Proc. Summer School on Diff. Geometry
Dep. de Matemática, Universidade de Coimbra
September 1999, 59-66.

A Matrix Lie Group of Carnot Type for Filiform Sub-Riemannian Structures and its Application to Control Systems in Chained Form[1]

Claudio Altafini[2]

Optimization and Systems Theory, Royal Institute of Technology
SE-10044, Stockholm, Sweden
e-mail: altafini@math.kth.se

Abstract. A Carnot group G is a simply connected graded nilpotent Lie group endowed a left-invariant distribution generating the Lie algebra $\mathfrak{g}$ of G. Here we show that the quotient manifold of a filiform Carnot group by the subgroup generated by its characteristic line field is projectively abelian. The result is used to show how a class of bilinear control systems have an intrinsic linear behavior.

1. Introduction and motivation

The kinematic model of an n-trailer i.e. a car-like vehicle pulling an arbitrary number of trailers is a typical example of a control system evolving on the mixed bundle as the vector fields corresponding to the control inputs of the system coexist with one forms corresponding to nonholonomic constraints that describe the assumption of rolling without slipping of the wheels, see [2, 12]. Systems like the n-trailer can be thought of as rank 2 distributions in n-dimensional space. Assume that the n-dimensional configuration space of the system is a smooth manifold M and that the two inputs of the system take values in the space $\mathcal{U}(\mathbb{R})$ of smooth functions over $\mathbb{R}$. Such a system is controllable and its Lie algebra is solvable. In [7] it

[1]This is an original research article and no version has been submitted for publication elsewhere.

[2]This work was supported by the Swedish Foundation for Strategic Research through the Center for Autonomous Systems at KTH.

was shown that it is also *differentially flat*, in the language of differential algebra [11]. Roughly speaking, a differentially flat system is "equivalent" to a linear system. Understanding this equivalence is the scope of this paper. Using the Lie-Palais theorem, we consider the Lie group of diffeomorphisms generated by the flows of the distribution and construct on it a suitable homogeneous space. The system induced on this homogeneous space is linear and the corresponding projection can be thought of as time elimination.

2. Chained form and rank 2 sub-Riemannian structures

In order to characterize the algebraic properties of a system which is differentially flat it is convenient to transform it into a form in which it looks simpler although not linear. The canonical form used for rank 2 distributions in n-dimensional space is a bilinear form called *chained form* [14]

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_2 u_1 \\ &\vdots \\ \dot{x}_n &= x_{n-1} u_1 . \end{aligned} \tag{1}$$

The distribution generated by the two input vector fields is then

$$\mathcal{D} = span \left\{ \begin{bmatrix} 1 \\ 0 \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\}.$$

It is possible to calculate explicitly the diffeomorphic transformation that, together with a static invertible change of feedback, transforms the n-trailer system mentioned above into (1), see [15].

The chained form can be considered as a sub-Riemannian structure i.e. as a triple

$$(M,\ \mathcal{D},\ \langle\cdot,\ \cdot\rangle) \tag{2}$$

where M is a smooth n-dimensional manifold, $\mathcal{D}$ is a rank 2 left-invariant distribution on M (generating the Lie algebra $\mathfrak{g}$ of M) and $\langle\cdot,\ \cdot\rangle$ is an inner product on $\mathcal{D}$ inducing a left-invariant Riemannian metric on $\mathfrak{g}$. We consider here only regular points in $p \in M$ i.e. such that the growth vector is constant in a neighborhood of p and the *minimal growth vector* case. This is equivalent to say that there exists a canonical flag of distributions

$$\mathcal{D} \subset \mathcal{D}^3 \subset \mathcal{D}^4 \subset \ldots \subset \mathcal{D}^n \tag{3}$$

having at p growth vector

$$\eta(p,\ \mathcal{D}) = \{2,\ 3,\ 4 \ldots n-1,\ n\}.$$

The filtration of $\mathcal{D}^i$ is calculated as

$$\begin{aligned} \mathcal{D}^{i+1} &= \mathcal{D}^i + \left[\mathcal{D},\, \mathcal{D}^i\right] \\ \mathcal{D}^2 &= \mathcal{D}. \end{aligned}$$

A Lie algebra with such a structure is $n-1$ step *graded nilpotent* and is called *filiform Lie algebra* in [8]. It can also be characterized by its lower (or descending) central series:

$$\mathfrak{g}_1 \supset \mathfrak{g}_3 \supset \mathfrak{g}_4 \supset \ldots \supset \mathfrak{g}_n \supset \mathfrak{g}_{n+1} = 0 \tag{4}$$

where

$$\begin{aligned} \mathfrak{g}_1 &= \mathfrak{g} \\ \mathfrak{g}_3 &= [\mathfrak{g},\, \mathfrak{g}] \\ \mathfrak{g}_{i+1} &= [\mathfrak{g},\, \mathfrak{g}_i] \qquad i = 3,\, 4,\, \ldots n \end{aligned}$$

and the position of subindex 2 is intentionally left empty as will be clarified below. The unconventional notation for the descending central series is meant to help keeping track of the dimensions of the corresponding subspaces. Since the level of bracketing needed to span the whole tangent space is maximal for given n, the minimal growth vector case can be intended as the *maximal nonabelian* situation one can encounter [1].

Such a sub-Riemannian system represents an higher order contact manifold and has been studied in different contexts leading to canonical structures that are called *Goursat normal form* [3] in the theory of exterior differential system or to their dual, the above described *chained forms* in the control theory literature. Classification results for these systems are reported in [5, 18].

A graded nilpotent Lie algebra admits as a vector space the following decomposition:

$$\mathfrak{g} = V \oplus V_3 \oplus V_4 \oplus \ldots \oplus V_r$$

where the generating subspace V is equipped with the operation:

$$[V\ V_i] = V_{i+1}.$$

The V_i can be expressed as $V_i = \mathcal{D}^i(p)/\mathcal{D}^{i-1}(p)$.

2.1. Characteristic line field

A system in Goursat normal form admits a line distribution [4]. Similarly to the Engel case, [13], for filiform distributions at a regular point p we can consider the map

$$\begin{aligned} \varphi\ :\ \mathcal{D}(p) &\rightarrow \mathfrak{gl}\left([\mathfrak{g},\mathfrak{g}]\right) \\ v &\mapsto \mathrm{ad}_v\big|_{[\mathfrak{g},\mathfrak{g}]}. \end{aligned}$$

Such map is onto but not 1-1; $\ker(\varphi) = \{v \in \mathcal{D} \mid \mathrm{ad}_v\xi = 0\ \forall \xi \in [\mathfrak{g},\mathfrak{g}]\}$ gives a one-dimensional subspace at each point p. Call $\mathfrak{g}_2$ the $(n-1)$-dimensional union of $\ker(\varphi)$ and $[\mathfrak{g},\mathfrak{g}]$ at g.

Its complementary subspace is called the *characteristic line field* $\mathcal{L} \subset \mathcal{D}$ of the graded sub-Riemannian structure and it is defined up to a group automorphism. Such a construction will be clear once we have introduced a matrix representation of $\mathfrak{g}$. In other words, we can prolong the flag (4) of a extra element $\mathfrak{g}_2$ of dimension $n-1$

$$\mathfrak{g}_1 \supset \mathfrak{g}_2 \supset \mathfrak{g}_3 \supset \ldots \supset \mathfrak{g}_n$$

such that, together with the generating condition $\mathcal{D} \oplus \mathfrak{g}_3 = \mathfrak{g}$ we have also the other semi-direct sum $\mathcal{L} \oplus_\pi \mathfrak{g}_2 = \mathfrak{g}$ (the function π is given below). If we take a basis $\{e_1, e_2, \ldots e_n\}$ that respects the gradings i.e. $V_i = \operatorname{span}\{e_i\}$ $i \geq 3$ and $\mathcal{D} = \operatorname{span}\{e_1, e_2\}$, then we have that $[e_1, e_i] = k\, e_{i+1}$, $i = 2, 3, \ldots n$. As the $\mathfrak{g}_i = V_i \oplus \ldots \oplus V_n$, are all ideals, the basis $\{e_1, e_2, \ldots e_n\}$ is a *strong Malcev basis.*

3. A unipotent matrix Lie group for the minimal growth vector case

By the Lie-Palais theorem, the Lie algebra $\mathfrak{g}$ gives the infinitesimal generators of a Lie group G. Such a group has to be also graded nilpotent. By Ado's theorem there exists a faithful unipotent representation $\eta : G \to GL(W)$ (over the field $\mathbb{R}$) which is a linear algebraic group. The reason for looking for a linear group is obviously that the exponential map $\mathfrak{g} \to G$ becomes the ordinary exponential and the adjoint map becomes the similarity transformation $A \to gAg^{-1}$, $A \in \mathfrak{g}$ and $g \in G$. A matrix unipotent representation of this abstract group was introduced recently in the control community by H. Struemper [16][1].

$X \in G$ has the following structure:

$$X = \begin{bmatrix} 1 & x_2 & x_3 & x_4 & x_5 & \ldots & x_n \\ 0 & 1 & x_1 & \frac{x_1^2}{2} & \frac{x_1^3}{6} & \ldots & \frac{x_1^{n-2}}{(n-2)!} \\ \vdots & \ddots & 1 & x_1 & \frac{x_1^2}{2} & \ddots & \vdots \\ & & & 1 & x_1 & \ddots & \frac{x_1^3}{6} \\ & & & & 1 & \ddots & \frac{x_1^2}{2} \\ \vdots & & & & \ddots & \ddots & x_1 \\ 0 & & \ldots & & \ldots & 0 & 1 \end{bmatrix}, \qquad x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$$

and a basis of a left-invariant representation of its Lie algebra $\mathfrak{g}$ is:

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 \\ & 0 & 1 & \ldots & 0 \\ \vdots & & & \ddots & 0 \\ & & & & 1 \\ 0 & & \ldots & & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ & 0 & 0 & \ldots & 0 \\ \vdots & & & \ddots & \vdots \\ & & & & 0 \\ 0 & & \ldots & & 0 \end{bmatrix}$$

[1] Struemper "baptized" it $SUP(n)$ the *Special Unipotent group* of dimension n. Notice that such a group is well-known in representation theory of nilpotent Lie groups. In [6] it is called just K_n. Here, following Gromov [9] we call it a Carnot group.

$$A_3 = \begin{bmatrix} 0 & 0 & 1 & \dots & 0 \\ & 0 & 0 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ & & & & 0 \\ 0 & & \dots & & 0 \end{bmatrix}, \quad \dots, \quad A_n = \begin{bmatrix} 0 & 0 & 0 & \dots & 1 \\ & 0 & 0 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ & & & & 0 \\ 0 & & \dots & & 0 \end{bmatrix}.$$

For $n = 3$, this group coincides with the Heisenberg group.

The main feature of G is that only $n-2$ brackets are different form 0, i.e. only the *minimal* number required to satisfy the Raschevskii-Chow theorem. The matrix commutators are in fact

$$[A_1, A_i] = A_1 A_i - A_i A_1 = -A_i A_1 = (-1)^{i-1} A_2 A_1^{i-1} = (-1)^{i-1} A_{i+1}.$$

As in all nilpotent graded Lie algebras, the bracket reduces to a (noncommutative, nonassociative) multiplication. Moreover, we have that the only nonnull structure constants give exactly (up to the sign) the remaining basis elements of $\mathfrak{g}$.

If $\mathfrak{g}$ is nilpotent, the exponential map $exp : \mathfrak{g} \to G$ is a global diffeomorphism and both the canonical coordinates of the first kind

$$(t_1, \dots t_n) \mapsto exp(t_1 e_1 + \dots + t_n e_n)$$

and the canonical coordinates of the second kind

$$(t_1, \dots t_n) \mapsto exp(t_1 e_1) \dots exp(t_n e_n)$$

define global diffeomorphisms.

A control system with the filiform sub-Riemannian structure (2) has the following left-invariant representation on the group G:

$$\dot{X} = X (u_1 A_1 + u_2 A_2) \qquad X \in G \tag{5}$$

where the parameters $u_i \in \mathbb{R}$ are called control inputs.

Considering the system (5) in the strong Malcev basis given by the canonical coordinates of the second kind and applying the Wei-Norman formula to the resulting product of exponentials [17], we obtain the chained form (1).

4. Semidirect sum in $\mathfrak{g}$

The series (4) is a succession of central extensions of abelian Lie algebras i.e. $Z(\mathfrak{g}/\mathfrak{g}_{i+1}) = \mathfrak{g}_i/\mathfrak{g}_{i+1} \quad i \geq 2$. The same consideration holds also for $\mathfrak{g}_2$ so that all the terms $\mathfrak{g}_i, \quad i = 2, 3, \dots, n$ are *nilpotent ideals*. Moreover, since $A_i A_j = 0 \quad \forall\ i, j > 1$, they are *abelian ideals* and $\mathfrak{g}_2$, the maximal abelian ideal of $\mathfrak{g}$, has codimension 1. The characteristic line field $\mathcal{L}$ is generated by A_1: $\mathcal{L} = span\{A_1\}$ and looking at the brackets, we have that the homomorphism

$$\pi : \mathcal{L} \to Der_{\mathbb{R}} \mathfrak{g}_2 \tag{6}$$

of the semidirect sum $\mathfrak{g} = \mathcal{L} \oplus_\pi \mathfrak{g}_2$ is a matrix multiplication which gives a "shift" in the A_i (up to sign):

$$\pi(A_1)(A_i) = (-1) A_{i+1}.$$

Correspondingly, it is possible to define a lower central series also on the group G and a semidirect product $G = G_{\mathcal{L}} \otimes_\pi G_2$ with both $G_{\mathcal{L}}$ and G_2 (abelian) subgroups of dimension respectively 1 and $n-1$ and with G_2 normal in G.

5. A homogeneous space for G preserving the graded structure

Consider $\mathcal{L} = span\,\{A_1\}$ and $\mathfrak{g}_2 = span\,\{A_2 \dots A_n\}$. As a consequence of the graded structure, $\mathfrak{g}_2$ is the maximal abelian ideal in $\mathfrak{g}$ and $\mathcal{L}$ is an abelian Lie subalgebra because of its dimension. On the corresponding subgroups $G_{\mathcal{L}}$ and G_2 we can form the homogeneous space $G/G_{\mathcal{L}}$ i.e. the left cosets of $G_{\mathcal{L}}$ in G defining the equivalence relation on G

$$g_1 \equiv g_2\,(mod\,G_{\mathcal{L}}) \quad \text{if} \;\; g_1^{-1} g_2 \in G_{\mathcal{L}}.$$

Similarly, if we take $A,\, B \in \mathfrak{g}$, we can define $A + \mathcal{L}$ as the equivalence class of A under the equivalence relation

$$A \equiv B(mod\,\mathcal{L}) \quad \text{if} \;\; A - B \in \mathcal{L}.$$

As $\mathcal{L}$ is not an ideal we cannot talk about quotient algebra but we can at least speak about quotient vector space. We have the following:

Proposition. *The graded structure is preserved under the projection*

$$\psi \;:\; \mathfrak{g} \to \mathfrak{g}/\mathcal{L}. \tag{7}$$

Proof. In fact, A and B belonging to the same left coset means $A,\, B \in span\,\{A_1,\, A_i\}$ for some i. The map (6) can be rewritten as:

$$\begin{array}{rcl} \pi \;:\; L \otimes V_i & \to & V_{i+1} \\ (A_1,\, A_i) & \mapsto & A_{i+1} \end{array}$$

so the composed map $\overline{\pi} = \psi \circ \pi$ is

$$\begin{array}{rcl} \overline{\pi} \;:\; V_i & \to & V_{i+1} \\ A_i & \mapsto & A_{i+1}. \end{array}$$ □

One could argue that the statement above is trivial because, by construction, the kernel of ψ has a trivial intersection with $\mathfrak{g}_i$, $i \geq 2$. Since $\psi(\mathfrak{g}) = \psi(\mathfrak{g}_2)$, $\mathfrak{g}/\mathcal{L}$ is isomorphic to $\mathfrak{g}_2$ as a vector space. When the component along A_1 is zero the projection (7) is identically zero.

The chained form in the projective space looks particularly nice: in the canonical coordinates of the second kind (1) the projection corresponds to the quotient of all the coordinates by $\dot{x}_1 = u_1$. Calling $v = u_2/u_1$ and rearranging the order we obtain:

$$\frac{d}{ds}\begin{bmatrix} x_n \\ \vdots \\ x_3 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ & & & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_n \\ \vdots \\ x_3 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} v \tag{8}$$

where $ds = \dot{x}_1 dt$. The system is *linear* and already in the classical *controllability canonical form*, see [10]. It is obtained by eliminating the time dependence and considering instead a moving frame parameterized by the *arclength* of the path followed by the point $p \in M$.

Acknowledgments. The author would like to thank the anonymous reviewer for pinpointing the numerous imperfections in the original manuscript.

References

[1] A. Bellaïche: *The tangent space in sub-Riemannian Geometry.* In A. Bellaïche and J.-J. Risler: Sub-Riemannian Geometry, Birkhäuser, Basel 1996.

[2] A. Bellaïche, F. Jean and J.-J. Risler: *Geometry of Nonholonomic Systems.* In J.P. Laumond: Robot Motion Planning and Control, Springer-Verlag 1998.

[3] R.L. Bryant, S.S. Chern, R.B. Gardner, H.L. Goldschmidt and P.A. Griffiths: *Exterior Differential Systems.* Springer-Verlag 1991.

[4] R.L. Bryant and L. Hsu: *Rigidity of integral curves of rank 2 distributions.* Inventiones mathematicae **114** (1993), 435–461.

[5] M. Cheaito, P. Mormul, W. Pasillas-Lèpine and W. Respondek: *On local classification of Goursat structure.* Comptes Rendus de l'Acadèmie des Sciences de Paris, 1998.

[6] L. Corwin and F.P. Greenleaf: *Representations of nilpotent Lie groups and their applications.* Cambridge University Press 1990.

[7] M. Fliess, J. Levine, P. Martin and P. Rouchon: *Flatness and defect of nonlinear systems: introductory theory and examples.* Int. Journal of Control **61** (1995), 1327–1361.

[8] V.V. Gorbatsevich, A.L. Onishchik and E.B. Vinberg: *Structure of Lie groups and Lie algebras.* In: Encyclopaedia of Mathematical Sciences, Lie groups and Lie algebras III, Springer-Verlag 1994.

[9] M. Gromov: *Carnot-Caratheodory spaces seen from within.* In A. Bellaïche and J.-J. Risler: Sub-Riemannian Geometry, Birkhäuser, Basel 1996.

[10] T. Kailath: *Linear Systems.* Prentice-Hall, Englewood Cliffs 1980.

[11] E. R. Kolchin: *Differential Algebraic Groups.* Academic Press 1985.

[12] J.P. Laumond, S. Sekhavat and F. Lamiraux: *Guidelines in Nonholonomic Motion Planning for Mobile Robot.* In J.P. Laumond: Robot Motion Planning and Control, Springer-Verlag 1998.

[13] R. Montgomery, M. Shapiro and M. Stolin: *A nonintegrable sub-Riemannian geodesic flow on a Carnot group.* Journal of Dynamical Control Systems **3** (1997), 519–530.

[14] R. Murray and S. Sastry: *Nonholonomic motion planning: steering with sinusoids.* IEEE Trans. on Automatic Control **38** (1993), 700–716.

[15] O. J. Sørdalen: *Conversion of the kinematics of a car with n trailers into chained form.* Proc. IEEE Int. Conf. on Robotics and Automation, Atlanta, Georgia 1993, 382–387.

[16] H. Struemper: *Nilpotent Approximation and Nilpotentization for Under-Actuated Systems on Matrix Lie Groups.* Proc. 37nd IEEE Conf. on Decision and Control, Tampa, FL 1998, 4188–4193.

[17] J. Wei and E. Norman: *On the global representations of the solutions of linear differential equations as a product of exponentials.* Proc. Amer. Math. Soc. **15** (1964), 327–334.

[18] M. Zhitomirskii: Rigid and abnormal line subdistributions of 2-distributions. Journal of Dynamical and Control systems **1** (1995), 281–307.