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Bifurcations, Singularities and Symmetries

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The aim of this lecture is to give a brief outline of some issues that arise in studying the geometry of bifurcation problems, and to indicate various links with other topics in differential geometry and topology.

By a *bifurcation problem* we mean a problem of finding the solutions $x \in X$ to an equation

$$F_\mu(x) \equiv F(x, \mu) = 0 \in Y$$

where μ is a multi-dimensional parameter. Here $F : X \times K \to Y$ is a map which is as smooth as necessary to enable appropriate tools of calculus to be used, and for simplicity the spaces X, Y and K are taken here to be finite-dimensional linear spaces.

Viewing the problem geometrically, we consider the *solution locus*

$$M = F^{-1}(0) \subset X \times K.$$

If $p_K : M \to K$ denotes projection of M to the parameter space K then we can rephrase the problem as follows:

Bifurcation problem. Study the changes in the geometrical structure of the set $p_K^{-1}(\mu)$ as μ varies in K.

See Figure 1 for a schematic representation.

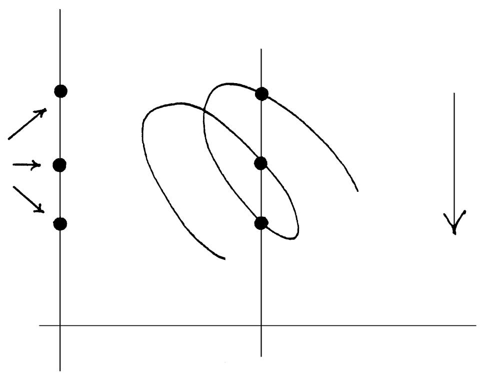

Figure 1: Schematic representation of solution locus M.

The set of points $\mu \in K$ at which changes in the solution set $p_K^{-1}(\mu)$ do actually occur is called the *bifurcation set.* Most points of K do not belong to the bifurcation set.

Of course, there is analogous problem posed using the projection p_X into the variable space X; here we have a possible solution x and want to know for which values of the parameter μ this solution can be attained, and how the answer varies with x.

Control problem. Study the changes in the geometrical structure of the set $p_X^{-1}(x)$ as x varies in X.

Let us return to the bifurcation problem, and focus on the case when $X = Y = \mathbf{R}^n$ and $K = \mathbf{R}^k$ with $n, k \geq 1$, as would arise for example if we were looking for equilibrium points for the dynamical system generated by a system of ordinary differential equations $\dot{x} = F(x, \mu), x \in \mathbf{R}^n$ for varying $\mu \in \mathbf{R}^k$. Here, under generic assumptions on F (namely that its Jacobian matrix have maximal rank on points of M) the solution locus is a smooth k-dimensional manifold, and so the set $p_K^{-1}(\mu)$ is usually a discrete set of points, hence a finite set if we restrict to some compact region of $\mathbf{R}^n$. Examples of typical local geometry for the projection $p_K : M \to \mathbf{R}^k$ are illustrated in Figure 2 for the cases $k = 1$ and $k = 2$. The integers attached to regions in $K = \mathbf{R}^k$ denote the number of solutions x corresponding to μ in those regions.

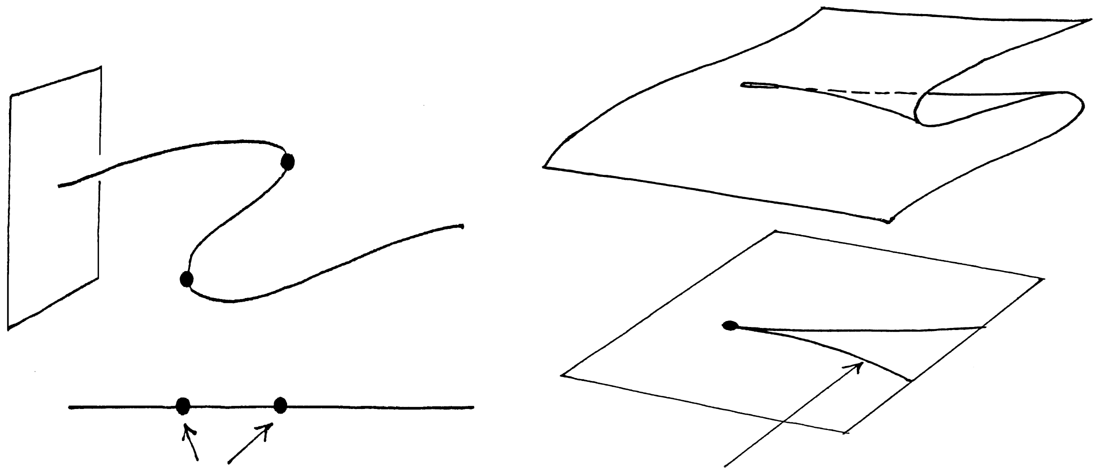

Figure 2: Generic local bifurcation geometry for $k = 1, 2$.

Results from singularity theory show that for $k = 1, 2$ these are in fact the *only* possibilities for the local structure of the bifurcation set in generic situations, that is without further constraints. For $k = 1$ the bifurcation points are *fold* points μ_0 where two solutions are born or annihilated at x_0 with $|x - x_0|$ varying locally as $\pm\sqrt{\mu - \mu_0}$ or $\pm\sqrt{\mu_0 - \mu}$; for $k = 2$ there are curves of folds with isolated *cusp* points where three solutions coalesce to one. In general (and disregarding any behaviour in $\mathbf{R}^n$ coming from infinity) the bifurcation set is the *discriminant* (the set of singular values) of the projection $p_K : M \to \mathbf{R}^k$.

Notice that in Figure 2b if we take a symmetrically-placed 'vertical' slice through M we obtain a 'pitchfork' configuration which is very commonly observed in bifurcation theory. A very simple example of a bifurcation problem yielding a pitchfork is obtained by taking $n = k = 1$ and $F : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ given by

$$F(x, \mu) = x^3 - \mu x.$$

See Figure 3.

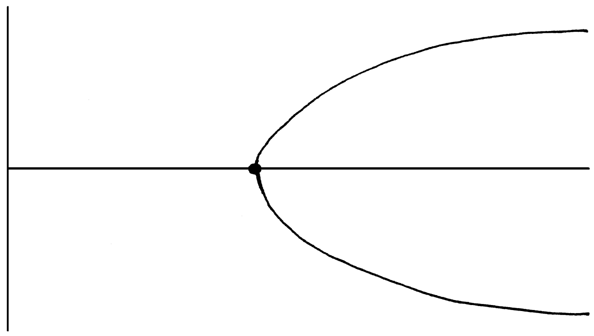

Figure 3: A pitchfork bifurcation.

Simple as it may be, this example is not typical (generic) because an arbitrarily small perturbation of the problem such as adding a constant term destroys the pitchfork and gives a solution locus with one fold and another branch with no bifurcation at all: see Figure 4.

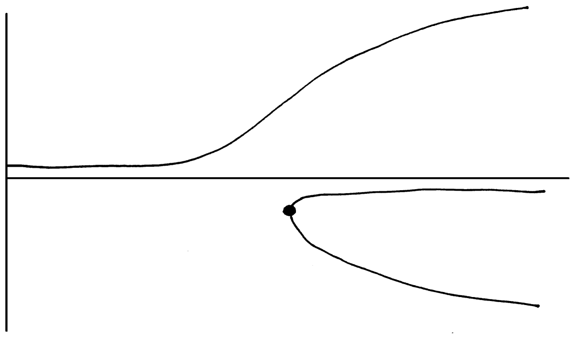

Figure 4: Perturbed pitchfork.

The key to understanding why the pitchfork is commonly observed although non-generic is to recognise the important role played here by *symmetry*. In the example given we see that $F(-x, \mu) = -F(x, \mu)$, and if we were to insist on keeping this reflection property then we could not get rid of the pitchfork by small perturbations. Thus in discussing typical or generic bifurcation phenomena it is crucial to take account of any symmetries that might be essential to the problem.

We return to discussing symmetries later, but conclude this section by stating some more results about generic bifurcations, this time for $k = 3$. It is not feasible to draw the solution locus in a space of dimension 4 or more, but instead we can sketch the local models for the bifurcation set itself in $\mathbf{R}^3$. There is one more local form corresponding to coalescence and annihilation or creation of four solutions: this is the *swallowtail* as illustrated in Figure 5.

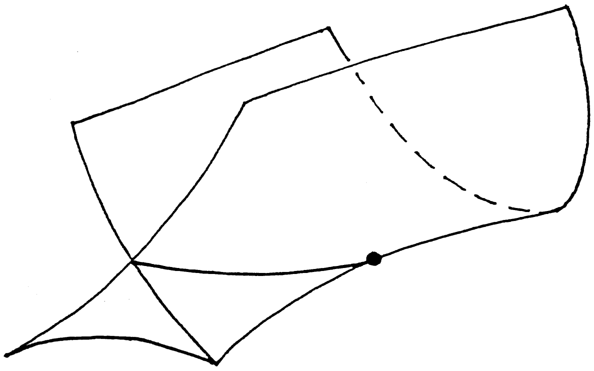

Figure 5: Swallowtail bifurcation set.

However, if the problems that we consider are restricted to variational problems, that is problems for which $F_\mu = \operatorname{grad} f_\mu$ for some $f_\mu : \mathbf{R}^n \to \mathbf{R}$, then there are two further examples called the *hyperbolic umbilic* and the *elliptic umbilic*: see Figure 6.

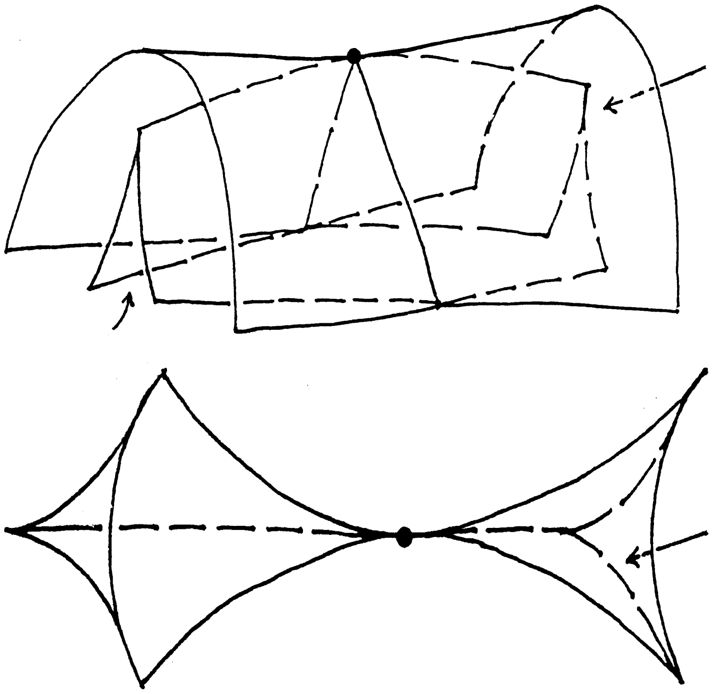

Figure 6: Hyperbolic and elliptic umbilic bifurcation sets.

These names, relating to focal sets near umbilic points of surfaces in $\mathbf{R}^3$, were coined by René Thom whose study of the generic bifurcation structures for equilibrium states in variational problems he called (elementary) *catastrophe theory* [RT], [PS]. Purported applications to biology, linguistics and many other fields were the subject of much controversy in the 1970s. The mathematics itself, fortified and extended by Mather [Ma], Golubitsky et al. [GS] and numerous others since the ground-breaking work of Thom, underlies almost the whole of bifurcation theory as it has since developed.

When $n = 1$ the solution locus M is generically a smooth k-manifold in $\mathbf{R}^{k+1}$. In this case the bifurcation set can be interpreted as the *apparent outline* (apparent contour) of M when viewed along the direction of the x-axis (compare Figure 2): bifurcations correspond to places where the lines parallel to the x-axis do not pierce the solution locus transversely, but are *tangent* to it. Fixing $k = 2$, we now look more closely at the structure of apparent outlines of surfaces in $\mathbf{R}^3$.

Apparent Outlines

Imagine a smooth 2-manifold (surface) S in $\mathbf{R}^3$, viewed in a particular direction. Think of it as made of semi-transparent material so that parts of it do not obscure other parts. We ask the natural question: What does the apparent outline look like generically?

To answer this we must define *generic*, which requires us to specify what are the allowable perturbations. In accordance with common practical situations, we take S to be fixed and permit only the direction of view to be varied (which differs from our discussion above, where the solution locus M could be varied but the view direction was fixed). It turns out that *fold curves* with isolated *cusp points* again give the generic description. But now imagine that we vary the view by rigidly rotating the surface, for example. Clearly the cusps will move and perhaps undergo changes. What are the local transitions in the outline that we can expect to see?

The generic 1-parameter transitions that occur in apparent outlines have been classified (see [A1], [B2]) and are as illustrated in Figure 7.

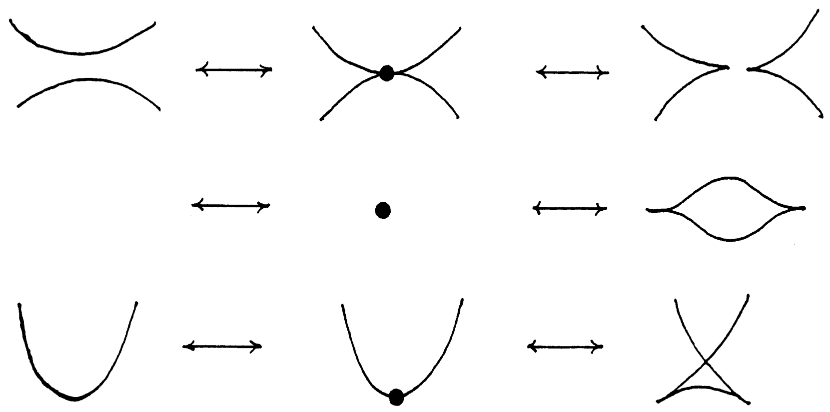

Figure 7: Generic 1-parameter transitions in apparent outlines.

The *beaks* and *lips* are fairly easy to visualise in terms of creating or removing two folds in a smooth sheet; the *swallowtail* transition can be seen by rotating a torus as in Figure 8.

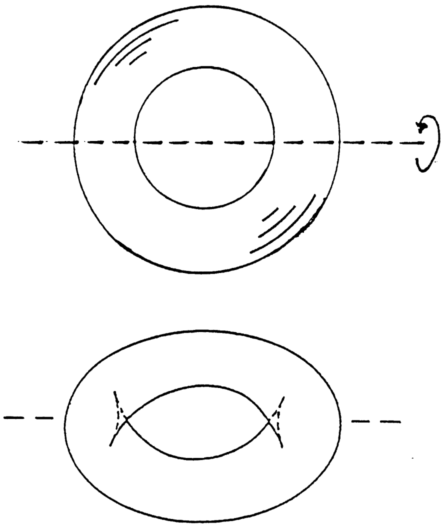

Figure 8: Swallowtails from a torus.

Finally, we can consider all possible views of a given surface from all directions: this yields a 2-parameter family of apparent outlines. What transitions can we now expect to see generically? This has also been answered using techniques from singularity theory by Bruce and Giblin [BG] to whom we refer for the pictures. It illustrates one way in which Thom's ideas on measuring degeneracy of a structure by counting the number of parameters needed to see that structure within a generic family have been fruitful both for mathematics and for its potential applications to fields such as computer vision [B1].

Symmetries

We now take up again the question of symmetries in a bifurcation problem, briefly touched upon with the pitchfork. Symmetries lead to persistence of degeneracies but at the same time they provide some vital scaffolding on which to pin the full geometry of the problem.

In this setting we suppose $K = \mathbf{R}$, as the most common symmetric bifurcation problems are those in which the key symmetric features of the problem are preserved as one parameter is varied, while other parameters may have the effect of breaking the symmetry. We need to introduce some terminology about group actions.

Let Γ be a compact Lie group acting by linear transformations on $\mathbf{R}^n$. If $F(x, \mu) = 0$ is a bifurcation problem with the property that

$$F(\gamma x, \mu) = \gamma F(x, \mu)$$

for every $\gamma \in \Gamma$ and all $x \in \mathbf{R}^n$ then we say that Γ acts as a group of *symmetries* of F or, more formally, that F is *equivariant* under the action of Γ. For example, with the pitchfork we have $\Gamma = \mathbf{Z}_2$ acting on $\mathbf{R}$ by $x \mapsto \pm x$ and we see that $x^3 - \mu x$ (being an odd function of x) is equivariant with respect to this action.

The *orbit* of x is $\Gamma x = \{\gamma x : \gamma \in \Gamma\}$. By the symmetry, anything that happens at and near x happens likewise at and near every point of the orbit of x. In particular, if x is a solution to $F(x, \mu) = 0$ then so is every point of the orbit of x. Another way of saying this is that the solution locus M is *invariant* under Γ, as for example the pitchfork solution locus is invariant under reflection $x \mapsto -x$.

Not every element of Γ necessarily moves every x; indeed the origin $0 \in \mathbf{R}^n$ is fixed under *every* element of Γ. More generally, we denote by Γ_x the subgroup of Γ consisting of those elements that fix a given x, that is

$$\Gamma_x = \{\gamma \in \Gamma : \gamma x = x\},$$

called the *isotropy subgroup* (sometimes the *stabilizer*) of x. If x is a solution to $F(x, \mu) = 0$ we think of Γ_x as the group of *symmetries of the solution* x.

An arbitrary subgroup H of Γ need not necessarily be the isotropy subgroup of any particular x. Nevertheless we can still study the set of points that *are* fixed under the action of H: this is a linear subspace of $\mathbf{R}^n$ which we denote by $\mathrm{Fix}(H)$. By definition we always have $x \in \mathrm{Fix}(\Gamma_x)$ although Γ_x may fix more elements than just x and its scalar multiples. The importance of the fixed-point subspaces results from the observation that if $x \in \mathrm{Fix}(H)$ and $\gamma \in H$ then

$$\gamma F_\mu(x) = F_\mu(\gamma x) = F_\mu(x)$$

and so $F_\mu(x)$ is fixed by γ. Hence

Proposition 1. *For every subgroup H of Γ we have*

$$F_\mu(\mathrm{Fix}(H)) \subset \mathrm{Fix}(H).$$

This means that for every subgroup H of Γ we can look at the restriction of the problem to the linear subspace $\mathrm{Fix}(H) \subset \mathbf{R}^n$, which (when $\mathrm{Fix}(H)$ is not zero or the whole of $\mathbf{R}^n$) gives us a useful smaller-dimensional problem to study: solutions to this problem will certainly be solutions to the original problem in $\mathbf{R}^n$. This is especially useful when $\mathrm{Fix}(H)$ has dimension 1 as we now see.

Proposition 2. *Suppose $\dim \mathrm{Fix}(H) = 1$ with $\mathrm{Fix}(H)$ generated by $v \in \mathbf{R}^n$. Assuming that $F(0,0) = 0$ and that the x-derivative of F in the direction of v passes through zero with nonzero speed as μ passes through 0 (that is $\frac{\partial^2 F}{\partial t \partial \mu}(tv, \mu) \neq 0$ at $(0,0)$) then there is a branch of solutions to $F(x, \mu) = 0$ in the direction of v, and therefore having symmetry at least H.*

This result is called the *Equivariant Branching Lemma*, essentially first stated formally by Vanderbauwhede [V] and Cicogna [Ci]. It illustrates the phenomenon of *spontaneous symmetry-breaking*: the problem as a whole retains full Γ-symmetry but as the parameter μ is varied one or more solutions with less symmetry are created.

The theory of spontaneous symmetry-breaking bifurcations has a long history in theoretical physics (see Michel [Mi]), and is now a large topic involving much technical investigation of specific group actions in $\mathbf{R}^n$. The main current approaches and many examples are studied in the two volumes by Golubitsky, Schaeffer and Stewart [GS] with other important aspects discussed by Field [F].

Forced symmetry breaking

We turn finally to some remarks on *forced symmetry-breaking.* Again we consider $F(x, \mu) = 0$ but now take $\mu \in \mathbf{R}^k$ and suppose the problem itself has less symmetry when $\mu \neq 0$ than it does when $\mu = 0$. Specifically, we suppose that the map F_0 is equivariant with respect to the action of a *continuous* group of symmetries, that is a Lie group Γ with $\dim \Gamma \geq 1$ such as a rotation group $\mathbf{SO}(2)$ or $\mathbf{SO}(3)$, but for $\mu \neq 0$ this symmetry is no longer present.

As before, if x is a solution to $F_0(x) = 0$ then so is the entire orbit Γx which in this case may be a compact manifold N of positive dimension. We ask: what happens to this manifold N of solutions as μ moves away from $0 \in \mathbf{R}^k$? See Figure 9.

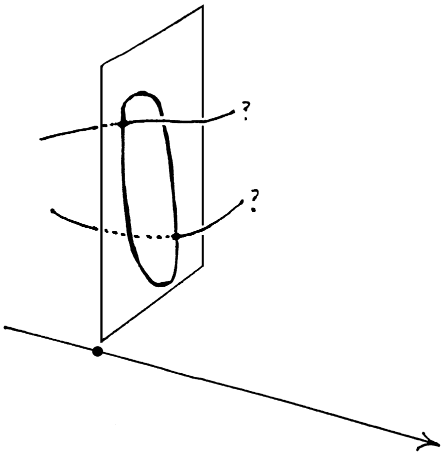

Figure 9: Bifurcation from a manifold.

The first step is to invoke the important principle that nondegeneracies imply persistence of local geometric structure. In this case the appropriate nondegeneracy assumption is that on N the derivative of F_0 (necessarily zero in directions *along* N) should be non-singular in the directions *normal* to N. An application of the implicit function theorem over the whole of N then shows that for $|\mu|$ sufficiently small there persists a manifold N_μ diffeomorphic to N and close to N on which the normal component of F_μ vanishes. This leaves only the component tangent to N_μ: it reduces the problem to the search for zeros of a vector field on N_μ itself (which we can think of as a copy of N) that vanishes identically when $\mu = 0$.

For ease of description, suppose the manifold N has trivial tangent bundle so we can regard the vector field on N simply as a map $F : N \to \mathbf{R}^m$ where $m = \dim N$. (The more general situation is easy to reconstruct from what follows.) The vanishing when $\mu = 0$ implies that F can be written in the form

$$F(x, \mu) \equiv A(x)\mu + O(|\mu|^2)$$

where $A(x)$ is an $m \times k$ matrix varying smoothly with x. The strategy now is as follows.

First we solve the *linear* (in μ) problem $A(x)\mu = 0$. Next we use methods from singularity theory to show that under generic assumptions the solution configuration is *structurally stable*, i.e. persists under small perturbations. Finally we deduce from this that if the generic assumptions hold we can disregard the $O(|\mu|^2)$ terms, at least for $|\mu|$ sufficiently small.

For the linear problem we first note that since $A(x)\alpha\mu = \alpha A(x)\mu$ for $\alpha \in \mathbf{R}$ the solution locus for this problem is invariant under multiplication of μ by scalars. Thus any interesting features in the parameter space $\mathbf{R}^k$ (such as the bifurcation set) have a *cone* structure with vertex at the origin.

Now to say that $(x, \mu) \in N \times \mathbf{R}^k$ solves the linear problem is just to say that $\mu \in \ker A$. Therefore we have to study the following natural question:

Question. *What is the generic behaviour of the kernels (whose dimensions of course may vary) of an m-parameter family of $m \times k$ matrices?*

The answers naturally depend on the choices of m and k.

When $m = 1$ the kernel of the $m \times 1$ matrix (row vector) $A(x)$ is the $(k-1)$-dimensional hyperplane orthogonal to the column vector $a(x) = A(x)^T$ and passing through the origin; the bifurcation set in $\mathbf{R}^k$ for the linear problem is the *envelope* of these hyperplanes. When $k = 2$ this envelope reduces to a number of lines through the origin (see results of Hale *et al.* [H], [HT] in this context). For $k = 3$ it is generically a cone on a locus consisting of smooth curves with isolated cusp points, with analogous results with higher order singularities when $k > 3$: see [C1]. For the full (nonlinear) problem the bifurcation set near $0 \in \mathbf{R}^k$ is a curvilinear version of the above, that is it can be obtained from the linear version by a C^1 diffeomorphism of $\mathbf{R}^k$ of the form $\mu \mapsto \mu + O(|\mu|^2)$.

Some different and interesting phenomena occur when $m = 2$. For $k = 2$ the matrix $A(x)$ is 2×2, and standard results show that a generic family of such $A(x)$ with $x \in N$ $(\dim N = 2)$ has rank 2 (hence zero kernel) everywhere except perhaps along a 1-manifold $N_1 \subset N$ where it has rank 1. Hence the linear problem has *no* solutions for $x \notin N_1$, and for $x \in N_1$ we are essentially back at the $n = 1$ case discussed above. Next we turn to the case $k = 3$, so the matrix $A(x)$ is 2×3.

Proposition 3. *Generically for most points $x \in N$ we have $\dim \ker A(x) = 1$. However, there may be a set of isolated points $\{x_i\} \in N$ (a finite set if N is compact) at which $\dim \ker A(x_i) = 2$.*

If $\dim \ker A = 1$ we can regard $\ker A$ as an element of the projective plane $\mathbf{R}P^2$, and so there is a (smooth) map

$$\kappa : N \setminus W \to \mathbf{R}P^2 : \kappa(x) \mapsto \ker A(x)$$

where $W = \bigcup_i \{x_i\}$. Therefore we can state the following result about the bifurcation set:

Proposition 4. *The bifurcation set for solutions away from the 'hot spots' $\{x_i\}$ is the cone on the double cover (in S^2) of the discriminant of the map κ.*

The generic structure of such a discriminant is well understood: it is (again) locally everywere a smooth curve with isolated cusp points and is stucturally stable. Hence for small $|\mu|$ the bifurcation set for the nonlinear problem is a curvilinear version of this cone.

This still leaves the question of what happens near the hot spots. How do the 1-dimensional kernels of $A(x)$ approach the 2-dimensional kernels of $A(x_i)$ as $x \to x_i$? (This seems an obvious but neglected geometrical problem arising from elementary linear algebra.) Also, what effect do the $O(|\mu|^2)$ terms have on the resulting bifurcation structure?

The answers are not easy to give briefly in words. We take the unit sphere S in $\mathbf{R}^3$ and look at the great circle $S_i = S \cap \ker A(x_i)$. There are two main types of behaviour, depending on the definiteness or otherwise of a certain quadratic form q_i associated with the matrix $A(x)$ near x_i. If q_i is definite then a neighbourhood M_i of $\{x_i\} \times S_i$ in M takes the form of an annulus projecting diffeomorphically to a neighbourhood U_i of S_i in S, so there are in fact no bifurcations here. (The control problem captures the geometry: a core curve of M_i projects by p_X to the single point x_i, as in Legendrian collapse of wavefronts [A2].) In contrast, if q_i is indefinite then M_i takes the form of a neighbourhood of a 1-complex consisting of the circle $\{x_i\} \times S_i$ with another 1-manifold attached to it at four points (two symmetrically placed pairs). The projection to U_i has four folds with images tangent to S_i, and so in this case bifurcations do occur.

This is just the linear problem. For the nonlinear problem we have to consider generic perturbations of this scenario. Pictures for the q_i definite case are in Figure 10. For descriptions and pictures for the indefinite case see [C2], where details of the above and a fuller discussion of bifurcation away from a manifold of solutions can be found.

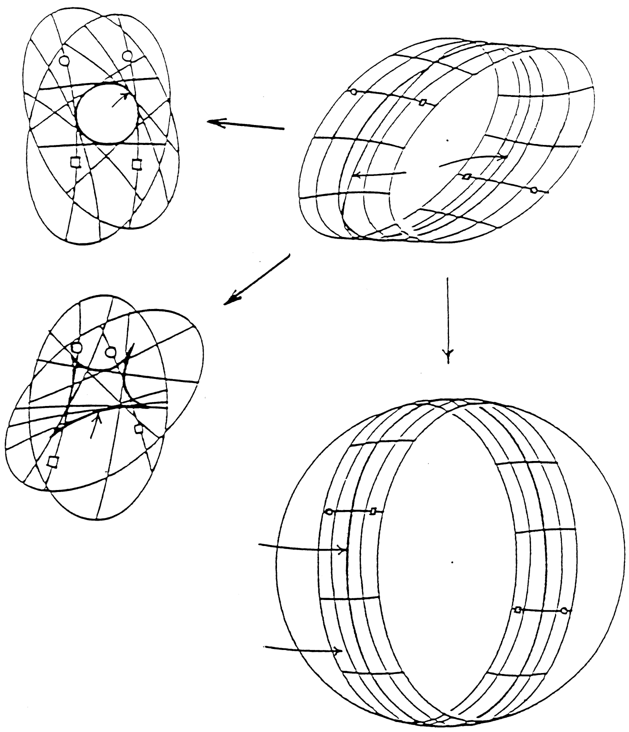

Figure 10: Geometry for solutions close to a 'hot spot' in N.
Here Σ is the singular set for p_N, with $\gamma = p_N(\Sigma)$.

In many cases of perturbation from a group orbit some residual symmetry persists, in which case the geometrical theory just described must be re-cast in the context of that remaining symmetry. Work in this direction is in progress.

References

[A1] V.I.Arnol'd: *Catastrophe Theory.* 2nd ed., Springer-Verlag, New York 1986.

[A2] V.I.Arnol'd: *Topological Invariants of Plane Curves and Caustics.* University Lecture Series Vol. 5, American Mathematical Society, Providence, RI 1994.

[B1] J.W.Bruce: *Seeing – the mathematical viewpoint.* Math. Intelligencer **6** (1984), 18–25.

[B2] J.W.Bruce: *Motion pictures: an application of singularity theory.* J. London Math. Soc. (2) **30** (1984), 160–170.

[BG] J.W.Bruce and P.J.Giblin: *Outlines and their duals.* Proc. London Math. Soc. (3) **50** (1985), 552–570.

[C1] D.R.J.Chillingworth: *Bifurcation from a Manifold.* Singularity Theory and its Applications (Warwick 1989, Part II) (M.Roberts and I.Stewart, eds.), Lecture Notes in Math. **1463**, Springer-Verlag, Berlin 1991, 22–37.

[C2] D.R.J.Chillingworth: *Generic multiparameter bifurcation from a manifold.* To appear in: Dynamics and Stability of Systems, 2000.

[Ci] G.Cicogna: *Symmetry breakdown from bifurcations.* Lettere al Nuovo Cimento **31** (1981), 600–602.

[F] M.J.Field: *Lectures on Bifurcations, Dynamics and Symmetry.* Pitman Research Notes in Math. **356**, Longman, Harlow UK 1996.

[GS] M.Golubitsky, D.G.Schaeffer and I.N.Stewart: *Singularities and Groups in Bifurcation Theory.* Vols.I,II, Springer-Verlag, New York 1985,1988.

[H] J.K.Hale: *Generic bifurcations with applications.* Nonlinear analysis and mechanics: Heriot-Watt Symposium Vol.I (R.J.Knops, ed.), Research Notes in Math. **17**, Pitman, London 1977, 59–157.

[HT] J.K.Hale and P.Z.Taboas: *Interaction of damping and forcing in a second order equation*, Nonlin. Anal., Theory, Methods and Applicns. **2** (1978), 77–84.

[Ma] J.Mather: *Stability of C^∞ mappings VI.* Proc. Liverpool Singularities Symposium, Vol.I (C.T.C.Wall, ed.), Lecture Notes in Math. **192**, Springer-Verlag, Berlin 1971, 207–253.

[Mi] L.Michel: *Symmetry defects and broken symmetry. Configurations. Hidden symmetry.* Rev. Mod. Phys. **52** (1980), 617–651.

[PS] T.Poston and I.Stewart: *Catastrophe Theory and its Applications.* Pitman, London 1978.

[RT] R.Thom: *Stabilité Structurelle et Morphogénèse.* W.A.Benjamin, Reading, Mass., 1972 (English transl. 1975).

[V] A.Vanderbauwhede: *Local Bifurcation and Symmetry.* Research Notes in Math. **75**, Pitman, London 1982.