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Parallels and Push-outs of Immersed Manifolds

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1. Introduction

It is a straightforward calculation to show that if $f : M^m \to \mathbf{R}^{m+1}$ is a smooth immersion of a connected, smooth manifold and $\xi : M^m \to \mathbf{R}^{m+1}$ is a normal field for f with constant length, then $f + \xi : M \to \mathbf{R}^{m+1}$ is an immersion if and only if, for all $p \in M$, $(f+\xi)(p)$ is not a focal point of f with base p. Such an immersion $f + \xi$ is then called a parallel immersion of f. Here I am going to survey results on parallel immersions and their generalizations.

2. Basic Definitions and Examples

Throughout this paper M^m will denote a boundaryless, connected, smooth (C^∞) m- dimensional manifold and $f : M^m \to \mathbf{R}^{m+k}$ will be a smooth immersion of M into Euclidean $(m+k)$-space.

The total space of the normal bundle of f is

$$N(f) = \{(p,x) \in M \times \mathbf{R}^{m+k} : x \in (f_* T_p M)^\perp\}$$

and the endpoint map $\eta : N(f) \to \mathbf{R}^{m+k}$ is defined by $\eta(p,x) = f(p) + x$. The set of singularities of η will be denoted by $\Sigma(f)$. The *focal* set of f is then $\eta(\Sigma(f)) \subset \mathbf{R}^{m+k}$.

At $p \in M$, let $N_p(f) = \{x \in \mathbf{R}^{m+k} : (p,x) \in N(f)\}$ and $\Sigma_p(f) = \{x \in \mathbf{R}^{m+k} : (p,x) \in \Sigma(f)\}$. Then $\Sigma_p(f) \subset N_p(f)$ can be thought of as the set of focal points of f with base p, although, strictly speaking, it is the subset $f(p)+\Sigma_p(f)$ of the affine normal plane $f(p)+N_p(f)$ which is the set of focal points with base p.

As in [8] the holonomy group of the normal bundle is used to compare normal planes and the focal sets in these planes.

Fix $p_0 \in M$ and for $p \in M$ and a (piecewise smooth) path γ from p_0 to p let $\varphi_{p,\gamma} : N_{p_0}(f) \to N_p(f)$ be the isometry defined by parallel transport in $N(f)$ along γ using the normal connection.

The *normal holonomy group* on $N_{p_0}(f)$ is

$$\mathcal{H}ol(f) = \{\varphi_{p_0,\gamma} : \gamma \text{ is a path from } p_0 \text{ to } p_0\}.$$

By taking an isometry from $N_{p_0}(f)$ to $\mathbf{R}^k$, $\mathcal{H}ol(f)$ can be thought of as a subgroup of $O(k)$ acting on $\mathbf{R}^k$. Crucial to this work are the results on the normal holonomy group by C.Olmos [14].

The *push-out space* of f

$$\Omega(f) = \{x \in N_{p_0}(f) : \forall p \in M \text{ and } \forall \text{ paths } \gamma \text{ from } p_0 \text{ to } p, \varphi_{p,\gamma}(x) \notin \Sigma_p(f)\}.$$

So $\Omega(f)$ is invariant under the action of $\mathcal{H}ol(f)$. It is the set of normals at p_0 which, when transported parallelly along all curves from p_0, do not meet focal points.

We shall see that each point of $\Omega(f)$ determines an immersion of some bundle into $\mathbf{R}^{m+k}$. For $x \in N_{p_0}$, let $W_x \subset N(f)$ be defined by

$$W_x = \{(p, \varphi_{p,\gamma}(x)) : p \in M, \gamma \text{ is a path from } p_0 \text{ to } p\}.$$

It is a subbundle of the normal bundle and its fibre is the orbit of $\mathcal{H}ol(f)$ through x. In [12] W_x is called the *holonomy subbundle* of $N(f)$ through x.

If $\mathcal{H}ol(f)$ is closed in $O(k)$ (that is, $\mathcal{H}ol(f)$ is compact) then W_x is a submanifold of $N(f)$. If it is not closed then this difficulty can be avoided by replacing M by its simply connected covering space $\widetilde{M}$ and replacing f by $\tilde{f} = f \circ \pi : \widetilde{M} \to \mathbf{R}^{m+k}$, where $\pi : \widetilde{M} \to M$ is the covering projection. Then $\Omega(\tilde{f})$ is isometric to $\Omega(f)$ and $\mathcal{H}ol(\tilde{f})$ is compact [8]. Now $\eta \mid W_x : W_x \to \mathbf{R}^{m+k}$ is an immersion

$$\iff W_x \cap \Sigma(f) = \emptyset$$

$$\iff x \in \Omega(f).$$

If $x \in \Omega(f)$ then $\eta \mid W_x : W_x \to \mathbf{R}^{m+k}$ is called a *push-out* of f. We shall see that the dimensions of the submanifolds W_x can vary with x.

In the special case where $\mathcal{H}ol(f)$ is trivial, there exists an orthonormal set of parallel normal fields $n_1, \ldots, n_k : M \to \mathbf{R}^{m+k}$. Then, if $x \in N_{p_0}(f)$, $x = \sum_{i=1}^k x_i n_i(p_0)$ for some $x_1, \ldots, x_k \in \mathbf{R}$. Put $\xi = \sum_{i=1}^k x_i n_i : M \to \mathbf{R}^{m+k}$. Then ξ is a parallel normal field and for any path γ from p_0 to $p \in M$, $\varphi_{p,\gamma}(x) = \xi(p)$. Thus in this special case, $W_x = \{(p, \xi(p)) : p \in M\}$ which is a section of $N(f)$ and can be identified with M, and $(\eta \mid W_x)(p, \xi(p)) = (f + \xi)(p)$. In this case, if $x \in \Omega(f)$ we say that $\eta \mid W_x$ is a *parallel* of f. So a parallel is a special case of a push-out. Whereas parallels of $f : M \to \mathbf{R}^{m+k}$ are immersions of M, a push-out can be an immersion of a manifold with dimension greater than m. The push-out space of an immersion was studied in [4] for the special case in which $\mathcal{H}ol(f)$ is trivial, and in [8] for the general case. Other general accounts and recent results can be found in [1, 2, 3, 5, 6].

If $\Sigma(f)$ is invariant under parallel transport (that is, $\Sigma_{p_0}(f)$ is invariant under the action of the normal holonomy group) then $\Omega(f)$ is isometric to $N_{p_0}(f) \setminus \Sigma_{p_0}(f)$. This situation can only happen if $f(M)$ is an isoparametric submanifold or a focal manifold of an isoparametric submanifold, that is, $f(M)$ has constant principal curvatures [12]. A description of a Veronese surface for which the push-out space has three path-connected components is described in [6]. Some of the push-outs are Cartan's 3-dimensional isoparametric submanifold.

Examples of the push-out space of immersed curves can be found in [1, 8].

Let K^2 denote the Klein bottle. In [3] examples are given of an immersion $f : K^2 \to \mathbf{R}^4$ and an embedding $g : K^2 \to \mathbf{R}^5$ for which $\Omega(f)$ and $\Omega(g)$ have four path-connected components. It is not known whether or not there exists an embedding of K^2 in $\mathbf{R}^4$ with this property.

3. Self-parallel immersions

In [10], H.Farran and S.A.Robertson defined two immersions $f, g : M^m \to \mathbf{R}^{m+k}$ to be *parallel* if, for all $p \in M$, $f(p) + N_p(f) = g(p) + N_p(g)$. A diffeomorphism $\delta : M \to M$ is called a *self-parallelism* of M with respect to f if f and $\delta \circ f$ are parallel. They studied the group of all self-parallelisms of M with respect to f.

It was shown by A.M. Flegmann [11] and B. Wegner [17] that f and g are parallel if and only if $g = f + \xi$ where ξ is a parallel normal field for f. So g is a parallel of f in the sense of Section 2. The *parallel rank*, $\rho(f)$, of an immersion $f : M^m \to \mathbf{R}^{m+k}$ is defined to be the dimension of the affine space spanned by the immersions which are parallels of f. Thus $0 \leq \rho(f) \leq k$ and $\rho(f) = k$ if and only if $\mathcal{H}ol(f)$ is trivial. For results on parallel rank and examples illustrating related concepts see [9, 10, 17]. Self-parallel curves are studied in [9, 13, 18]. Effectively, a self-parallel curve is a push-out from a central curve [18].

Some of the above ideas are extended in [21] to parallel curves on surfaces, in [22] to parallel immersions into spaces of constant curvature, and in [19, 20] to parallelism for polygons and polyhedra. A general account of parallelism for both smooth curves and polygons can be found in [23] which includes two animated examples on self-parallel polygons.

4. Theorems on the push-out space

As usual, $f : M^m \to \mathbf{R}^{m+k}$ denotes an immersion and $\Omega(f)$ denotes the push-out space of f. The following theorems are about the number of path-connected components of $\Omega(f)$ and the topology of $\Omega(f)$.

Theorem 1. [8] $\Omega(f)$ has only a finite number of path-connected components.

Theorem 2. [4, 6, 8, 16] Let M be compact and let $k = 1$, then

(1) if M is orientable, $\Omega(f)$ has at most two components,

(2) if M is nonorientable, $\Omega(f)$ is connected.

Essentially the first part of this theorem was proved by B. Smyth who considered the connected components of the space

$$\{\lambda \in \mathbf{R} : \text{for some } p \in M,\ \lambda \text{ is a principal curvature at } p\}.$$

He also considered the corresponding problem for immersions $f : M^m \to S^{m+1}$ [16]. It is not known whether or not there exists a complete embedding $f : M^m \to \mathbf{R}^{m+1}$ of a noncompact manifold for which $\Omega(f)$ has more than two components.

Theorem 3. [8] Let M be compact then $\Omega(f)$ is a neighbourhood of the origin in $N_{p_0}(f)$.

An example is given in [8] in which $\Omega(f)$ is not open. However, it is shown in [4] that if M is compact and $\mathcal{H}ol(f)$ is trivial then $\Omega(f)$ is open.

Theorem 4. [8] The path-connected component of $\Omega(f)$ to which $\mathbf{0}$ belongs is star-like.

It was shown in [4] that if $\mathcal{H}ol(f)$ is trivial then the path-connected components of $\mathcal{H}ol(f)$ are convex. This can be generalised by considering $\mathcal{H}ol_0(f)$, the component of the identity in $\mathcal{H}ol(f)$, that is

$$\mathcal{H}ol_0(f) = \{\varphi_{p_0,\gamma} \in \mathcal{H}ol(f) : \gamma \text{ is homotopically trivial } \}.$$

It is a consequence of the work of C.Olmos [14] that through each $x \in N_{p_0}(f)$ there is an affine plane $\Pi \subset N_{p_0}(f)$ which intersects all the orbits of $\mathcal{H}ol_0(f)$ orthogonally and is an affine normal plane to the regular orbits. Using this comment $\Omega(f)/\mathcal{H}ol_0(f)$ can be identified with a subset of a vector space.

Theorem 5. [8] Each connected component of $\Omega(f)/\mathcal{H}ol_0(f)$ is convex.

Theorem 6. [8] Let M be compact and let $\mathcal{H}ol(f)$ act transitively on $S^{k-1} \subset \mathbf{R}^k \equiv N_{p_0}(f)$. Then $\Omega(f)$ is an open ball in $\mathbf{R}^k$.

The second part of Theorem 2 is a special case of this since M being nonorientable is equivalent to $\mathcal{H}ol(f)$ being transitive on $S^0 = \{-1, 1\}$.

Theorem 7. [8] Let $\mathcal{H}ol(f)$ have an orbit that is dense in S^{k-1} but $\mathcal{H}ol_0(f)$ is not transitive on $S^{k-1} \subset \mathbf{R}^k \equiv N_{p_0}(f)$. Then, either $\Omega(f) = \{\mathbf{0}\}$ or the interior of $\Omega(f)$ is an open ball in $\mathbf{R}^k$ whose closure contains $\Omega(f)$.

Examples illustrating these theorems can be found in [8].

There are still many interesting problems to be solved in the case in which $\mathcal{H}ol(f)$ is trivial. In this case the focal set in each normal plane consists of at most m hyperplanes (where $m = \dim M$). Put

$$\alpha(m,k) = \begin{cases} 2^m & \text{for } m \le k \\ \Sigma_{r=0}^{k}\binom{m}{r} & \text{for } m > k \end{cases}$$

and

$$\gamma(m,k) = \begin{cases} 2^m & \text{for } m \le k \\ 2\Sigma_{r=0}^{k-1}\binom{m-1}{r} & \text{for } m > k \end{cases}$$

so that $\alpha(m,k)$ (resp. $\gamma(m,k)$) is the number of components (resp. unbounded components) in the complement of m hyperplanes in general position in $\mathbf{R}^k$. Let $d(f)$ denote the number of path-connected components of $\Omega(f)$.

Theorem 8. [4] Let $\mathcal{H}ol(f)$ be trivial then $d(f) \le \alpha(m,k)$.

Conjecture. *Let M be compact and let $\mathcal{H}ol(f)$ be trivial then $d(f) \leq \gamma(m,k)$.*

This is trivially true if $m \leq k$ since then $\gamma(m,k) = \alpha(m,k)$. The first part of Theorem 2 shows that the conjecture is true for $k = 1$, as $\gamma(m,1) = 2$. Also the conjecture has been proved recently in the case $k = 2$ where $\gamma(m,2) = 2m$ [15]. For all $m \geq 1, k \geq 1$ it has been shown that there exists an embedding $f : S^1 \times \ldots \times S^1 \to \mathbf{R}^{m+k}$ of the product of m copies of S^1, for which $d(f) = \gamma(m,k)$, [1, 7]. In order to get large $d(f)$ the manifold seems to need a large number of generators for its homology.

Evidence which supports the conjecture is the *unboundedness property.*

Theorem 9. [4] Let M be compact, let $\mathcal{H}ol(f)$ be trivial and let A be a path-connected component of $\Omega(f) \subset N_{p_0}$. Then there exists $p \in M$ such that the path-connected component of $N_p(f) \setminus \Sigma_p(f)$ which contains the set $\{\varphi_{p,\gamma}(A) : \gamma \text{ is a path from } p_0 \text{ to } p\}$ is unbounded.

More information can be obtained about $\Omega(f)$ by assigning an integer to each of its path-connected components, called the *index* of the component. Roughly, if $x \in \Omega(f) \subset N_{p_0}$, the index of x is the number of focal points of f with base p_0 (counted with a multiplicity) on the line segment from x to p_0. This number is constant over a path-connected component of $\Omega(f)$. The possible indices that can occur are related to the topology of the manifold.

Theorem 10. [2] Let $f : M^m \to \mathbf{R}^{m+k}$ be an immersion of a compact manifold and let $\mathcal{H}ol(f)$ be trivial then if $\Omega(f)$ has a component with odd index, the Euler characteristic of M is zero.

This was proved for the case $k = 1$ in [4] where there are also results relating the index and the ranks of the homology groups of M. See also [5].

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