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TRANSFERRING FOURIER MULTIPLIERS ON ADMISSIBLE SPACES

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To José Guadalupe, in memoriam

Abstract. Proved is a theorem that allows to transfer Fourier multipliers from the discrete to the continuous case in spaces more general than the Lebesgue spaces, for instance the weighted Lorentz spaces.

1. Introduction and statement of the result

The essential aim of this note is to furnish a generalization of the following theorem proved by Igari.

Theorem ([6]). *Let $1 < p < \infty$ and assume that m is a bounded function on $\mathbb{R}^d$, continuous except on a set of Lebesgue measure zero. If $\{m(\varepsilon n)\} \in M_p(\mathbb{Z}^d)$ for all sufficiently small $\varepsilon > 0$ and $\liminf_{\varepsilon\to 0^+} \|m(\varepsilon n)\|_{M_p(\mathbb{Z}^d)} < \infty$, then $m \in M_p(\mathbb{R}^d)$ and*

$$\|m\|_{M_p(\mathbb{R}^d)} \le \liminf_{\varepsilon\to 0^+} \|m(\varepsilon n)\|_{M_p(\mathbb{Z}^d)}.$$

This theorem is a well-known converse to a multiplier restriction result of de Leeuw, [8]. Here, $M_p(\mathbb{Z}^d)$ and $M_p(\mathbb{R}^d)$ denote the spaces of L^p-Fourier multipliers on $\mathbb{Z}^d$ and $\mathbb{R}^d$, respectively, and $\|m_n\|_{M_p(\mathbb{Z}^d)}$ or $\|m\|_{M_p(\mathbb{R}^d)}$ denote the multiplier norms of m_n or m, that is the norms of relevant operators associated to m_n or m that act on $L^p((-\pi,\pi)^d)$ or $L^p(\mathbb{R}^d)$.

Igari's result was generalized in several directions, cf. [7], [3], [1], [2], [5], mainly to orthogonal expansions other than the trigonometric system. The generalization we consider in this note deals back with Fourier multipliers but in the spaces more general than the Lebesgue spaces L^p. We tried to formulate this generalization in an abstract, to some extent, form, but an immediate application of our main result, Theorem 1.5, gives a transference result for Fourier multipliers in weighted Lorentz spaces, cf. Section 3 for details. Due to this "abstract form" a straightforward argument from the proof of the aforementioned Igari's paper, [6], does not apply to the present situation. More sophisticated argument from [7] does, however, the job. We use it in the proof of our result.

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In the sequel we fix $d \in \mathbb{N}$ and use a fairly standard notation. Thus: $\mathcal{D} = \mathcal{D}(\mathbb{R}^d)$ is the space of all C^∞ functions with compact support; given $f \in L^1(\mathbb{R}^d)$, its Fourier transform is defined by

$$\hat{f}(x) = \int_{\mathbb{R}^d} f(y) e^{-ixy}\, dy$$

and its inverse transform by

$$\check{f}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} f(y) e^{ixy}\, dy;$$

also, given $f \in L^1(\mathbb{T}^d)$, its Fourier coefficients are defined by

$$\hat{f}(k) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(y) e^{-iky}\, dy, \qquad k \in \mathbb{Z}^d,$$

and the Fourier series associated to f is the series

$$f(x) \sim \sum_{k \in \mathbb{Z}^d} \hat{f}(k) e^{ikx}.$$

Throughout the paper we will consequently use the following convention: given an object T on $\mathbb{R}^d$, by $\tilde{T}$ we will denote its "periodic" counterpart, i.e. a "corresponding" object on $\mathbb{T}^d$. Thus, if $\mathcal{M}$ denotes the linear space of measurable functions on $\mathbb{R}^d$ (two functions are identified when they differ on a set of Lebesgue measure zero) then $\widetilde{\mathcal{M}}$ is the linear space of (equivalence classes of) measurable functions on $\mathbb{T}^d$. The family of dilations δ_λ, $\lambda > 0$, on functions on $\mathbb{R}^d$ is given by $\delta_\lambda \varphi(x) = \varphi(\lambda x)$. For functions φ, ψ, we write $\langle \varphi, \psi \rangle$ to denote the integral $\int_{\mathbb{R}^d} \varphi(x)\psi(x)\, dx$ whenever it makes sense.

Definition 1.1. *We call a quasi-normed space $(X, \|\cdot\|_X)$ admissible if*

(a) *X, as a subset, is contained in $\mathcal{M}$;*

(b) *$(X, \|\cdot\|_X)$ is δ-homogeneous in the sense that for every $\lambda > 0$, δ_λ maps X into X and*

$$\|\delta_\lambda f\|_X = \lambda^A \|f\|_X, \qquad \lambda > 0,\ f \in X,$$

for a homogeneity constant $A \in \mathbb{R}$;

(c) *$\mathcal{D} \subset X$.*

We will consider a fixed family $\{h_r\}_{r>1}$ of elements of $\mathcal{D}$ satisfying

$$\text{(1)} \qquad 0 \le h_r \le 1, \quad h_r = 1 \quad \text{on} \quad [-\pi, \pi]^d, \qquad \operatorname{supp} h_r \subset [-\pi - 1/r, \pi + 1/r]^d$$

On the torus $\mathbb{T}^d \sim [-\pi, \pi)^d$, (with the usual identification of the boundary points) we consider the space of all C^∞ functions, $\widetilde{\mathcal{D}} = \mathcal{D}(\mathbb{T}^d)$, which are identified with 2π-periodic, C^∞ functions on $\mathbb{R}^d$ (2π-periodicity means $\varphi(x + 2\pi k) = \varphi(x)$, $x \in \mathbb{R}^d$, $k = (k_1, k_2, \ldots, k_d) \in \mathbb{Z}^d$).

Definition 1.2. *Given an admissible space $(X, \|\cdot\|_X)$ we define $(\widetilde{X}, \|\cdot\|_{\widetilde{X}})$ by*

$$\widetilde{X} = \{g \in \widetilde{\mathcal{M}} : g = f\chi_{\mathbb{T}^d} \text{ for an } f \in X\}$$

and

$$\|g\|_{\widetilde{X}} = \inf\{\|f\|_X : g = f\chi_{\mathbb{T}^d}\}.$$

It may be checked that $\widetilde{X}$ equipped with $\|\cdot\|_{\widetilde{X}}$ becomes a quasi-normed space. Moreover, $\widetilde{\mathcal{D}} \subset \widetilde{X}$ and

$$\|g\|_{\widetilde{X}} \leq \|g\|_X$$

for any $g \in \mathcal{D}$ with $\operatorname{supp} g \subset (-\pi, \pi)^d$.

Definition 1.3. *Let $(Y, \|\cdot\|_Y)$ be an admissible space. We say $(Y, \|\cdot\|_Y)$ satisfies the property (W_1) provided that for every $g \in \widetilde{\mathcal{D}}$ and the family $\{h_r\}_{r>1} \subset \mathcal{D}$ that satisfies (1)*

$$(W_1) \qquad \|g\|_{\widetilde{Y}} = \lim_{r\to\infty} \|h_r g\|_Y,$$

where the last g is treated as a 2π-periodic function on $\mathbb{R}^d$. We say $(Y, \|\cdot\|_Y)$ satisfies the property (W_2) provided that from every sequence $\{y_j\}_{j\in\mathbb{N}} \subset \mathcal{D}$, which is uniformly bounded in Y, $\sup_j \|y_j\|_Y \leq R < \infty$, one can choose a subsequence $\{y_{j_i}\}$ and an element $y \in Y \cap L^1_{\mathrm{loc}}(\mathbb{R}^d)$ such that $\|y\|_Y \leq R$ and, for every $\varphi \in \mathcal{D}$,

$$(W_2) \qquad \lim_{i\to\infty} \langle y_{j_i}, \varphi\rangle = \langle y, \varphi\rangle.$$

Definition 1.4. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be admissible spaces and assume $\mathcal{D}$ lies densely in X. We call a bounded measurable function $m(x)$ on $\mathbb{R}^d$ an (X, Y)-Fourier multiplier provided for every $\varphi \in \mathcal{D}$, $(m\hat{\varphi})\check{} \in Y$ and*

$$|(m\hat{\varphi})\check{}\,|_Y \leq C\|\varphi\|_X$$

with a constant $C \geq 0$ independent of $\varphi \in \mathcal{D}$. The least constant C for which the above holds is then defined to be the multiplier norm of $m(x)$ and is denoted $\|m(x)\|_{\mathrm{mult}(X,Y)}$. In the same way a bounded sequence $\{m_k\}_{k\in\mathbb{Z}^d}$ is called an $(\widetilde{X}, \widetilde{Y})$-Fourier multiplier provided for every $\varphi \in \widetilde{\mathcal{D}}$

$$\Big\| \sum_{k\in\mathbb{Z}^d} m_k \hat{\varphi}(k) e^{ikx} \Big\|_{\widetilde{Y}} \leq C\|\varphi\|_{\widetilde{X}}.$$

As before the least C is then called the multiplier norm of m_k and is denoted $\|m_k\|_{\mathrm{mult}(\widetilde{X},\widetilde{Y})}$.

Note that the last series is absolutely convergent for every $x \in \mathbb{R}^d$, defines an element of $\widetilde{\mathcal{D}}$, hence it also belongs to $\widetilde{Y}$ (let us agree to use spherical summation to sum the above series and those that follow). Due to the assumption on the density of $\mathcal{D}$ in X the operator $\varphi \to (m\hat{\varphi})\check{}$, initially defined on $\mathcal{D}$, can be extended in a unique way to a bounded operator from X to Y. The same remark applies to the operator $\varphi \to \sum m(k)\hat{\varphi}(k)e^{ikx}$, initially defined on $\widetilde{\mathcal{D}}$ provided $\widetilde{\mathcal{D}}$ lies densely in $\widetilde{X}$.

Theorem 1.5. *Assume $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are two admissible spaces with the δ-homogeneity constants A and B respectively, $\mathcal{D}$ lies densely in X, and a bounded function $m(x)$ on $\mathbb{R}^d$, continuous except on a set of Lebesgue measure zero, is such that for all sufficiently small $\varepsilon > 0$ the sequence $\{m(\varepsilon k)\}_{k\in\mathbb{Z}^d}$ is an $(\widetilde{X}, \widetilde{Y})$-Fourier multiplier and, moreover,*

$$L = \liminf_{\varepsilon\to 0^+} \varepsilon^{B-A} \|m(\varepsilon k)\|_{\mathrm{mult}(\widetilde{X},\widetilde{Y})} < \infty.$$

Assume, in addition, that Y *satisfies the properties* (W_1) *and* (W_2). *Then* $m(x)$ *is an* (X,Y)*-Fourier multiplier and* $\|m(x)\|_{\mathrm{mult}(X,Y)} \le L$.

2. Proof of the theorem

Fix $g \in \mathcal{D}$ and consider λ's sufficiently large; in particular so large that $\delta_\lambda g \in \widetilde{\mathcal{D}}$. Take the dilation $\delta_\lambda g$ against the multiplier sequence $\{m(k/\lambda)\}$ and write

$$\|\sum m(k/\lambda)\widehat{\delta_\lambda g}(k)e^{ikx}\|_{\widetilde{Y}} \le \|m(k/\lambda)\|_{\mathrm{mult}(\widetilde{X},\widetilde{Y})}\|\delta_\lambda g\|_{\widetilde{X}}$$
$$\le \lambda^A\|m(k/\lambda)\|_{\mathrm{mult}(\widetilde{X},\widetilde{Y})}\|g\|_X.$$

Since $m(x)$ is bounded, $|m(x)| \le K$, by Parseval's identity,

$$\|\sum m(k/\lambda)\widehat{\delta_\lambda g}(k)e^{ikx}\|_{L^2(\mathbb{T}^d)} \le K\lambda^{-d/2}\|g\|_{L^2(\mathbb{R}^d)}.$$

Letting to denote

$$G_\lambda(x) = \sum m(k/\lambda)\widehat{\delta_\lambda g}(k)e^{ikx}$$

we rewrite the above as

$$\|G_\lambda(x)\|_{\widetilde{Y}} \le \lambda^A\|m(k/\lambda)\|_{\mathrm{mult}(\widetilde{X},\widetilde{Y})}\|g\|_X$$

and

$$\|G_\lambda(x)\|_{L^2(\mathbb{T}^d)} \le K\lambda^{-d/2}\|g\|_{L^2(\mathbb{R}^d)}.$$

Since, by assumption, Y and, evidently, $L^2(\mathbb{R}^d)$ both satisfy (W_1), for every $\eta > 0$, given λ sufficiently large we can find $r = r(\lambda) > \lambda$ such that

$$\|h_{r(\lambda)}(x)G_\lambda(x)\|_Y \le (1+\eta)\|G_\lambda\|_{\widetilde{Y}}$$

and

$$\|h_{r(\lambda)}(x)G_\lambda(x)\|_{L^2(\mathbb{R}^d)} \le (1+\eta)\|G_\lambda\|_{L^2(\mathbb{T}^d)}.$$

Consequently,

$$\|h_{r(\lambda)}(x)G_\lambda(x)\|_Y \le (1+\eta)\lambda^A\|m(k/\lambda)\|_{\mathrm{mult}(\widetilde{X},\widetilde{Y})}\|g\|_X$$

and

$$\|h_{r(\lambda)}(x)G_\lambda(x)\|_{L^2(\mathbb{R}^d)} \le (1+\eta)K\lambda^{-d/2}\|g\|_{L^2(\mathbb{R}^d)},$$

and the dilation property next gives

$$\|h_{r(\lambda)}(x/\lambda)G_\lambda(x/\lambda)\|_Y \le (1+\eta)\lambda^{A-B}\|m(k/\lambda)\|_{\mathrm{mult}(\widetilde{X},\widetilde{Y})}\|g\|_X$$

and

$$\|h_{r(\lambda)}(x/\lambda)G_\lambda(x/\lambda)\|_{L^2(\mathbb{R}^d)} \le (1+\eta)K\|g\|_{L^2(\mathbb{R}^d)}.$$

Choosing a sequence $0 < \lambda_1 < \lambda_2 < \dots$, $\lambda_j \to \infty$, such that

$$\lambda_j^{A-B}\|m(k/\lambda_j)\|_{\mathrm{mult}(\widetilde{X},\widetilde{Y})} \le (1+\eta)L$$

and letting to denote

$$F_{\lambda_j}(x) = h_{r(\lambda_j)}(x/\lambda_j)G_{\lambda_j}(x/\lambda_j)$$

allows to write

$$\|F_{\lambda_j}\|_Y \le (1+\eta)^2L\|g\|_X, \qquad j \in \mathbb{N},$$

and

$$\|F_{\lambda_j}\|_{L^2(\mathbb{R}^d)} \le (1+\eta)K\|g\|_{L^2(\mathbb{R}^d)}, \qquad j \in \mathbb{N}.$$

In what follows, from a given sequence we will frequently choose a subsequence. To avoid multiplication of subscripts let us agree to denote the resulting subsequence in the same way as the sequence we started from. We believe this agreement will not lead to a confusion.

Since Y satisfies (W_2) one can choose a subsequence of $\{\lambda_j\}$ and an element $F \in Y$ such that

$$\|F\|_Y \le (1+\eta)^2 L\|g\|_X \tag{2}$$

and, for every $\varphi \in \mathcal{D}$,

$$\langle F_{\lambda_j}, \varphi\rangle = \int_{\mathbb{R}^d} F_{\lambda_j}(x)\varphi(x)\,dx \to \langle F, \varphi\rangle$$

with $j \to \infty$. Next, by weak compactness of a ball of arbitrary radius in $L^2(\mathbb{R}^d)$, from $\{\lambda_j\}$ one can choose a subsequence and an element $F_o \in L^2(\mathbb{R}^d)$ such that $\|F_o\|_{L^2(\mathbb{R}^d)} \le (1+\eta)K\|g\|_{L^2(\mathbb{R}^d)}$ and, for every $\psi \in L^2(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} F_{\lambda_j}(x)\psi(x)\,dx \to \langle F_o, \psi\rangle \tag{3}$$

with $j \to \infty$. From the above it follows that $F = F_o \in L^2(\mathbb{R}^d)$. Relying solely on (3), the weak convergence of F_{λ_j} to F_o in $L^2(\mathbb{R}^d)$, we will now show that

$$F_o = (m\hat{g})\check{}\,. \tag{4}$$

This, the fact that $F = F_o$ and arbitrariness of $\eta > 0$ in (2) then show

$$\|(m\hat{g})\check{}\|_Y \le L\|g\|_X$$

and finish the proof of the theorem.

Let's start proving (4). For any given $N = 1, 2, \dots$ we will use the decomposition

$$F_\lambda(x) = h_{r(\lambda)}(x/\lambda)\Bigg(\sum_{|k|\le N[\lambda]} + \sum_{|k|>N[\lambda]}\Bigg) m(k/\lambda)\widehat{\delta_\lambda g}(k)e^{ikx/\lambda} = F_\lambda^N(x) + R_\lambda^N(x), \tag{5}$$

where $|k| = (k_1^2 + \cdots + k_d^2)^{1/2}$ and $[\cdot]$ denotes the greatest integer function. We start with estimating $\|R_\lambda^N\|_{L^2(\mathbb{R}^d)}$. Applying the identity

$$\widehat{\delta_\lambda g}(k) = -(\lambda/|k|)^2(\delta_\lambda(\Delta g))\hat{}(k),$$

where Δ denotes the Laplacean on $\mathbb{R}^d$, and using Parseval's identity, gives

$$\begin{aligned}
\int_{\mathbb{R}^d} |R_\lambda^N(x)|^2\,dx &= \int_{\mathbb{R}^d} h_{r(\lambda)}(x/\lambda)^2 \cdot \Big| \sum_{|k|>N[\lambda]} m(k/\lambda)\widehat{\delta_\lambda g}(k)e^{ikx/\lambda}\Big|^2\,dx \\
&= \lambda^d \int_{\mathbb{R}^d} h_{r(\lambda)}(y)^2 \cdot \Big| \sum_{|k|>N[\lambda]} m(k/\lambda)\widehat{\delta_\lambda g}(k)e^{iky}\Big|^2\,dy \\
&\le (3\lambda)^d \int_{\mathbb{T}^d} \Big| \sum_{|k|>N[\lambda]} m(k/\lambda)\widehat{\delta_\lambda g}(k)e^{iky}\Big|^2\,dy \\
&\le K^2(3\lambda)^d(2\pi)^d \sum_{|k|>N[\lambda]} |\widehat{\delta_\lambda g}(k)|^2 \\
&\le \frac{K^2(3\lambda)^d}{N^4}(2\pi)^d \sum_{k\in\mathbb{Z}^d} |(\delta_\lambda(\Delta g))\widehat{\ }(k)|^2 \\
&\le \frac{K^2 3^d}{N^4}\|\Delta g\|_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

Hence, we conclude that

$$\|R_\lambda^N\|_{L^2(\mathbb{R}^d)} = O(N^{-2}), \tag{6}$$

uniformly in $\lambda \to \infty$. By invoking the diagonal argument we can find a subsequence of $\{\lambda_j\}$ such that for every $N \in \mathbb{N}$, $\{R_{\lambda_j}^N\}$ is weakly convergent to a function R^N in $L^2(\mathbb{R}^d)$. Clearly, $\|R^N\|_{L^2(\mathbb{R}^d)} = O(N^{-2})$, hence, for an increasing sequence $\{N_n\}_{n\in\mathbb{N}}$ of positive integers, $\{R^{N_n}\}_{n\in\mathbb{N}}$ converges to zero a.e. on $\mathbb{R}^d$. By defining $F^{N_n} = F_o - R^{N_n}$, $n \in \mathbb{N}$, it is clear that $F_{\lambda_j}^{N_n} \to F^{N_n}$ weakly in $L^2(\mathbb{R}^d)$ for every $n \in \mathbb{N}$. Moreover, $\{F^{N_n}\}_{n\in\mathbb{N}}$ converges to F_o a.e. on $\mathbb{R}^d$. In a moment we will check that

$$\lim_{j\to\infty} F_{\lambda_j}^{N_n}(x) = (2\pi)^{-d} \int_{\|y\|\le N_n} m(y)\hat{g}(y)e^{ixy}\,dy \tag{7}$$

for every $x \in \mathbb{R}^d$. Having this, first note, that for fixed $n \in \mathbb{N}$ the sequence $\{F_{\lambda_j}^{N_n}\}_{j\in\mathbb{N}}$ is uniformly bounded on $\mathbb{R}^d$:

$$\begin{aligned}
|F_{\lambda_j}^{N_n}(x)| &\le \frac{K}{\lambda_j^d} \sum_{|k|\le N_n[\lambda_j]} |\hat{g}(k/\lambda_j)| \le \frac{CK}{\lambda_j^d} \sum_{|k|\le N_n[\lambda_j]} (|k|/\lambda_j)^{-2} \\
&\le \frac{CK}{\lambda_j^{d-2}} \sum_{|k|\le N_n[\lambda_j]} |k|^{-2} \le C' N_n^{d-2}.
\end{aligned}$$

Therefore, when integrating the sequence $\{F_{\lambda_j}^{N_n}\}_{j\in\mathbb{N}}$ over any rectangle $P = [r_1, s_1] \times \cdots \times [r_d, s_d]$, $-\infty < r_i < s_i < \infty$, using the Lebesgue dominated convergence theorem is possible. On the other hand, by weak convergence in $L^2(\mathbb{R}^d)$,

$$\int_P F_{\lambda_j}^{N_n}(x)\,dx \to \int_P F^{N_n}(x)\,dx,$$

when $j \to \infty$. This, combined with (7), shows that for a.e. $x \in \mathbb{R}^d$

$$F^{N_n}(x) = (2\pi)^{-d} \int_{\|y\| \le N_n} m(y)\hat{g}(y)e^{ixy}\, dy$$

and letting $n \to \infty$, (4) will then follow. Consequently, the proof of Theorem 1.5 will be finished.

We now return to (7). Fix $N \in \mathbb{N}$ and $x \in \mathbb{R}^d$. For large λ

$$F_\lambda^N(x) = \frac{1}{(2\pi\lambda)^d} \sum_{|k| \le N[\lambda_j]} m(k/\lambda)\hat{g}(k/\lambda)e^{ikx/\lambda}$$

tends with $\lambda \to \infty$ to $(2\pi)^{-d} \int_{\|y\| \le N} m(y)\hat{g}(y)e^{ixy}\, dy$ since, by assumption made on $m(y)$, the function under the last integral is Riemann integrable on the region $\{y \in \mathbb{R}^d : \|y\| \le N\}$.

3. Examples and comments

Let m_f^α denote the distribution function of an $f \in \mathcal{M}$ with respect to the measure $d\mu_\alpha(x) = \|x\|^\alpha\, dx$ (dx means the Lebesgue measure on $\mathbb{R}^d$),

$$m_f^\alpha(t) = \mu_\alpha(\{x \in \mathbb{R}^d : |f(x)| > t\}), \qquad t > 0.$$

It is easily seen that the weighted Lorentz spaces $L_\alpha^{p,q}(\mathbb{R}^d)$, $0 < p < \infty$, $0 < q \le \infty$, $\alpha > -d$, that consist of those $f \in \mathcal{M}$ for which the quantity

$$\|f\|_{p,q;\alpha} = \begin{cases} \left(\int_0^\infty \left[t m_f^\alpha(t)^{1/p}\right]^q \frac{dt}{t}\right)^{1/q}, & q < \infty, \\ \sup_{0<t<\infty} \left[t m_f^\alpha(t)^{1/p}\right], & q = \infty, \end{cases}$$

is finite, is an admissible space with the homogeneity constant $-(d+\alpha)/p$. Moreover, $L_\alpha^{p,q}(\mathbb{R}^d)$ satisfies the (W_1) property, $\mathcal{D}$ is dense in $L_\alpha^{p,q}(\mathbb{R}^d)$ if $q < \infty$, and, when $1 < p < \infty$, $1 < q < \infty$, $L_\alpha^{p,q}(\mathbb{R}^d)$ satisfies the (W_2) property. The last statement is easily explained by the fact that the quasi-norm $\|\cdot\|_{p,q;\alpha}$ is then equivalent to a norm and with this norm $L_\alpha^{p,q}(\mathbb{R}^d)$ becomes a reflexive Banach space (with the above assumptions on p and q, the dual space to $L_\alpha^{p,q}(\mathbb{R}^d)$ can be identified with $L_\alpha^{p',q'}(\mathbb{R}^d)$, $p' = p/(p-1)$, $q = q/(q-1)$). Needless to say, that if $X = L_\alpha^{p,q}(\mathbb{R}^d)$ then $\widetilde{X} = L_\alpha^{p,q}(\mathbb{T}^d)$ and $\|\cdot\|_{\widetilde{X}}$ coincides with the usual quasi-norm on $L_\alpha^{p,q}(\mathbb{T}^d)$ given now in terms of

$$\widetilde{m}_f^\alpha(t) = \widetilde{\mu}_\alpha(\{x \in \mathbb{T}^d : |f(x)| > t\}).$$

The result from the proposition that follows (under stronger assumptions and with additional restrictions) has been used in an outline of proof of a result stated in p.267 of [4]. In the case of $1 < p_2 < \infty$, $1 < q_2 < \infty$ the proposition is a direct consequence of Theorem 1.5. Since the (unweighted) weak-L^p case, that is the case of $L^{p_2,\infty}$, $1 \le p_2 < \infty$, is important for possible applications, we decided to include it into consideration.

Proposition 3.1. *Let $m(x)$ be a bounded function on $\mathbb{R}^d$, continuous except on a set of Lebesgue measure zero. Let $0 < p_1 < \infty$, $0 < q_1 < \infty$, $\alpha > -d$ and either $1 < p_2 < \infty$ and $1 < q_2 < \infty$ or $\alpha = 0$, $1 \le p_2 < \infty$ and $q_2 = \infty$.*

Assume further that for all sufficiently small $\varepsilon > 0$ *the sequence* $\{m(\varepsilon k)\}_{k\in\mathbb{Z}^d}$ *is an* $(L^{p_1,q_1}_\alpha(\mathbb{T}^d), L^{p_2,q_2}_\alpha(\mathbb{T}^d))$*-Fourier multiplier and, moreover,*

$$L = \liminf_{\varepsilon\to 0^+} \varepsilon^{(d+\alpha)(1/p_1-1/p_2)}\|m(\varepsilon k)\|_{\mathrm{mult}} < \infty.$$

Then $m(x)$ *is an* $(L^{p_1,q_1}_\alpha(\mathbb{R}^d), L^{p_2,q_2}_\alpha(\mathbb{R}^d))$*-Fourier multiplier and* $\|m(x)\|_{\mathrm{mult}} \le L$*.*

Proof. Consider the additional case $1 \le p_2 < \infty$ and $q_2 = \infty$. The idea of proof that follows belongs to Connett and Schwartz and the proof itself can be easily read off from their paper [3], cf. also [1]. We include it here only for a sake of completeness. Let us also add that using the (W_2) property is now simply replaced by applying Fatou's lemma. To simplify the notation, in what follows we write μ (the Lebesgue measure) in place of μ_0 and $L^{p,q}$ in place of $L^{p,q}_0$.

Fix $g \in \mathcal{D}$ and $t > 0$. By assumption, for λ's sufficiently large,

$$\begin{aligned} t\mu(\{x\in\mathbb{T}^d : |\textstyle\sum m(k/\lambda)\widehat{\delta_\lambda g}(k)e^{ikx}| > t\})^{1/p_2} &\le \|m(k/\lambda)\|_{\mathrm{mult}}\|\delta_\lambda g\|_{L^{p_1,q_1}(\mathbb{T}^d)} \\ &\le \lambda^{-d/p_1}\|m(k/\lambda)\|_{\mathrm{mult}}\|g\|_{L^{p_1,q_1}(\mathbb{R}^d)}. \end{aligned}$$

This gives

$$t\mu(\{x\in\mathbb{T}^d : |F_\lambda(x)| > t\})^{1/p_2} \le \lambda^{-d/(1/p_1-1/p_2)}\|m(k/\lambda)\|_{\mathrm{mult}}\|g\|_{L^{p_1,q_1}(\mathbb{R}^d)},$$

where

$$F_\lambda(x) = \chi_{\mathbb{T}^d}(x/\lambda)\sum m(k/\lambda)\widehat{\delta_\lambda g}(k)e^{ikx/\lambda}.$$

We now choose a sequence $0 < \lambda_1 < \lambda_2 < \dots$, $\lambda_j \to \infty$, and a function $F_o \in L^2(\mathbb{R}^d)$ such that

$$t\mu(\{x\in\mathbb{R}^d : |F_{\lambda_j}(x)| > t\})^{1/p_2} \le (L + 1/j)\|g\|_{L^{p_1,q_1}(\mathbb{R}^d)}, \tag{8}$$

and $\{F_{\lambda_j}\}_{j\in\mathbb{N}}$ converges weakly to $F_o \in L^2(\mathbb{R}^d)$. The argument for the proof that $F_o = (m\hat{g})\check{}$ is the same as in the proof of (4) in Section 2. Thus, it remains only to check that

$$t\mu(\{x\in\mathbb{R}^d : |F_o(x)| > t\})^{1/p_2} \le L\|g\|_{L^{p_1,q_1}(\mathbb{R}^d)}. \tag{9}$$

We will use again the decomposition $F_\lambda(x) = F^N_\lambda(x) + R^N_\lambda(x)$ from the proof of Theorem 1.5, cf. Section 2. Arguing as before we have to our disposal a subsequence of $\{\lambda_j\}_{n\in\mathbb{N}}$ (let us recall that the agreement concerning avoiding multiplication of subscripts is still valid!), an increasing sequence $\{N_n\}_{n\in\mathbb{N}}$ of positive integers, functions $F^{N_n} \in L^2(\mathbb{R}^d)$ and the decomposition

$$F_{\lambda_j}(x) = F^N_{\lambda_j}(x) + R^N_{\lambda_j}(x)$$

that satisfy:

- F^{N_n} converges to F_o a.e., $n \to \infty$;
- for every $n = 1, 2, \dots$, $F^{N_n}_{\lambda_j}$ converges to F^{N_n} a.e., $j \to \infty$;
- $\|R^{N_n}_{\lambda_j}\|_{L^2(\mathbb{R}^d)} = O(N_n^{-2})$, uniformly in $j = 1, 2, \dots$.

We use the above properties to show (9). Fix $\eta > 0$. Fatou's lemma then gives

$$\mu(\{|F_o(x)| > t\})^{1/p_2} \leq \liminf_{n\to\infty} \mu(\{|F^{N_n}(x)| > t\})^{1/p_2}.$$

Hence, for a subsequence of $\{N_n\}$,

$$\mu(\{|F_o(x)| > t\})^{1/p_2} \leq \mu(\{|F^{N_n}(x)| > t\})^{1/p_2} + \eta. \tag{10}$$

Fix $n = 1, 2, \ldots$. Fatou's lemma again gives

$$\mu(\{|F^{N_n}(x)| > t\})^{1/p_2} \leq \liminf_{j\to\infty} \mu(\{|F^{N_n}_{\lambda_j}(x)| > t\})^{1/p_2}.$$

Hence, for a subsequence of $\{\lambda_j\}$,

$$\mu(\{|F^{N_n}(x)| > t\})^{1/p_2} \leq \mu(\{|F^{N_n}_{\lambda_j}(x)| > t\})^{1/p_2} + \eta. \tag{11}$$

By invoking the diagonal argument, we can assume that (11) holds for every $n, j \in \mathbb{N}$. Combining (10) and (11) then gives

$$\mu(\{|F_o(x)| > t\})^{1/p_2} \leq \mu(\{|F^{N_n}_{\lambda_j}(x)| > t\})^{1/p_2} + 2\eta. \tag{12}$$

We now have

$$\begin{aligned} \mu(\{|F^{N_n}_{\lambda_j}(x)| > t\})^{1/p_2} &= \mu(\{|F_{\lambda_j}(x) - R^{N_n}_{\lambda_j}(x)| > t\})^{1/p_2} \\ &\leq \mu(\{|F_{\lambda_j}(x)| > t(1-\eta)\})^{1/p_2} + \mu(\{|R^{N_n}_{\lambda_j}(x)| > t\eta\})^{1/p_2}. \end{aligned} \tag{13}$$

By Chebyshev's inequality

$$\mu(\{|R^{N_n}_{\lambda_j}(x)| > t\eta\}) \leq \|R^{N_n}_{\lambda_j}\|^2_{L^2(\mathbb{R}^d)}/(t\eta)^2.$$

Hence $\mu(\{|R^{N_n}_{\lambda_j}(x)| > t\eta\})$ can be made arbitrarily small for sufficiently large n, uniformly in $j = 1, 2, \ldots$. Let

$$\mu(\{|R^{N_n}_{\lambda_j}(x)| > t\eta\})^{1/p_2} \leq \eta \tag{14}$$

for $n \geq n_o$ and $j = 1, 2, \ldots$. Combining (12), (13) and (14) now gives

$$\mu(\{|F_o(x)| > t\})^{1/p_2} \leq \mu(\{|F_{\lambda_j}(x)| > t(1-\eta)\})^{1/p_2} + 3\eta.$$

By using arbitrariness of η and (8) and letting $j \to \infty$ shows (9) and finishes the proof of Proposition 3.1. □

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