\mathcal{F}}$ (a full subcategory of $\mathbf{Vect}_{\mathcal{F}}^{fd}$ such that each object of $\mathbf{Vect}_{\mathcal{F}}^{fd}$ is isomorphic to exactly one object of $\mathbf{Mat}_{\mathcal{F}}$). The category $\mathbf{Mat}_{\mathcal{F}}$ still provides an example of symmetric self-adjunction. We can envisage this category as the category whose objects are natural numbers (the dimensions of finite dimensional vector spaces) and an arrow $M: m \rightarrow n$ is an $n \times m$ matrix with entries from \mathcal{F} . Composition of such arrows becomes matrix multiplication.

For every $p \in N$, the functor $p \otimes _$, which maps an object n of $\mathbf{Mat}_{\mathcal{F}}$ to the product $m \cdot n$ and an arrow M of $\mathbf{Mat}_{\mathcal{F}}$ to the Kronecker product $I_p \otimes M$, is the part of a symmetric self-adjunction $\langle \mathbf{Mat}_{\mathcal{F}}, p \otimes _, \varphi, \gamma, \chi \rangle$ for some indexed sets φ, γ and χ of matrices.

Suppose now that p is equal to the p we used in the definition of composition of m - n -diagrams. We have the following picture in which \mathcal{K} is the category of the symmetric self-adjunction freely generated by a single object,

$$Br_p \xrightarrow{\cong} \mathcal{K} \xrightarrow{R} \mathbf{Mat}_{\mathcal{F}}$$

and R is the functor which strictly preserves the structure of symmetric self-adjunction, and which extends the function that maps the unique generator of \mathcal{K} to the object 1 of $\mathbf{Mat}_{\mathcal{F}}$. The functor R exists by the freedom of \mathcal{K} .

The above composition of functors (which we also denote by R) has the following properties: for every pair of m - n -diagrams D_1 and D_2 we have that $R(D_2 \circ D_1) = R(D_2) \cdot R(D_1)$ (by the functoriality of R) and $R(p^k \cdot D_1) = p^k \cdot R(D_1)$. These properties show that R can serve as a core for a matrix representation of Brauer's algebras. It is not difficult to check that this representation coincides with Brauer's representation, which is now properly justified via symmetric self-adjunctions.

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