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On Indicatrices[1]

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1. Introduction

We start by considering the space H of half-lines in R^{n+1} which can be identified with $S^n \times R^{n+1}$ in a natural way. Let $f : M \to R^{n+1}$ be a smooth, bounded map and assume that $g : M \to H$ is a smooth map such that, for $x \in M$, $g(x)$ is a half-line starting at $f(x)$. Now surround $f(M)$ by a round n-sphere S and define $F : M \to S$ by taking $F(x)$ to be the intersection point of $g(x)$ with S. We will say that F is a *spherical indicatrix*. The purpose of the present note is to study the map F in some particular cases. Maps of this type have cropped up in some of our work [3], [4].

2. Normal indicatrices of a codimension 1 immersion

In what follows M will denote a boundaryless, compact, connected, oriented, n-dimensional manifold. Let $f : M \to R^{n+1}$ be a smooth immersion, smooth here meaning C^∞. Assume that S is a round n-sphere surrounding $f(M)$. For what follows there is no loss of generality in assuming that the sphere is centred at the origin. Let now $g : M \to S^n \times R^{n+1}$ be of the form (U, f). The *indicatrix* F_U is defined as follows. For $x \in M$, consider the half-line $f(x) + \alpha U(x), \alpha \geq 0$. Then $F_U(x)$ is the intersection of the half-line with S. We can write $F_U = f + \lambda U$, where λ is a positive, smooth map.

Let us now denote by $N : M \to R^{n+1}$ the normal unit vector field determined in the following way. If, for $x \in M$, θ_x is the orientation for the tangent space T_xM, then $[f_{*x}(\theta_x), N(x)]$

[1]This is an original research article and no version has submitted for publication elsewhere.

is the usual orientation of R^{n+1}. Here f_{*x} denotes the induced linear map and the tangent space to R^{n+1} at x will be identified with R^{n+1} itself. The maps F_N and F_{-N} are the *normal indicatrices.*

Maps of the type $f_\xi : M \to R^{n+k}$, with $f_\xi(x) = f(x) + \xi(x)$, where ξ is a parallel normal field where studied by Carter and Şentürk [1] among other people.

If M is not diffeomorphic to S^n then any indicatrix will have critical points. We shall next characterise the critical points of the normal indicatrices.

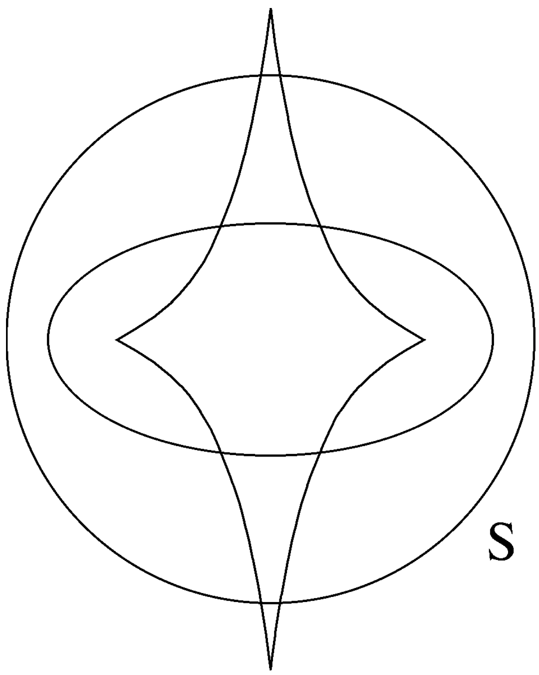
Proposition 1. *x is a critical point of F_N iff $F_N(x)$ is a focal point of f with x as base point.*

Proof. Let $F_N = f + \lambda N$ and $x \in M$. Use a chart ϕ at x and conclude that F_{N*x} is injective iff the matrix $I + \lambda(x)A$ is nonsingular, where I is the identity and $A = [a_{ij}]$ is given by

$$N_{*x}((\frac{\partial}{\partial \phi_i})_x) = \sum_{j=1}^{n} a_{ji}\ f_{*x}((\frac{\partial}{\partial \phi_j})_x).$$

Next it can be shown that $F_N(x)$ is a focal point of f with x as base point iff $I + \lambda(x)A$ is singular. This can be done using the square-distance function $L_z(y) = \| f(y) - z \|^2$, with $z = F_N(x)$, of which x is a critical point and the characterization of critical point degeneracy in terms of focal points as given in [6]. □

If $f : S^n \to R^{n+1}$ is an smooth immersion such that its Gaussian curvature does not vanish then one of the maps F_N,F_{-N} is an immersion. Since nonvanishing Gaussian curvature is equivalent to focal set boundedness it is clear that if the radius of S is sufficiently large then both maps are immersions. A simple example where one of the normal indicatrices is not an immersion is shown below.



3. Degree of an indicatrix

Let us have the same assumptions and notations as in Section 2. We start with a result on *mod* 2 degrees.

Proposition 1. *Let $f : M \to R^{n+1}$ be an immersion. Then $degree_2\ F_N + degree_2\ F_{-N} \equiv e(M)\ mod\ 2$, where $e(M)$ stands for the Euler number of M.*

Proof. Choose $p \in S$ such that p is a regular value for both F_N and F_{-N}. Consider $L_p : M \to R$ given by $L_p(x) = \| f(x) - p \|^2$. The result follows easily from the fact that L_p is a Morse function [6] and the number of its critical points is congruent with $e(M)$ *mod* 2. □

Proposition 2. *Let M be even-dimensional. If F_U is an indicatrix such that, for $x \in M$, $\angle(U(x), N(x)) \leq \frac{\pi}{2}$ then degree $F_U = \frac{1}{2}\ e(M)$.*

Proof. This follows from an old result of Heinz Hopf [5] and the fact that there is a homotopy between F_U and the map rN, where r is the radius of S. In fact, for $t \in [0,1], x \in M$, $H(x,t) = rN(x) + t(F_U(x) - rN(x))$ is different from zero. If, for some $t \neq 0, 1, x \in M$, $H(t,x)$ were zero we would have $F_U(x) = -rN(x)$ and consequently $f(x) = -\lambda(x)U(x) - rN(x)$. Since $\angle(U(x), N(x)) \leq \frac{\pi}{2}$, that would imply $\| f(x) \| > r$. We can then use H to define a homotopy between F_U and the map rN. □

If M is odd-dimensional there is still a homotopy but the result is no longer true. For instance, for $M = S^n$, odd n, we can have arbitrary odd degree. We refer the reader to the results in [5].

Assume now that $M = S^1$ and that we are immersing it into R^2.

Proposition 3. *Let $F_U : S^1 \to S$ be an indicatrix such that, for $s \in S^1$, $\angle(U(s), N(s)) < \pi$. Then F_U is homotopic to F_N.*

Proof. Let $\widetilde{T} : S^1 \times [0,1] \to S^1$ be such that $\widetilde{T}(s,t) = \frac{U(s)+t(N(s)-U(s))}{\|U(s)+t(N(s)-U(s))\|}$. Use now $\widetilde{T}$ to obtain $H : S^1 \times [0,1] \to S$ given by $H(s,t) = f(s) + \mu(s,t)\widetilde{T}(s,t)$, where $\mu(s,t)$ is obtained after finding the intersection of the half-line $f(s) + \alpha\widetilde{T}(s,t), \alpha \geq 0$, with S. □

We see that the *tangential indicatrix*, F_T, with $T(x)$ the tangent vector to the curve at x, is homotopic to F_N. Using a rotation of angle πt in R^2 we can show that F_T and F_N are homotopic to F_{-T} and F_{-N} respectively.

Let $f : S^1 \to R^2$ be an immersion and $p \in R^2 \setminus f(S^1)$. We recall the following definitions. The *rotation number* of f is 2π *degree* T, $T(x)$ being the tangent vector to f at x. On the other hand the *winding number* of f with respect to p is the degree of the map from S^1 into itself given by $\frac{f(x)-p}{\|f(x)-p\|}$.

In the next two propositions both circles S^1 and S are to be oriented in a similar way.

Proposition 4. *Let $f : S^1 \to R^2$ be such that no tangent line passes through 0. Then, for U as in Proposition 3, degree F_U is the winding number of f with respect to O.*

Proof. It is enough to consider the case $U = T$. Now define $H : S^1 \times [0,1] \to S$ by $H(x,t) = r\frac{f(x)+t\lambda(x)T(x)}{\|f(x)+t\lambda(x)T(x)\|}$, where r is the radius of S. □

Proposition 5. *Let $f : S^1 \to R^2$ be such that its curvature does not vanish. Then, for U as in Proposition 3, 2π degree $F_U = rot\ f$, where $rot\ f$ stands for the rotation number of f.*

Proof. Again we consider he case $U = T$. From $F_T = f + \lambda T$ it is clear that f and F_T are regularly homotopic and consequently $rot\ f = rot\ F_T$. Since $rot\ F_T = 2\pi\ degree\ F_T$ the result follows. □

4. Applications

There is no reason to consider just immersions with codimension 1. An interesting situation occurs with curves in 3-space.

A. Curves with small total torsion

In [2] it was convenient at some stage to indicate how curves with small total torsion could be obtained. There we used a convenient non-degenerate homotopy as suggested by [7]. Here we will use another type of homotopy for a similar purpose.

Let $f : S^1 \to R^3$ be a closed curve with nonvanishing torsion. Consider $F_T : S^1 \to S$ given by $F_T(x) = f(x) + \lambda(x)T(x)$, where $T(x)$ is the unit tangent vector at x. For $0 \le t \le 1$, $g_t = f + t\lambda T$ gives rise to a non-degenerate homotopy, that is one that at every stage t the corresponding curve g_t has curvature which vanishes nowhere. Since under a non-degenerate deformation the total torsion varies continuously and the total torsion of a spherical curve is zero it follows that curves with very small, nonzero total torsion can be obtained.

B. Linking numbers

Let $f, g : S^1 \to R^3$ be curves with disjoint images. The *linking number* $L(f,g)$ is the degree of the map $\phi : S^1 \times S^1 \to S^2$ given by $\phi(x,y) = \frac{f(x)-g(y)}{\|f(x)-g(y)\|}$.

Proposition 1. *Let $f, g : S^1 \to R^3$ be curves such that the image of one of them does not intersect any tangent line to the other. Then the linking number $L(f,g)$ is zero.*

Proof. Assume that no tangent line to f meets $g(S^1)$. Consider $F_T = f + \lambda T$. Then f is homotopic to F_T and the homotopy induces a homotopy between $\phi : S^1 \times S^1 \to S^2$, given by $\phi(x,y) = \frac{f(x)-g(y)}{\|f(x)-g(y)\|}$ and $\psi : S^1 \times S^1 \to S^2$, given by $\psi(x,y) = \frac{F_T(x)-g(y)}{\|F_T(x)-g(y)\|}$. Therefore $L(f,g) = L(F_T, g)$. If we choose the 2-sphere S for the definition of F_T sufficiently big it follows that $L(F_T, g) = 0$. □

Let us recall that two space curves f and g are *athwart* if no tangent line to f intersects a tangent line to g [4]. Athwart curves are examples of curves in the conditions of Proposition 1. It is known [4] that there are curves which cannot be athwart to any other curve. We are going to show that on the other hand given a curve we can always find another one such that the conditions of Proposition 1 are satisfied.

Proposition 2. *Let $f : S^1 \to R^3$ be a curve. Then, for every sphere containing $f(S^1)$ in its interior, there is a spherical curve $g : S^1 \to R^3$ such that no tangent line to f meets $g(S^1)$.*

Proof. We follow [3] where we showed that the tangent lines to f do not fill R^3. Consider F_T, F_{-T}. Then $X = F_T(S^1) \cup F_{-T}(S^1)$ is a set of which the complement in S^2 is open and nonempty. Any curve $g : S^1 \to R^3$ with image in $S^2 \setminus X$ will do. □

In the statement above spherical may be replaced by plane as we show next.

Proposition 2′. *Let $f : S^1 \to R^3$ be a curve. Then, for every plane which does not meet $f(S^1)$, there is a plane curve $g : S^1 \to R^3$ such that no tangent line to f meets $g(S^1)$.*

Proof. Take a plane π such that $f(S^1) \cap \pi = \emptyset$. Consider the projective closure RP^2 of π and define $F : S^1 \to RP^2$ by letting $F(x)$ be the intersection point of the tangent line to f at x with π. Such a map F is smooth and again by Sard's Theorem it is possible to find a non-empty open subset of π which avoids $F(S^1)$. Take for g any curve with image in that open subset. □

Obviously similar results can be obtained replacing tangent by principal normal or binormal if the extra assumption of nonvanishing curvature is imposed where necessary.

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